BULL. AUSTRAL. MATH. SOC. 16N60
VOL. 51 (1995) [377–381]

ON ADDITIVE MAPS OF PRIME RINGS

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Let R be a prime ring of characteristic not 2, C be the extended centroid of R, and $f\colon R \to R$ be an additive map. Suppose that $[f(x), x^2] = 0$ for all $x \in R$. Then there exist $\lambda \in C$ and an additive map $\zeta\colon R \to C$ such that $f(x) = \lambda x + \zeta(x)$ for all $x \in R$. In particular, if $f(x)^2 = x^2$ for all $x \in R$, then $\zeta = 0$ and either $\lambda = 1$ or $\lambda = -1$.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In the present paper we continue the series of papers concerning arbitrary additive maps of prime rings satisfying certain identities (see, for example [**1**, **2**, **3**, **4**] and references given there).

Throughout, R will be a prime ring with extended centroid C, and $f\colon R \to R$ will be an additive map. Let us mention three results from the recent papers [**1**, **2**, **4**]:

(I) If $[f(x), x] = 0$ for all $x \in R$, then there exist $\lambda \in C$ and an additive map $\zeta\colon R \to C$ such that $f(x) = \lambda x + \zeta(x)$, $x \in R$.

(II) If the characteristic of R is not 2 and $f(x)x + xf(x) = 0$ for all $x \in R$, then $f = 0$.

(III) If $[f(x), f(y)] = [x, y]$ for all $x, y \in R$, then there exists an additive map $\zeta\colon R \to C$ such that either $f(x) = x + \zeta(x)$, or $f(x) = -x + \zeta(x)$, $x \in R$.

The main goal of this paper is to prove

THEOREM 1. *If the characteristic of R is not 2 and $[f(x), x^2] = 0$ for all $x \in R$, then $[f(x), x] = 0$ for all $x \in R$. Therefore, there exist $\lambda \in C$ and an additive map $\zeta\colon R \to C$ such that $f(x) = \lambda x + \zeta(x)$, $x \in R$.*

Thus, we consider an identity that is certainly more general then those considered in (I) and (II). In fact, (II) can be derived at once from Theorem 1. Indeed, assuming that $f(x)x + xf(x) = 0$, $x \in R$, it follows from Theorem 1 that $f(x)x - xf(x) = 0$ and therefore $f(x)x = xf(x) = 0$, $x \in R$. Whence $f(x)y + f(y)x = 0$, $x, y \in R$; multiplying from the right by $f(x)$ we get $f(x)Rf(x) = 0$, $x \in R$, which yields $f = 0$.

As an application of Theorem 1 we shall obtain

Received 6 June 1994

Theorem 2. *If the characteristic of R is not 2 and $f(x)^2 = x^2$ for all $x \in R$, then either $f = I$ or $f = -I$, where I is the identity on R.*

Clearly, the condition $f(x)^2 = x^2$ is (at least when the characteristic of R is not 2) equivalent to the condition $f(x)f(y) + f(y)f(x) = xy + yx$. Therefore, Theorem 2 can be considered as a Jordan analogue of a Lie - type result (III).

2. Proofs

We shall make extensive use of the following well known result: If $a_i, b_i \in RC + C$ satisfy $\sum a_i x b_i = 0$ for all $x \in R$, then the a_i's as well as the b_i's are C-dependent, unless all $a_i = 0$ or all $b_i = 0$.

Defining $B(x, y) = [f(x), y]$, we see that $[f(x), x^2] = 0$ can be written as $B(x, x)x + xB(x, x) = 0$, $x \in R$. In the next lemma, motivated by some analogous considerations in [**3**], we treat a more general situation.

Lemma. *Let $n \geqslant 2$ and suppose that the characteristic of R is different from $2, 3, \ldots, n$. Let $B\colon R \times \ldots \times R \to R$ be a map, additive in each of the n arguments. If*

$$B(x, \ldots, x)x + xB(x, \ldots, x) = 0 \tag{1}$$

for all $x \in R$, then $x^{2n+2}B(x, \ldots, x) = B(x, \ldots, x)x^{2n+2} = 0$ for all $x \in R$.

Proof: Introducing $\widetilde{B}\colon R \times \ldots \times R \to R$ by

$$\widetilde{B}(x_1, \ldots, x_n) = \sum_{\pi \in S_n} B\big(x_{\pi(1)}, \ldots, x_{\pi(n)}\big)$$

and noting that $\widetilde{B}(x, \ldots, x) = n!B(x, \ldots, x)$, we see that there is no loss of generality in assuming that B is symmetric (that is, $B(x_1, \ldots, x_n) = B\big(x_{\pi(1)}, \ldots, x_{\pi(n)}\big)$ for each $\pi \in S_n$). Now set

$$B_i(y, x) = B\Bigg(\underbrace{y, \ldots, y}_{i}, \underbrace{x, \ldots, x}_{n-i}\Bigg),$$
$$b_i(x) = B_i\big(x^2, x\big) \qquad i = 0, \ldots, n.$$

Replacing x by $x + ky$, $k \in \mathbb{N}$, in (1), we get

$$ka_1(x, y) + \ldots + k^n a_n(x, y) = 0, \qquad x, y \in R,\ k \in \mathbb{N}$$

where

$$a_i(x, y) = \binom{n}{i}\big(B_i(y, x)x + xB_i(y, x)\big) + \binom{n}{i-1}\big(B_{i-1}(y, x)y + yB_{i-1}(y, x)\big) \tag{2}$$

https://doi.org/10.1017/S0004972700014209 Published online by Cambridge University Press

for $i = 1, \dots n$. Since the characteristic of R is different from $2, \dots, n$, it follows that $a_i(x, y) = 0$ [5, Lemma 1]. Taking x^2 for y in (2) and x^2 for x in (1) we obtain

$$(3)\qquad \binom{n}{i}(b_i(x)x + xb_i(x)) + \binom{n}{i-1}(b_{i-1}(x)x^2 + x^2 b_{i-1}(x)) = 0,$$

$$(4)\qquad b_n(x)x^2 + x^2 b_n(x) = 0.$$

Next, let us prove by induction on k that

$$(5)\qquad \binom{n}{n+1-k}\sum_{i=0}^{k}\binom{k}{i}x^{2i}b_{n+1-k}(x)x^{2k-2i} = 0, \qquad k = 1, \dots, n+1.$$

For $k = 1$ this is just relation (4). Suppose that (5) holds for some $k < n+1$. Multiply (5) first from the left and then from the right by x, sum up the identities so obtained, and use (3) to conclude that (5) holds for $k+1$.

Thus, in the case when $k = n+1$, we have

$$\sum_{i=0}^{n+1}\binom{n+1}{i}x^{2i}B(x, \dots, x)x^{2(n+1)-2i} = 0.$$

Since $B(x, \dots, x)$ commutes with x^2, we get $2^{n+1}B(x, \dots, x)x^{2n+2} = 0$, proving the lemma. $\square$

Proof of Theorem 1: Replacing x by $x \pm y$ in $[f(x), x^2] = 0$ we get

$$(6)\qquad [f(x), xy + yx] + [f(y), x^2] = 0, \qquad x, y \in R$$

and hence

$$(7)\qquad [f(x), yz + zy] + [f(y), zx + xz] + [f(z), xy + yx] = 0, \qquad x, y, z \in R.$$

Pick $z \in R$ such that $z^2 = 0$. Our intention is to prove that there exist $\lambda, \mu \in C$ such that $f(z) = \lambda z + \mu$. By (6) we have

$$(8)\qquad [f(z), zy + yz] = 0$$

for each $y \in R$. Replacing y by yz we obtain $f(z)zyz - zyzf(z) = 0$, $y \in R$. Therefore, $f(z)z = \mu z = zf(z)$ for some $\mu \in C$. Using this in (8), we get $zy(\mu - f(z)) + (f(z) - \mu)yz = 0$ for all $y \in R$. Consequently, there is $\lambda \in C$ such that $f(z) - \mu = \lambda z$, as desired.

Define $q(x) = [f(x), x]$ and note that

$$(9)\qquad q(x)x + xq(x) = 0, \qquad x \in R.$$

https://doi.org/10.1017/S0004972700014209 Published online by Cambridge University Press

Suppose that $x \in R$ is such that $q(x)x^k = x^k q(x) = 0$ for some $k > 1$. Let us show that this yields $q(x)x^{k-1} = x^{k-1}q(x) = 0$. Set $z = q(x)x^{k-1}$ and note that $z^2 = xz = zx = 0$ and $f(z) = \lambda z + \mu$ for some $\lambda, \mu \in C$. Substituting xr for y in (7), where $r \in R$, we obtain

$$[f(x), xrq(x)x^{k-1}] + \lambda[q(x)x^{k-1}, x^2r + xrx] = 0,$$

that is,

$$\big(f(x)x - \lambda x^2\big)rq(x)x^{k-1} - xrq(x)x^{k-1}f(x) = 0.$$

Therefore, either $q(x)x^{k-1} = 0$ or $f(x)x - \lambda x^2$ and x are C-dependent. But in the latter case we clearly have $[f(x)x, x] = 0$, that is $q(x)x = 0$. Thus, $q(x)x^{k-1} = x^{k-1}q(x) = 0$ in any case.

Note that (9) and the Lemma tell us that $q(x)x^6 = x^6 q(x) = 0$ for all $x \in R$. But then, by the arguments just given, we have $q(x)x = xq(x) = 0$ for all $x \in R$. Replacing x by $x \pm y$ in $q(x)x = 0$ we arrive at

$$q(x)y + [f(x), y]x + [f(y), x]x = 0, \qquad x, y \in R.$$

Multiplying from the right by $q(x)$ it follows that $q(x)yq(x) = 0$ for all $x, y \in R$, and hence $q(x) = 0$. Apply (I) and proof is complete. ∎

Proof of Theorem 2: Obviously, $[f(x), x^2] = 0$, so that $f(x) = \lambda x + \zeta(x)$ by Theorem 1. Therefore, $f(x)^2 = x^2$ can be written as

(10) $$\big(\lambda^2 - 1\big)x^2 + 2\lambda\zeta(x)x + \zeta(x)^2 = 0 \qquad \text{for all } x \in R.$$

Suppose first that $\lambda^2 = 1$. Then we have $\zeta(x)(2\lambda x + \zeta(x)) = 0$, $x \in R$. Thus, either $\zeta(x) = 0$ or x lies in the centre of R. Since both the centre of R and the kernel of ζ are additive subgroups of R, it follows that either $\zeta = 0$ or R is commutative. In any case the result follows immediately.

Thus, the proof will be completed by showing that the possibility $\lambda^2 \neq 1$ cannot occur.

If $\lambda^2 \neq 1$, then (10) shows that for any $x \in R$ there is a polynomial $X^2 + \alpha X + \beta \in C[X]$ satisfied by x (that is, R is algebraic of bounded degree 2 over C). It is known by standard PI theory that this is equivalent to the condition that either R is commutative or R embeds in $M_2(F)$ for a field F containing C. Therefore, without loss of generality we may assume that R is a subring of $M_2(F)$. Let $\operatorname{tr} x$ denote the trace of the matrix x and $\det x$ its determinant. We have $x^2 - x \operatorname{tr} x + \det x = 0$. Clearly, if the matrix x is not scalar and $x^2 + \alpha x + \beta = 0$, then $\alpha = -\operatorname{tr} x$ and $\beta = \det x$. According to (10), for each nonscalar matrix $x \in R$ we have

$$\frac{2\lambda\zeta(x)}{\lambda^2 - 1} = -\operatorname{tr} x \qquad \text{and} \qquad \frac{\zeta(x)^2}{\lambda^2 - 1} = \det x,$$

which gives

$$\operatorname{tr}^2(x) = \gamma \det x, \tag{11}$$

where $\gamma = 4\lambda^2(\lambda^2 - 1)^{-1}$. Let $\mu \in R$ be a scalar matrix. For each nonscalar matrix $x \in R$ the matrix $x + \mu$ is also nonscalar and so $\operatorname{tr}^2(x + \mu) = \gamma \det(x + \mu)$. Since $\operatorname{tr}(x + \mu) = \operatorname{tr} x + 2\mu$ and $\det(x + \mu) = \det x + \mu \operatorname{tr} x + \mu^2$ and since (11) holds for x, we get

$$\mu(4 - \gamma)(\operatorname{tr} x + \mu) = 0 \tag{12}$$

for all nonscalar matrices $x \in R$. Note that $\gamma = 4\lambda^2(\lambda^2 - 1)^{-1} \neq 4$. Thus, we have either $\mu = -\operatorname{tr} x$ for all nonscalar $x \in R$, or $\mu = 0$. If $\operatorname{tr} x = 0$ for all such x, then we clearly have $\mu = 0$. If $\operatorname{tr} x \neq 0$, then $\operatorname{tr} 2x \neq \operatorname{tr} x$ and so it follows $\mu = 0$ again. These arguments show that 0 is the only scalar matrix in R, whence (11) holds for all $x \in R$.

If $\det x = 0$ for all $x \in R$, then also $\operatorname{tr} x = 0$, which leads to $x^2 = 0$, contrary to the primeness of R. Thus, we may assume that $\det x \neq 0$ for some $x \in R$. Note that $\operatorname{tr} x^2 = \operatorname{tr}^2 x - 2\det x$ and $\operatorname{tr} x^3 = \operatorname{tr}^3 x - 3\operatorname{tr} x \det x$. Whence, applying (11) we see that

$$\gamma \det^2 x = \gamma \det x^2 = \operatorname{tr}^2 x^2 = \left(\operatorname{tr}^2 x - 2\det x\right)^2 = (\gamma - 2)^2 \det^2 x,$$
$$\gamma \det^3 x = \gamma \det x^3 = \operatorname{tr}^2 x^3 = \left(\operatorname{tr}^3 x - 3\operatorname{tr} x \det x\right)^2 = \gamma(\gamma - 3)^2 \det^3 x.$$

As $\det x \neq 0$, it follows that $\gamma = (\gamma - 2)^2 = \gamma(\gamma - 3)^2$. This gives $\gamma = 4$, which is impossible as noticed above. Thus we have proved indeed that $\lambda^2 = 1$. The proof of Theorem 2 is complete. $\square$

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