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COMPOSITION OPERATORS ON WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

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Abstract

We give algebraic criteria for distinguishing composition operators among all continuous linear operators on spaces of continuous functions with topologies generated by seminorms that are weighted analogues of the supremum norm. In another direction, we also characterize those self maps of the underlying topological space which induce composition operators on such weighted spaces, as well as determine conditions on these self maps which correspond to various basic properties of the induced composition operator. Our results are applied to a question concerning translation invariance which arises in the context of topological dynamics.

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If $L(T)$ is a topological vector space of scalar valued functions on a non-void set T, then each function $\varphi\colon T \to T$ induces a natural linear mapping C_φ from $L(T)$ into the vector space $\mathsf{F}(T)$ of all scalar valued functions on T defined by

$$C_\varphi f = f \circ \varphi,$$

$f \in L(T)$. In case the range of C_φ is contained in $L(T)$ and $C_\varphi\colon L(T) \to L(T)$ is continuous, C_φ is said to be a composition operator on $L(T)$. More generally, if M is any vector subspace of $L(T)$ which happens to be invariant under C_φ, the corresponding map $\widehat{C}_\varphi\colon L(T)/M \to \mathsf{F}(T)/M$, where $\widehat{C}_\varphi(f+M) = C_\varphi f + M$ for each $f \in L(T)$, is said to be a composition operator on the quotient space $L(T)/M$ whenever $\widehat{C}_\varphi$ is a continuous map from $L(T)/M$ into $L(T)/M$.

Interest in composition operators at least traces back to an article on classical mechanics by B. O. Koopman [6], but the concept was not singled out for systematic study until about two decades ago. In view of the early applications (for example, see [6], [7]), it is not surprising that these investigations initially focused on the context provided by the classical L_2-spaces and functional Hilbert spaces such as $H_2(\mathbf{D})$; we refer to the survey by E. A. Nordgren [10] for details. Composition operators on general L_p-spaces and Banach spaces of analytic functions have also been the subject of considerable attention in a second phase of the study which has import for such topics as ergodic theory, Markov processes, and statistical mechanics (for example, see [4], [8]). Spaces of continuous functions equipped with topologies induced by weighted analogues of the supremum norm offer yet another context in which composition operators hold interest (for example, see [5], [13]). However, while many of the results for Hilbert space carry over with minor modification to the L_p-spaces or, as the case may be, some other natural Banach space counterpart (see [10, page 37]), the Hilbert space case offers much less insight into what might be expected in this third setting, and only limited progress in that direction has heretofore been realized.

In this paper, we take up the study of composition operators on two related classes of weighted continuous function spaces with topologies deriving from the supremum norm; namely, the spaces of type $CV_0(T)$ and $CV_b(T)$. The framework provided by these classes is broad enough to include many of the standard continuous function spaces encountered in analysis (see [14, page 123] for a partial list of explicit instances), as well as more specialized spaces that arise in connection, say, with age-dependent population models (see [15]) or the theory of analytically uniform spaces (see [1]), and our development in the sequel will thus apply in a wide range of situations. Following a preliminary section, we begin in Section 2 by characterizing those mappings on the underlying topological space T which serve to induce composition operators on the corresponding weighted spaces $CV_0(T)$ and $CV_b(T)$. The problem of identifying composition operators on $CV_0(T)$ among all continuous linear operators on this space together with the companion problem for $CV_b(T)$ will then be resolved in Section 3, while basic properties of composition operators in the setting at hand will be considered in the fourth and final section.

1. Preliminaries

In what follows, T will always denote a completely regular Hausdorff space. Any upper semicontinuous function $v\colon T \to \mathbf{R}^+$ will be called a *weight* on T. If V is a set of weights on T such that, given any $t \in T$, there is some $v \in V$ for which $v(t) \succ 0$, we write $V \succ 0$. A set V of weights on T is said to be *directed* *upward* provided that, for every pair $v_1, v_2 \in V$ and each $\lambda \succ 0$, there exists $v \in V$ so that $\lambda v_i \preccurlyeq v$ (pointwise on T) for $i = 1, 2$. Since there is no loss of generality, we hereafter assume that sets of weights are directed upward; a set V of weights on T which additionally satisfies $V \succ 0$ will be referred to as a *system of weights* on T.

We will write $C(T)$ to indicate the collection of all scalar valued continuous functions on T, where the scalar field $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$; there will be no loss of generality in tacitly assuming that $\mathbf{K} = \mathbf{C}$. Now, taking a system V of weights on T, we put

$$CV_0(T) = \{f \in C(T) \colon vf \text{ vanishes at infinity on } T \text{ for all } v \in V\}$$

and

$$CV_b(T) = \{f \in C(T) \colon vf \text{ is bounded for all } v \in V\}.$$

Obviously, $CV_0(T)$ and $CV_b(T)$ are vector spaces (over $\mathbf{K}$), while the upper semicontinuity of the weights yields that $CV_0(T) \subseteq CV_b(T)$. For $v \in V$ and $f \in C(T)$, let us now put

$$p_v(f) = \sup\{v(t)|f(t)| \colon t \in T\}.$$

Then p_v can be regarded as a seminorm on either $CV_b(T)$ or $CV_0(T)$, and we assume that each of these two spaces is equipped with the Hausdorff locally convex topology induced by $\{p_v \colon v \in V\}$. As a matter of notational convenience, given $v \in V$, the closed unit ball corresponding to the seminorm p_v in either $CV_0(T)$ or $CV_b(T)$ will be denoted by B_v; this ambiguity should occasion no difficulty since the setting will always be clear from context.

If U and V are two systems of weights on T, we write $U \preccurlyeq V$ whenever, given $u \in U$, there exists $v \in V$ such that $u \preccurlyeq v$. In this case, we then clearly have that $CV_0(T) \subseteq CU_0(T)$ and $CV_b(T) \subseteq CU_b(T)$, as well as that the inclusion map is continuous in both instances. When $U \preccurlyeq V$ and $V \preccurlyeq U$, consequently, we say that U and V are equivalent systems of weights on T and indicate this by writing $U \sim V$.

As noted earlier, many standard spaces of continuous functions can be realized in the context set forth above (see [14]), and we will pause to consider various (other) special cases as we go along and the occasion demands. At this point, however, let us only mention that if $v \colon T \to \mathbf{R}^+$ is any weight satisfying $v(t) \succ 0$ for every $t \in T$, then $V = \{\lambda v \colon \lambda \succ 0\}$ is a system of weights on T for which both $CV_0(T)$ and $CV_b(T)$ are normed vector spaces with topologies finer than that of pointwise convergence. In keeping with standard usage, we shall indicate the special case where $v(t) = 1$ for each $t \in T$ by simply writing $C_0(T)$ and $C_b(T)$ in place of $CV_0(T)$ and $CV_b(T)$, respectively.

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2. Functions inducing composition operators

In developing our characterizations of those functions $\varphi\colon T \to T$ which induce composition operators on weighted spaces of type $CV_0(T)$ and $CV_b(T)$, we work under the modest requirement that

(2.a) T is a completely regular Hausdorff space,

(2.b) V is a system of weights on T, and

(2.c) corresponding to each $t \in T$, there exists $f_t \in CV_0(T)$ such that $f_t(t) \neq 0$;

these basic assumptions will be in force throughout the present section and those that follow.

Conditions (2.a), (2.b), and (2.c) primarily serve to exclude trivial cases along with certain (often unnecessary) pathological situations (for example, see [3]). Of course, (2.c) is automatically fulfilled if T happens to be locally compact. In view of (2.c), moreover, it readily follows from [9, Lemma 2, page 69] that if $\varphi\colon T \to T$ is any function for which the corresponding composition map C_φ induced on $CV_0(T)$ has its range contained in $C(T)$, then φ is necessarily continuous. For a continuous function $\varphi\colon T \to T$, furthermore,

$$V(\varphi) = \{v \circ \varphi \colon v \in V\}$$

is clearly a system of weights on T.

2.1. LEMMA. *If $\varphi\colon T \to T$ induces a continuous composition map $C_\varphi\colon CV_0(T) \to CV_b(T)$, then φ is continuous, $V(\varphi)$ is a system of weights on T, and $V \preccurlyeq V(\varphi)$.*

PROOF. As observed above, φ is necessarily continuous whence $V(\varphi)$ is a system of weights on T, and so it will suffice to show that $V \preccurlyeq V(\varphi)$. To this end, fixing $v \in V$, let us choose $u \in V$ such that $C_\varphi(B_u) \subseteq B_v$; we claim that $v \preccurlyeq 2u \circ \varphi$. For this, we take $t_0 \in T$ and set $s = \varphi(t_0)$. In case $u(s) \succ 0, G = \{t \in T\colon u(t) \prec 2u(s)\}$ is an open neighborhood of $s = \varphi(t_0)$. Thus, according to [9, Lemma 2, page 69], there exists $f \in CV_0(T)$ such that $0 \preccurlyeq f \preccurlyeq 1, f(s) = 1$, and $f(T \backslash G) = 0$, and therefore $g = (2u(s))^{-1} f \in B_u$. Since this yields that $g(\varphi(t))v(t) \preccurlyeq 1$ for every $t \in T$, we conclude that $v(t_0) \preccurlyeq 2u(\varphi(t_0))$. On the other hand, suppose that $u(s) = 0$ and $v(t_0) \succ 0$. If we then put $\varepsilon = v(t_0)/2$ and set $G = \{t \in T\colon u(t) \prec \varepsilon\}, G$ would be an open neighborhood of s, and we could again find $f \in CV_0(T)$ such that $0 \preccurlyeq f \preccurlyeq 1, f(s) = 1$, and $f(T \backslash G) = 0$. But this would mean that $C_\varphi(\varepsilon^{-1} f) \in B_v$, which is clearly impossible. Having thus established our claim, the proof is complete.

We now proceed to characterize those mappings $\varphi\colon T \to T$ which induce composition operators on $CV_b(T)$.

2.2. THEOREM. *A function $\varphi: T \to T$ induces a composition operator C_φ on $CV_b(T)$ if, and only if, φ is continuous and $V \preccurlyeq V(\varphi)$.*

PROOF. Since necessity follows from Lemma 2.1, let us suppose that φ is continuous and $V \preccurlyeq V(\varphi)$. Then, given $f \in CV_b(T)$ and $v \in V$, if we choose $u \in V$ such that $v \preccurlyeq u \circ \varphi$, we have that

$$p_v(f \circ \varphi) \preccurlyeq p_{u\circ\varphi}(f \circ \varphi) \preccurlyeq p_u(f),$$

which immediately yields the desired conclusion.

Assuming that $\varphi: T \to T$ is continuous and $V \preccurlyeq V(\varphi)$, the foregoing result shows that φ would as well induce a composition operator on $CV_0(T)$ if only $CV_0(T)$ were to be invariant under the composition operator $C_\varphi: CV_b(T) \to CV_b(T)$. This need not be the case, however, as can be seen by taking T to be the natural numbers $\mathbb{N}$ with the discrete topology and V to be the set of all nonnegative constant functions on T, whereby $CV_0(T)$ is the classical Banach sequence space c_0. For a constant function $\varphi: T \to T$, say $\varphi(t) = 1$ for every $t \in T$, it is obvious that $V \preccurlyeq V(\varphi)$, but $f \circ \varphi \in c_0$ in this case only when $f: \mathbb{N} \to \mathbb{K}$ is a sequence such that $f(1) = 0$.

Something more must therefore be required of a function $\varphi: T \to T$ if φ is to induce a composition operator on $CV_0(T)$. For the purpose of formulating a suitable condition, if $v: T \to \mathbb{R}^+$ is any weight on T and $\varepsilon \succ 0$, we put $N(v, \varepsilon) = \{t \in T: v(t) \succcurlyeq \varepsilon\}$. Of course, since v is upper semicontinuous, $N(v, \varepsilon)$ is a closed subset of T.

2.3. THEOREM. *For a function $\varphi: T \to T$, the following are equivalent:*

1. *φ induces a composition operator on $CV_0(T)$;*

2. *φ induces a composition operator C_φ on $CV_b(T)$ and $CV_0(T)$ is invariant under C_φ;*

3. (i) *φ is continuous,* (ii) *$V \preccurlyeq V(\varphi)$, and* (iii) *for each $v \in V$, $\varepsilon \succ 0$, and compact set $K \subseteq T$, $\varphi^{-1}(K) \cap N(v, \varepsilon)$ is compact;*

4. (i) *φ is continuous,* (ii) *$V \preccurlyeq V(\varphi)$, and* (iii) *for each $v \in V$, $\varepsilon \succ 0$, and $u \in V$ such that $v \preccurlyeq u \circ \varphi$, $\varphi^{-1}(K) \cap N(v, \varepsilon)$ is compact whenever K is a compact subset of $N(u, \varepsilon)$.*

PROOF. Let us first assume that 1. holds. In this case, 3(i) and 3(ii) follow from Lemma 2.1. To see that 3(iii) is also satisfied, fix $\varepsilon \succ 0$, take $v \in V$, and let K be a compact subset of T. According to [9, Lemma 2, page 69], there exists $f \in CV_0(T)$ such that $0 \preccurlyeq f \preccurlyeq 1$ and $f(K) = 1$. Since $f \circ \varphi \in CV_0(T), C = \{t \in T: v(t) f(\varphi(t)) \succcurlyeq \varepsilon\}$ is compact, and this serves to establish 3(iii) in view of the fact that $\varphi^{-1}(K) \cap N(v, \varepsilon) \subseteq C$. Noting that 4 is an obvious consequence of 3, we now assume that 4 holds. This being the case, φ induces

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a composition operator C_φ on $CV_b(T)$ by Theorem 2.2, and we would further show that $CV_0(T)$ is invariant under C_φ. To this end, given $f \in CV_0(T)$, $v \in V$, and $\varepsilon \succ 0$, let us consider the set $F = \{t \in T\colon v(t)|f(\varphi(t))| \succcurlyeq \varepsilon\}$, which we may suppose to be nonvoid. Next, choosing $u \in V$ so that $v \preccurlyeq u \circ \varphi$, we put $K = \{t \in T\colon u(t)|f(t)| \succcurlyeq \varepsilon\}$. Then K is compact, $\varphi(F) \subseteq K$, $M = \sup\{|f(t)|\colon t \in K\} \succ 0$, and $K \subseteq N(u, \varepsilon/M)$. In view of 4.(iii), therefore, $\varphi^{-1}(K) \cap N(v, \varepsilon/M)$ is compact, whereby F must as well be compact since $F \subseteq N(v, \varepsilon/M)$, and we thus conclude that $C_\varphi f \in CV_0(T)$. Since 2 clearly implies 1, the argument is now complete.

REMARK. In case $\varphi\colon T \to T$ is a (surjective) homeomorphism, for example, Condition 3.(iii) of Theorem 2.3 is necessarily satisfied, and such a function φ will therefore induce composition operators on both $CV_0(T)$ and $CV_b(T)$ as soon as $V \preccurlyeq V(\varphi)$.

Theorem 2.3 provides some insight into a question that arises in the context of topological dynamics. The setting is a special case of that under consideration, and in the remainder of the section we shall take T to be the real line $\mathbf{R}$ (with the usual topology) and V to be a system of weights on $\mathbf{R}$ generated by a single *continuous* weight $v\colon \mathbf{R} \to \mathbf{R}^+$ such that $v(t) \succ 0$ for each $t \in \mathbf{R}$ (that is, $V = \{\lambda v\colon \lambda \succ 0\}$). Further, for each $\omega \in \mathbf{R}$, we put $\varphi_\omega(t) = t + \omega, t \in \mathbf{R}$. If $\pi\colon \mathbf{R} \times CV_0(\mathbf{R}) \to C(\mathbf{R})$ is the function defined by setting $\pi(\omega, f) = f \circ \varphi_\omega$ for $\omega \in \mathbf{R}$ and $f \in CV_0(\mathbf{R})$, there is a question (for example, see [11, Example 3.2]) as to when π is a dynamical system on the weighted Banach space $CV_0(\mathbf{R})$. For this to be the case, of course, each translation $\varphi_\omega\colon \mathbf{R} \to \mathbf{R}, \omega \in \mathbf{R}$, must necessarily induce a composition operator on $CV_0(\mathbf{R})$, and the foregoing remark asserts that this will occur if $V \preccurlyeq V(\varphi_\omega)$ for each $\omega \in \mathbf{R}$. We now proceed to show that this condition is also sufficient.

To begin, if $V \preccurlyeq V(\varphi_\omega)$ for some $\omega \in \mathbf{R}$, we put

$$\lambda(\omega) = \inf\{\lambda \succcurlyeq 0\colon v \preccurlyeq \lambda v \circ \varphi_\omega\},$$

which is the same as setting $\lambda(\omega) = \sup\{v(t)/v(t+\omega)\colon t \in \mathbf{R}\}$.

2.4. LEMMA. *If $V \preccurlyeq V(\varphi_\omega)$ for every $\omega \in \mathbf{R}$, then there exist $\delta \succ 0$ and a function $M\colon \mathbf{R} \to \mathbf{R}^+$ such that, given $\omega \in \mathbf{R}, \lambda(\tau) \preccurlyeq M(\omega)$ whenever $\tau \in \mathbf{R}$ with $|\tau - \omega| \prec \delta$.*

PROOF. For each $n \in \mathbf{N}$, put $F_n = \{\tau \in \mathbf{R}\colon \lambda(\tau) \preccurlyeq n\}$. Since F_n is closed in $\mathbf{R}$ for every $n \in \mathbf{N}$ and $\mathbf{R} = \bigcup_{n=1}^\infty F_n$, we invoke the Baire category theorem to obtain $m \in \mathbf{N}$ for which $\operatorname{int}(F_m) \neq \varnothing$, and accordingly choose $\sigma \in F_m$ and $\delta \succ 0$ so that $(\sigma - \delta, \sigma + \delta) \subseteq F_m$. Fixing $\omega \in \mathbf{R}$, we now set $M(\omega) = m\lambda(\omega - \sigma)$. For

$\tau \in \mathbf{R}$ such that $|\omega - \tau| \prec \delta$ and any $t \in \mathbf{R}$, we then have that

$$\frac{v(t)}{v(t+\tau)} = \frac{v(t)}{v(t+\tau-\omega+\sigma)} \cdot \frac{v(t+\tau-\omega+\sigma)}{v(t+\tau)}$$
$$\preccurlyeq \lambda(\tau-\omega+\sigma)\lambda(\omega-\sigma) \preccurlyeq M(\omega),$$

whereby $\lambda(\tau) \preccurlyeq M(\omega)$.

As remarked above, if $V \preccurlyeq V(\varphi_\omega)$ for each $\omega \in \mathbf{R}$, then $\pi(\omega, f) = f \circ \varphi_\omega \in CV_0(\mathbf{R})$ for every $\omega \in \mathbf{R}$ and all $f \in CV_0(\mathbf{R})$.

2.5. LEMMA. *Assume that $V \preccurlyeq V(\varphi_\omega)$ for each $\omega \in \mathbf{R}$. Then, given $f \in CV_0(\mathbf{R})$, the function $\pi(\cdot, f)\colon \mathbf{R} \to CV_0(\mathbf{R})$ is continuous.*

PROOF. We being by fixing $\omega \in \mathbf{R}$ and $\varepsilon \succ 0$. Applying Lemma 2.4, we next choose $\delta_1 \succ 0$ and $M(\omega) \succ 0$ so that $\lambda(\tau) \preccurlyeq M(\omega)$ for all $\tau \in \mathbf{R}$ with $|\omega - \tau| \prec \delta_1$. Since $vf \in C_0(\mathbf{R})$, moreover, there exists $\delta_2 \succ 0$ such that

$$|v(s)f(s) - v(t)f(t)| \prec \varepsilon/2M(\omega)$$

for $s, t \in \mathbf{R}$ such that $|s - t| \prec \delta_2$, and there also exists $t_0 \in \mathbf{R}^+$ so that

$$v(t)|f(t)| \prec \varepsilon/4M(\omega) \tag{2.d}$$

when $t \in \mathbf{R}$ and $|t| \succ t_0$. Finally, setting $\eta = \inf\{v(t)\colon |t| \preccurlyeq t_0\}$, there exists $\delta_3 \succ 0$ such that if $s, t \in \mathbb{R}$ with $|s|, |t| \preccurlyeq t_0 + 1$ and $|s - t| \prec \delta_3$, then

$$|v(s) - v(t)| \prec \eta\varepsilon/[2(p_v(f)+1)M(\omega)]. \tag{2.e}$$

We now put $\delta = \min\{\delta_1, \delta_2, \delta_3, 1\}$, and take $\tau \in \mathbf{R}$ so that $|\tau - \omega| \prec \delta$. For any $t \in \mathbf{R}$, since it follows from either (2.d) or (2.e) that

$$v(t+\omega)|f(t+\omega)| \left| \frac{v(t)}{v(t+\tau)} - \frac{v(t)}{v(t+\omega)} \right| \prec \varepsilon/2,$$

we then have that

$$v(t)|f(t+\tau) - f(t+\omega)| \preccurlyeq \frac{v(t)}{v(t+\tau)}|v(t+\tau)f(t+\tau) - v(t+\omega)f(t+\omega)| + \frac{\varepsilon}{2} \prec \varepsilon,$$

as required to complete the proof.

2.6. THEOREM. *Given $v \in C(\mathbf{R})$ such that $v(t) \succ 0$ for each $t \in \mathbf{R}$, let $V = \{\lambda v\colon \lambda \succ 0\}$, and consider the function $\pi\colon \mathbf{R} \times CV_0(\mathbf{R}) \to C(\mathbf{R})$ defined by setting $\pi(\omega, f) = f \circ \varphi_\omega$ for $\omega \in \mathbf{R}$ and $f \in CV_0(\mathbf{R})$, where $\varphi_\omega(t) = t + \omega, t \in \mathbf{R}$. The following are then equivalent:*

1. *π is a (linear) dynamical system on $CV_0(\mathbf{R})$;*
2. *$V \preccurlyeq V(\varphi_\omega)$ for every $\omega \in \mathbf{R}$;*
3. *$C_{\varphi_\omega}(CV_0(\mathbf{R})) \subseteq CV_b(\mathbf{R})$ for each $\omega \in \mathbf{R}$.*

PROOF. Assertion 3 obviously follows from 1., while the implication $2 \Rightarrow 1$ is an immediate consequence of Lemma 2.5 and (the remark following) Theorem 2.3 since π clearly satisfies the requisite group properties. In order to show that 3 implies 2, let us suppose that 2 fails to hold. There would then exist $\omega \in \mathbf{R}$ and a sequence (t_n) in $\mathbf{R}$ such that $v(t_n) \succcurlyeq n^2 v(t_n + \omega)$ for each $n \in \mathbf{N}$, and we may further suppose that there exists $g \in C_0(\mathbf{R})$ for which $g(t_n + \omega) = 1/n$, $n \in \mathbf{N}$. If we now set $f = g/v$, then $f \in CV_0(\mathbf{R})$. Since

$$v(t_n)|f(t_n + \omega)| \succcurlyeq n^2 g(t_n + \omega) = n$$

for each $n \in \mathbf{N}$, however, $f \circ \varphi_\omega \notin CV_b(\mathbf{R})$, and so the proof is complete.

In bringing the section to an end, we note two examples which help to illustrate the situation:

(i) If we put $v(t) = \exp(-|t|)$ for $t \in \mathbf{R}$ and take any $\omega \in \mathbf{R}$, then $V \preccurlyeq V(\varphi_\omega)$ since $v(t) \preccurlyeq \exp(|\omega|)v(t + \omega)$ for all $t \in \mathbf{R}$ (see [11, Example 3.2]);

(ii) If $v(t) = \exp(-t^2)$ for $t \in \mathbf{R}$, however, it is obvious that $V \preccurlyeq V(\varphi_\omega)$ only if $\omega = 0$.

3. Characterization of composition operators

In view of Condition (2.b), the point evaluation $\delta(t)$ corresponding to any $t \in T$ defines a continuous linear functional on either $CV_0(T)$ or $CV_b(T)$. Thus, putting $\Delta(T) = \{\delta(t): t \in T\}$, we may regard $\Delta(T)$ as a subset of either the continuous dual $CV_0(T)'$ or $CV_b(T)'$. This observation readily leads to a description of the continuous composition maps from $CV_0(T)$ into $CV_b(T)$ which parallels a standard result for functional Hilbert spaces (see [10, page 46]).

3.1. THEOREM. *For a linear transformation $\Phi: CV_0(T) \to CV_b(T)$, there exists $\varphi: T \to T$ such that $\Phi = C_\varphi$ if, and only if, the transpose mapping Φ^* from $CV_b(T)'$ into the algebraic dual $CV_0(T)^*$ leaves $\Delta(T)$ invariant. In case $\Phi^*(\Delta(T)) \subseteq \Delta(T)$, moreover, φ is necessarily continuous, and $\Phi = C_\varphi$ is continuous if, and only if, $V \preccurlyeq V(\varphi)$.*

PROOF. Assume that $\Phi = C_\varphi$ for some $\varphi: T \to T$, if $t \in T$ and $f \in CV_0(T)$. Then

$$\langle f, \Phi^*(\delta(t))\rangle = \langle \Phi(f), \delta(t)\rangle = \langle f \circ \varphi, \delta(t)\rangle = \langle f, \delta(\varphi(t))\rangle,$$

whereby $\Phi^*(\delta(t)) = \delta(\varphi(t))$. Conversely, let us assume that $\Phi^*(\Delta(T)) \subseteq \Delta(T)$. For $t \in T$, if we take $\varphi(t)$ to be the (unique) element of T such that $\Phi^*(\delta(t)) = \delta(\varphi(t))$ and consider any $f \in CV_0(T)$, then

$$\Phi(f)(t) = \langle f, \Phi^*(\delta(t))\rangle = \langle f, \delta(\varphi(t))\rangle = f(\varphi(t)),$$

as desired. Furthermore, as noted previously, φ is continuous since $C_\varphi\colon CV_0(T) \to CV_b(T) \subseteq C(T)$, while $\Phi = C_\varphi$ is continuous exactly when $V \preccurlyeq V(\varphi)$ in view of Lemma 2.1 and Theorem 2.2.

REMARK. If $CV_0(T)$ is everywhere replaced by $CV_b(T)$ in the statement of Theorem 3.1, this yields a companion result which can be established by (essentially) the same argument.

Theorem 2.6 provides one instance where the condition $V \preccurlyeq V(\varphi)$ is automatically satisfied if $C_\varphi\colon CV_0(T) \to CV_b(T)$, and the question arises as to whether this would always be the case. As we note in the following example, however, the answer is negative.

3.2. EXAMPLE. Let $\mathbb{N}^*$ denote the one-point compactification of $\mathbb{N}$, and consider the compact Hausdorff space $T = \mathbb{N}^* \times \mathbb{N}^*$. For each $\gamma \in \mathbb{N}^*$, setting

$$u_\gamma(\alpha,\beta) = \begin{cases} 1, & \beta = \gamma \\ 0, & \beta \neq \gamma \end{cases}$$

for each $(\alpha, \beta) \in T$ defines a weight on T; we put

$$V = \left\{ \lambda \sum_{\gamma\in F} u_\gamma \colon \lambda \succ 0; F \subseteq \mathbb{N}^*, F \text{ finite} \right\}.$$

Then V is a system of weights on T. As sets, moreover, both $CV_0(T)$ and $CV_b(T)$ coincide with $C(T)$, and thus the conditions (2.a), (2.b), and (2.c) are all satisfied, while every continuous function $\varphi\colon T \to T$ induces a map $C_\varphi\colon CV_0(T) \to CV_b(T)$. At this point, let us consider the function $\varphi\colon T \to T$ defined by setting $\varphi(\alpha,\beta) = (\beta,\alpha)$, $(\alpha,\beta) \in T$. Clearly, φ is continuous (indeed, even a surjective homeomorphism), but it is not the case that $V \preccurlyeq V(\varphi)$. In fact, since $u_1(k,1) = 1$ for every $k \in \mathbb{N}$, we need only observe that, corresponding to each $v \in V$, there exists $k \in \mathbb{N}$ for which $v(1,k) = 0$.

One description of those continuous linear operators on $CV_0(T)$ which happen to be composition operators immediately follows from Theorem 3.1. From several points of view, however, a characterization entirely in terms of the operator itself would be preferable. We next give a deeper result which will serve to resolve the problem in a manner reflecting the classical work in this direction (for example, see [2, page 142]).

3.3. THEOREM. *Let $\Phi\colon CV_0(T) \to CV_b(T)$ be a continuous linear operator. Then there exists $\varphi\colon T \to T$ such that $\Phi = C_\varphi$ if, and only if, the following two conditions are satisfied:*

(i) *for each $t \in T$, there exists $g \in CV_0(T)$ such that $\Phi(g)(t) \neq 0$;*

(ii) *$\Phi(fg) = \Phi(f)\Phi(g)$ whenever $f, g \in CV_0(T) \cap C_b(T)$.*

PROOF. Since (i) and (ii) are obviously necessary, we need only check sufficiency, and so let us assume that both (i) and (ii) are satisfied. In any event, moreover, we have that $A = CV_0(T) \cap C_b(T)$ is a selfadjoint subalgebra of $C_b(T)$ which separates the points of T. Now, fixing $t \in T$, we put

$$W(t) = \{f \in A \colon \Phi(f)(t) = 0\}.$$

Then $W(t)$ is a vector subspace of $CV_0(T)$, and it follows from (ii) that $W(t)$ is a module over A. According to (i), furthermore, $\mathrm{cl}(W(t))$ is a proper subset of $CV_0(T)$. Thus, since we have an instance of the bounded case of the weighted approximation problem, we conclude from [9, Theorem 1, page 106] that there exists $s \in T$ such that $f(s) = 0$ for every $f \in W(t)$. Suppose that there as well exists $r \in T$ with $r \neq s$ such that $f(r) = 0$ for all $f \in W(t)$. Then, taking $g \in A$ so that $g(r) = 1$, we also choose $h \in A$ such that $h(r) = 0$ and $h(s) = 1$. Noting that $\Phi(g)(t) \neq 0$ in this case, let us now put

$$f = h - \frac{\Phi(h)(t)}{\Phi(g)(t)} g.$$

Since $f \in A$ and $\Phi(f)(t) = 0$, we have that $f \in W(t)$ whence $f(r) = 0$. This would imply that $\Phi(h)(t) = 0$, however, which certainly does not hold, and so we see that there is a unique element $\varphi(t) \in T$ such that $f(\varphi(t)) = 0$ for every $f \in W(t)$. Consequently, another application of [9, Theorem 1, page 106] even yields that

$$\mathrm{cl}(W(t)) = \{f \in CV_0(T) \colon f(\varphi(t)) = 0\}.$$

In order to show that $\Phi(f)(t) = f(\varphi(t))$ for every $f \in CV_0(T)$, let us fix $g \in A$ such that $g(\varphi(t)) = 1$. Setting

$$h = [\Phi(g)(t)]^{-1} g^2 - g,$$

we then have that $h \in W(t)$, and the fact that $h(\varphi(t)) = 0$ immediately implies that $\Phi(g)(t) = 1$. Now, given any $f \in CV_0(T)$, let us put $h = f - f(\varphi(t))g$. Then $h \in \mathrm{cl}(W(t))$ since $h(\varphi(t)) = 0$, and therefore $\Phi(h)(t) = 0$. From this, however, it follows that $\Phi(f)(t) = f(\varphi(t))$, as was required to conclude the proof.

In particular, Theorem 3.3 serves to distinguish the composition operators among all continuous linear operators on $CV_0(T)$. As will subsequently be demonstrated, however, a continuous linear operator $\Phi \colon CV_b(T) \to CV_b(T)$ which satisfies (i) and (ii) of Theorem 3.3 can fail to be a composition operator on $CV_b(T)$ (Example 3.5), and hence something more is needed in this case.

3.4. THEOREM. *Let $\Phi \colon CV_b(T) \to CV_b(T)$ be a continuous linear operator. Then there exists $\varphi \colon T \to T$ such that $\Phi = C_\varphi$ if, and only if, the following two*

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conditions are satisfied:

(i) *for each* $t \in T$, *there exists* $g \in CV_0(T)$ *such that* $\Phi(g)(t) \neq 0$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ *whenever* $f \in CV_b(T)$ *and* $g \in CV_0(T) \cap C_b(T)$.

PROOF. Since necessity is again obvious, let us assume that (i) and (ii) both hold. According to Theorem 3.3, therefore, there exists $\varphi\colon T \to T$ such that $\Phi(f) = f \circ \varphi$ for every $f \in CV_0(T)$. Now, consider any $f \in CV_b(T)$, and fix $t \in T$. Then $G = \{s \in T\colon |f(s)| \prec |f(\varphi(t))| + 1\}$ is an open neighborhood of $\varphi(t)$, and so there exists $g \in CV_0(T) \cap C_b(T)$ such that $g(\varphi(t)) = 1$ and $g(s) = 0$ for every $s \in T \backslash G$. In view of (ii), since $g^2 \in CV_0(T) \cap C_b(T)$, it now follows that

$$\Phi(fg^2)(t) = \Phi(f)(t)\Phi(g^2)(t) = \Phi(f)(t).$$

On the other hand, since $fg^2 \in CV_0(T)$, we have that

$$\Phi(fg^2)(t) = (fg^2)(\varphi(t)) = f(\varphi(t)),$$

whereby the proof is complete.

We shall conclude the section with two examples. The first shows that condition (ii) of Theorem 3.4 cannot be replaced by the corresponding condition from Theorem 3.3, while the second demonstrates that (i) of Theorem 3.4 cannot be rephrased in terms of functions belonging to $CV_b(T)$.

3.5. EXAMPLE. Let $T = \mathbf{R}^+$ with the usual topology induced by $\mathbf{R}$, put $v(t) = \exp(-t)$ for $t \in T$, and set $V = \{\lambda v\colon \lambda \succ 0\}$. Fixing $p \in \beta T \backslash T$, we now define an operator Φ on $CV_b(T)$ by setting

$$\Phi(f)(t) = f(t) + \beta(fv)(p),$$

where $f \in CV_b(T), t \in T$, and $\beta(fv)$ denotes the Stone extension of fv to the Stone-Čech compactification βT of T. Then Φ is clearly a bounded linear operator on $CV_b(T)$. Moreover, given $t \in T$, there exists $g \in CV_0(T)$ such that $\Phi(g)(t) = g(t) \neq 0$. Since $C_b(T) \subseteq CV_0(T)$, we also have that

$$\Phi(fg)(t) = f(t)g(t) = \Phi(f)(t)\Phi(g)(t)$$

for every $t \in T$ whenever $f, g \in C_b(T) = C_b(T) \cap CV_0(T)$. However, Φ is not a composition operator on $CV_b(T)$. To see this, let us suppose that there does indeed exist some $\varphi\colon T \to T$ such that $\Phi(f) = f \circ \varphi$ for each $f \in CV_b(T)$. Setting $f(t) = \exp(t)$ for $t \in T$, since $f \in CV_b(T)$ and $f(\varphi(0)) = \Phi(f)(0) = 2$, it would then follow that $\varphi(0) = \log 2$. Thus, for any $g \in C_b(T)$, we would have that

$$g(\log 2) = \Phi(g)(0) = g(0),$$

and this contradiction serves to establish our claim.

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3.6. EXAMPLE. Let $T = \{0\} \cup [1, \infty)$ with the relative topology induced by $\mathbb{R}$, put $v(t) = 1$ for $t \in T$, and set $V = \{\lambda v \colon \lambda \succ 0\}$. Again fixing $p \in \beta T \backslash T$, we now put

$$\Phi(f)(t) = \begin{cases} f(t), & t \in [1, \infty) \\ \beta(f)(p), & t = 0, \end{cases}$$

where $f \in CV_b(T) = C_b(T)$ and $\beta(f)$ denotes the Stone extension of f to βT. Then Φ is obviously a bounded linear operator on $CV_b(T)$ such that $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in CV_b(T)$. Moreover, $v \in CV_b(T)$ and $\Phi(v)(t) = 1$ for each $t \in T$, but $\Phi(g)(0) = 0$ for every $g \in CV_0(T) = C_0(T)$, whereby Φ is not a composition operator on $CV_b(T)$.

4. Basic properties of composition operators

If we consider the composition map $C_\varphi \colon C(T) \to C(T)$ induced by a continuous function $\varphi \colon T \to T$, then it is obvious that

$$\ker(C_\varphi) = \{f \in C(T) \colon N(f) \subseteq T \backslash \varphi(T)\},$$

where $N(f) = \{t \in T \colon f(t) \neq 0\}$. Since $N(f)$ is open when $f \in C(T)$, we see that C_φ is therefore injective if $\varphi(T)$ is dense in T. On the other hand, if even the restriction $C_\varphi \,|\, CV_0(T)$ of C_φ to some $CV_0(T)$ is injective, then $\varphi(T)$ must be dense in T in view of condition (2.c). We collect these straightforward observations in the following statement.

4.1. THEOREM. *Let $C_\varphi \colon C(T) \to C(T)$ be the composition map induced by a continuous function $\varphi \colon T \to T$, and let V be any system of weights on T for which* (2.c) *is satisfied. Then the following are equivalent*:
1. *$\varphi(T)$ is dense in T*;
2. *C_φ is injective*;
3. *$C_\varphi \,|\, CV_b(T)$ is injective*;
4. *$C_\varphi \,|\, CV_0(T)$ is injective.*

Some preparation is needed before stating the expected "dual" result.

4.2. LEMMA. *Let V be a system of weights on T for which* (2.c) *is satisfied, let U be a system of weights on T such that $U \preccurlyeq V$, and assume that $\varphi \colon T \to T$ induces a composition map $C_\varphi \colon CV_0(T) \to CU_0(T)$. Then φ is injective if, and only if, $C_\varphi(CV_0(T))$ is dense in $CU_0(T)$.*

PROOF. Assume, first of all, that $C_\varphi(CV_0(T))$ is dense in $CU_0(T)$, and fix $s, t \in T$ with $s \neq t$. We now choose $g \in CV_0(T)$ so that $g(s) = 0$ and

$g(t) = 1$, as well as $u \in U$ for which $u(s)$, $u(t) \succcurlyeq 1$. Since $g \in CU_0(T)$, there exists $f \in CV_0(T)$ such that $p_u(f \circ \varphi - g) \prec \frac{1}{2}$. Thus, $|f(\varphi(s))| \prec 1/2$ and $|f(\varphi(t)) - 1| \prec 1/2$, whereby $\varphi(s) \neq \varphi(t)$. Conversely, assuming that φ is injective, $A = \{f \circ \varphi\colon f \in C_b(T)\}$ is a selfadjoint subalgebra of $C_b(T)$ which separates the points of T. Moreover, the vector subspace $W = \{f \circ \varphi\colon f \in CV_0(T)\}$ is a module over A. Since we again have an instance of the bounded case of the weighted approximation problem, the density of $C_\varphi(CV_0(T)) = W$ in $CU_0(T)$ now follows as an immediate consequence of [9, Theorem 1, page 106], and the proof is thereby complete.

Given any closed subset F of T, the characteristic function 1_F of F is then a weight on T; we put

$$\mathscr{F} = \mathscr{F}(T) = \{\lambda 1_F\colon \lambda \succ 0; F \subseteq T, F \text{ finite }\}.$$

Then $\mathscr{F}$ is a system of weights on T, and

$$C\mathscr{F}_0(T) = C\mathscr{F}_b(T) = (C(T), \omega(\mathscr{F})),$$

where $\omega(\mathscr{F})$ denotes the topology of pointwise convergence on T. Thus, (2.c) is trivially satisfied, while $\mathscr{F}$ is the smallest system of weights on T in the sense that $\mathscr{F} \preccurlyeq U$ for any system U of weights on T.

4.3. THEOREM. *Let $C_\varphi\colon C(T) \to C(T)$ be the composition map induced by a continuous function $\varphi\colon T \to T$, and let V be any system of weights on T for which* (2.c) *is satisfied. Then the following are equivalent*:

1. *φ is injective*;
2. *$C_\varphi(CV_0(T))$ is $\omega(\mathscr{F})$-dense in $C(T)$*;
3. *$C_\varphi(CV_b(T))$ is $\omega(\mathscr{F})$-dense in $C(T)$*;
4. *$C_\varphi(C(T))$ is $\omega(\mathscr{F})$-dense in $C(T)$.*

PROOF. Since $\mathscr{F} \preccurlyeq V$, if φ is injective, then 2 holds by Lemma 4.2. The implications $2 \Rightarrow 3$ and $3 \Rightarrow 4$ are obvious, while another application of Lemma 4.2, this time taking $U = V = \mathscr{F}$, shows that 4 implies 1.

REMARK. One additional consequence of Lemma 4.2 is the fact that $CV_0(T)$ is always $\omega(\mathscr{F})$-dense in $C(T)$, which follows by setting $U = \mathscr{F}$ and $\varphi(t) = t$, $t \in T$.

In case $\varphi\colon T \to T$ happens to induce a composition map $C_\varphi\colon CV_0(T) \to CV_0(T)$, then taking $U = V$ in Lemma 4.2 gives us that φ is injective if and only if $C_\varphi(CV_0(T))$ is dense in $CV_0(T)$. Furthermore, even if φ only induces a composition map $C_\varphi\colon CV_b(T) \to CV_b(T)$, it is still the case that φ will be injective when $C_\varphi(CV_b(T))$ is dense in $CV_b(T)$; indeed, 3 of Theorem 4.3 would then be satisfied in view of the preceding Remark. However, as we next demonstrate, the converse assertion can fail to hold in this case.

4.4. EXAMPLE. Let $T = \mathbf{R}$ with the usual topology, put $v(t) = 1$ for $t \in T$, and set $V = \{\lambda v : \lambda \succ 0\}$, whereby $CV_b(T) = C_b(T)$. If we now put $\varphi(t) = \tan^{-1} t, t \in T$, then φ is injective and certainly induces a composition map $C_\varphi : CV_b(T) \to CV_b(T)$. Setting $f(t) = \sin t, t \in T$, let us suppose that there exists $g \in C_b(T)$ such that $p_v(g \circ \varphi - f) \prec 1/3$. Since $|g(\varphi(k\pi))| \prec 1/3$ for each $k \in \mathbf{N}$, we have that $|g(\pi/2)| \preccurlyeq 1/3$. Similarly, however, we would also have that $|g(\pi/2) - 1| \preccurlyeq 1/3$, which is a contradiction. Thus, $C_\varphi(CV_b(T))$ is not dense in $CV_b(T)$.

Turning to the question as to when a composition map $C_\varphi : CV_b(T) \to CV_b(T)$, say, will actually be surjective, we begin by noting a necessary condition on the function $\varphi : T \to T$.

4.5. LEMMA. *Let $C_\varphi : C(T) \to C(T)$ be the composition map induced by a continuous function $\varphi : T \to T$, and let V be any system of weights on T for which* (2.c) *is satisfied. If $CV_0(T) \subseteq C_\varphi(C(T))$, then $\varphi : T \to \varphi(T)$ is a homeomorphism.*

PROOF. The fact that φ is injective follows from Theorem 4.3. Now, given any $g \in CV_0(T)$, since there exists $f \in C(T)$ such that $g = f \circ \varphi$, $g \circ \varphi^{-1} = f|\varphi(T)$ is continuous on $\varphi(T)$. Consequently, $\varphi^{-1} : \varphi(T) \to T$ is continuous in view of (2.c), and the proof is thus complete.

As Example 4.4 makes plain, however, even when $\varphi : T \to \varphi(T) \subsetneq T$ is a homeomorphism which induces a composition map $C_\varphi : CV_b(T) \to CV_b(T)$, something else is needed if C_φ is to be surjective. The missing ingredient is a requirement that the range of φ be what one might term "$CV(\varphi^{-1})_b$-embedded" in T. Indeed, we note in passing that if $\varphi : T \to \varphi(T) \subsetneq T$ is a homeomorphism and V is any system of weights on T for which (2.c) is satisfied, then $V(\varphi^{-1}) = \{v \circ \varphi^{-1} : v \in V\}$ is a system of weights on $\varphi(T)$ for which (2.c) is satisfied. The next result extends [2, Theorem 10.3(b), page 141].

4.6. THEOREM. *Assume that the function $\varphi : T \to T$ induces a composition map $C_\varphi : CV_b(T) \to CV_b(T)$. Then C_φ is surjective if and only if*

(i) *$\varphi : T \to \varphi(T)$ is a homeomorphism, and*

(ii) *given $g \in CV(\varphi^{-1})_b(\varphi(T))$, there exists $f \in CV_b(T)$ such that $f\,|\,\varphi(T) = g$.*

PROOF. Assuming that C_φ is surjective, (i) then holds by Lemma 4.5. Further, given $g \in CV(\varphi^{-1})_b(\varphi(T)), v \in V$, and any $t \in T, g \circ \varphi \in C(T)$ and

$$|g \circ \varphi(t)|v(t) = |g(\varphi(t))|v \circ \varphi^{-1}(\varphi(t)) \preccurlyeq p_{v \circ \varphi^{-1}}(g).$$

Since this implies that $g \circ \varphi \in CV_b(T)$, there exists $f \in CV_b(T)$ such that $f \circ \varphi = g \circ \varphi$, which is to say that $f|\varphi(T) = g$. Conversely, let us assume

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that (i) and (ii) are both satisfied. Fixing $g \in CV_b(T)$, we then have that $g \circ \varphi^{-1} \in C(\varphi(T))$. Moreover, given $v \in V$ and $t \in T$,

$$|g \circ \varphi^{-1}(\varphi(t))| v \circ \varphi^{-1}(\varphi(t)) = |g(t)| v(t) \preccurlyeq p_v(g),$$

whereby $g \circ \varphi^{-1} \in CV(\varphi^{-1})_b(\varphi(T))$. Thus, according to (ii), there exists $f \in CV_b(T)$ so that $f|\varphi(T) = g \circ \varphi^{-1}$, and hence $f \circ \varphi = g$ as required.

REMARK. For a homeomorphism $\varphi: T \to \varphi(T) \subseteq T$, the foregoing argument also shows that setting $\Phi(f) = f \circ \varphi^{-1}$ for each $f \in CV_b(T)$ actually defines a topological isomorphism between $CV_b(T)$ and $CV(\varphi^{-1})_b(\varphi(T))$, and it readily follows that $\Phi(CV_0(T)) = CV(\varphi^{-1})_0(\varphi(T))$. Moreover, in any situation where a function $\varphi: T \to T$ induces a composition map $C_\varphi: CV_0(T) \to CV_0(T)$, this latter fact can then be combined with Lemma 4.5 to show that C_φ will be surjective if, and only if, $\varphi: T \to \varphi(T)$ is a homeomorphism and $\varphi(T)$ is "$CV(\varphi^{-1})_0$-embedded" in the sense that each $g \in CV(\varphi^{-1})_0(\varphi(T))$ has an extension belonging to $CV_0(T)$.

Since they are difficult to verify in practice, the "embedding" conditions of Theorem 4.6 and its analogue for $CV_0(T)$ (as described in the preceding Remark) are less than satisfactory. There is a functional analytic approach to surjectivity, however, which can sometimes offer a way around this problem while simultaneously providing additional information. According to Ptàk's open mapping theorem (see [12, page 163]), if $CV_0(T)$, say, happens to be B-complete (or fully complete), and if $\varphi: T \to T$ induces a nearly open composition operator $C_\varphi: CV_0(T) \to CV_0(T)$ such that $C_\varphi(CV_0(T))$ is dense in $CV_0(T)$, then C_φ is necessarily an open surjection. Consequently, when V is a system of weights generated by a single continuous weight on T, for example, C_φ will be (open and) surjective as soon as $C_\varphi: CV_0(T) \to CV_0(T)$ is a continuous nearly open operator with dense range. We shall bring the paper to an end by characterizing those composition maps $C_\varphi: CV_0(T) \to CV_0(T)$ which are nearly open in the following sense: given any $v \in V$, there exists $u \in V$ such that $B_u \subseteq \operatorname{cl}(C_\varphi(B_v))$.

4.7. THEOREM. *Assume that the function $\varphi: T \to T$ induces a composition map $C_\varphi: CV_0(T) \to CV_0(T)$. Then C_φ is nearly open if, and only if, φ is injective and $V(\varphi) \preccurlyeq V$.*

PROOF. Assume that C_φ is nearly open. Then, fixing $f \in CV_0(T)$, let $v \in V$, choose $u \in V$ so that $B_u \subseteq \operatorname{cl}(C_\varphi(B_v))$, and put $\lambda = p_u(f)+1$. In this case, there exists $g \in B_v$ such that $g \circ \varphi \in \lambda^{-1}(f + B_v)$, whereby $C_\varphi(\lambda g) = \lambda g \circ \varphi \in f + B_v$. This shows that $C_\varphi(CV_0(T))$ is dense in $CV_0(T)$, and so φ is injective by Lemma 4.2. To show that $V(\varphi) \preccurlyeq V$, we fix $v \in V, t \in T$, and $\varepsilon \succ 0$. Again choosing $u \in V$ so that $B_u \subseteq \operatorname{cl}(C_\varphi(B_v))$, we now put $G = \{s \in T: u(s) \prec u(t)+\varepsilon/2\}$. Since G

is open and $t \in G$, there exists $g \in CV_0(T)$ with $0 \preccurlyeq g \preccurlyeq 1$ for which $g(t) = 1$ and $g(T \backslash G) = 0$ by [9, Lemma 2, page 69]. Setting $\alpha = (u(t)+\varepsilon/2)^{-1}, \alpha g(s)u(s) \prec 1$ for each $s \in T$, and hence $\alpha g \in B_u$. For $w \in V$ such that $w(t) \succcurlyeq 1$, we can therefore find $f \in B_v$ so that $p_w(f \circ \varphi - \alpha g) \preccurlyeq \alpha\varepsilon[2(v(\varphi(t))+1)]^{-1}$, which means that $|f(\varphi(t)) - \alpha| \preccurlyeq |f(\varphi(t)) - \alpha g(t)|w(t) \preccurlyeq \alpha\varepsilon[2(v(\varphi(t))+1)]^{-1}$. Consequently, we have that

$$1 \succcurlyeq |f(\varphi(t))|v(\varphi(t)) \succcurlyeq \alpha v(\varphi(t)) - |f(\varphi(t)) - \alpha|v(\varphi(t)) \succcurlyeq \alpha(v(\varphi(t)) - \varepsilon/2).$$

Since this gives us that $v(\varphi(t)) \preccurlyeq \alpha^{-1}+\varepsilon/2 = u(t)+\varepsilon$, we see that $v(\varphi(t)) \preccurlyeq u(t)$ for each $t \in T$, and thus $V(\varphi) \preccurlyeq V$.

Turning to the converse, let us now assume that φ is injective and $V(\varphi) \preccurlyeq V$. We then fix $v \in V$, choose $u \in V$ so that $v \circ \varphi \preccurlyeq u$, and take any $f \in CV_0(T)$ for which $p_u(f \circ \varphi) \prec 1$. Since φ is injective, another application of Lemma 4.2 yields that $C_\varphi(CV_0(T))$ is dense in $CV_0(T)$. This being the case, it will suffice to show that $f \circ \varphi \in \mathrm{cl}(C_\varphi(B_v))$ in order to establish that $B_u \subseteq \mathrm{cl}(C_\varphi(B_v))$. To that end, given $w \in V$, we put $K_1 = \{t \in T\colon |f(\varphi(t))|w(t) \succcurlyeq 1\}$ and $K_2 = \{t \in T\colon |f(t)|v(t) \succcurlyeq 1\}$. Then K_1 and K_2 are both compact. For any $t \in T$, moreover, since $|f(\varphi(t))|v(\varphi(t)) \preccurlyeq |f(\varphi(t))|u(t) \prec 1$, $\varphi(t) \notin K_2$. Thus, because φ is necessarily continuous, $\varphi(K_1)$ is compact and $\varphi(K_1) \cap K_2 = \varnothing$. Choosing $g \in C_b(T)$ such that $0 \preccurlyeq g \preccurlyeq 1, g(K_2) = 1$, and $g(\varphi(K_1)) = 0$, we now put $h = f - gf$. Since $h \in CV_0(T)$ and

$$|h(t)|v(t) = (1 - g(t))|f(t)|v(t) \prec 1$$

for every $t \in T$, $h \in B_v$, and so we need only observe that

$$|f(\varphi(t)) - h(\varphi(t))|w(t) = g(\varphi(t))|f(\varphi(t))|w(t) \prec 1$$

for any $t \in T$ in order to conclude the argument.

The next result is an immediate consequence of Theorem 4.7 taken together with Theorem 2.3.

4.8. COROLLARY. *A function $\varphi\colon T \to T$ induces a nearly open composition operator on $CV_0(T)$ if, and only if,* (i) *φ is a continuous injection,* (ii) *$V(\varphi) \sim V$, and* (iii) *for each $v \in V$, $\varepsilon \succ 0$, and compact set $K \subseteq T$, $\varphi^{-1}(K) \cap N(v,\varepsilon)$ is compact.*

REMARK. If $\varphi\colon T \to T$ induces a nearly open composition map $C_\varphi\colon CV_b(T) \to CV_b(T)$, then the necessity argument from the proof of Theorem 4.7 can be adapted (by using Theorem 4.3 in place of Lemma 4.2) to as well show that φ

must be injective and $V(\varphi) \preccurlyeq V$. As can be seen from Example 4.4, however, the converse assertion is false. Example 4.4 also shows that condition (iii) cannot be omitted from the statement of Corollary 4.8.

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