Canad. J. Math. Vol. **66** (2), 2014 pp. 284–302
http://dx.doi.org/10.4153/CJM-2013-029-7

Random Harmonic Functions in Growth Spaces and Bloch-type Spaces

Kjersti Solberg Eikrem

Abstract. Let $h_v^\infty(\mathbf{D})$ and $h_v^\infty(\mathbf{B})$ be the spaces of harmonic functions in the unit disk and multi-dimensional unit ball admitting a two-sided radial majorant $v(r)$. We consider functions v that fulfill a doubling condition. In the two-dimensional case let

$$u(re^{i\theta},\xi)=\sum_{j=0}^{\infty}(a_{j0}\xi_{j0}r^j\cos j\theta+a_{j1}\xi_{j1}r^j\sin j\theta),$$

where $\xi=\{\xi_{ji}\}$ is a sequence of random subnormal variables and a_{ji} are real. In higher dimensions we consider series of spherical harmonics. We will obtain conditions on the coefficients a_{ji} that imply that u is in $h_v^\infty(\mathbf{B})$ almost surely. Our estimate improves previous results by Bennett, Stegenga, and Timoney, and we prove that the estimate is sharp. The results for growth spaces can easily be applied to Bloch-type spaces, and we obtain a similar characterization for these spaces that generalizes results by Anderson, Clunie, and Pommerenke and by Guo and Liu.

1 Introduction

1.1 Spaces of Harmonic Functions

Let v be a positive increasing continuous function on $[0,1)$, assume that $v(0)=1$ and $\lim_{r\to 1}v(r)=+\infty$. We study growth spaces of harmonic functions in the unit disk $\mathbf{D}$ and also in the multidimensional unit ball $\mathbf{B}$ in $\mathbf{R}^n$. We let

$$h_v^\infty(\mathbf{D})=\{u\colon \mathbf{D}\to\mathbf{R}\mid \Delta u=0, |u(x)|\le Kv(|x|)\text{ for some }K>0\},$$

and define $h_v^\infty(\mathbf{B})$ similarly. The study of harmonic growth spaces on the disk and the corresponding spaces of analytic functions A_v^∞ was initiated by L. Rubel and A. Shields in [11] and by A. Shields and D. Williams in [14, 15]. Recently multidimensional analogs were considered in [1, 6]. Various results on the coefficients of functions in growth spaces were obtained in [4]. Hadamard gap series in growth spaces have been studied by a number of authors; see [5] and references therein.

Examples of functions in $h_v^\infty(\mathbf{D})$ can be constructed by lacunary series; see [5]. Another way to construct examples is by using random series, and such functions will be the main focus of this paper. We consider

$$u(re^{i\theta},\xi)=\sum_{j=0}^{\infty}(a_{j0}\xi_{j0}r^j\cos j\theta+a_{j1}\xi_{j1}r^j\sin j\theta),\tag{1.1}$$

Received by the editors July 24, 2012; revised May 23, 2013.
Published electronically August 26, 2013.
AMS subject classification: **30B20**, 31B05, 30H30, 42B05.
Keywords: harmonic functions, random series, growth space, Bloch-type space.

where $\xi = \{\xi_{ji}\}$ is a sequence of independent random variables and

$$\mathbf{a}_j := (a_{j0}, a_{j1}) \in \mathbf{R}^2.$$

We will also study random harmonic functions on $\mathbf{B}$; such functions can be written as

$$u(x,\xi) = \sum_{m=0}^{\infty}\sum_{l=0}^{L_m} a_{ml}\xi_{ml} r^m Y_{ml}\Big(\frac{x}{r}\Big), \tag{1.2}$$

where $r = |x|$, $\{L_m\}$ depends on n and Y_{ml} are spherical harmonics of degree m normalized to fulfill $\|Y_{ml}\|_\infty \leq 1$. Our main results will be proven in several dimensions.

We always assume that the weights satisfy the doubling condition

$$v(1-d) \leq Dv(1-2d). \tag{1.3}$$

Typical examples are

$$v(r) = \Big(\frac{1}{1-r}\Big)^{\alpha} \quad \text{and} \quad v(r) = \max\Big\{1, \Big(\log\frac{1}{1-r}\Big)^{\alpha}\Big\}$$

for $\alpha > 0$. For convenience we define a new function $g\colon [1,\infty) \to [1,\infty)$ such that $g(x) = v(1-\frac{1}{x})$. Then (1.3) is equivalent to

$$g(2x) \leq Dg(x). \tag{1.4}$$

We will use v and g interchangeably.

The Bloch space is the space of analytic functions f on $\mathbf{D}$ satisfying

$$|f(0)| + \sup_{z\in\mathbf{D}}\big(1-|z|^2\big)\,|f'(z)| < \infty.$$

The generalizations of this space where $1-|z|^2$ is replaced by another weight $w(|z|)$ that is decreasing and fulfills $\lim_{r\to 1^-} w(r) = 0$ are called Bloch-type spaces. A harmonic function u is in the Bloch-type space $\mathcal{B}_w$ if

$$\|u\|_{\mathcal{B}_w} = |u(0)| + \sup_{z\in\mathbf{D}} w\big(|z|\big)\,|\nabla u(z)| < \infty.$$

Random Bloch functions have been studied by J. M. Anderson, J. Clunie, and Ch. Pommerenke in [2] and by F. Gao in [7].

1.2 Known Results

Let $\mathbf{a}_j = (a_{j0}, a_{j1}) \in \mathbf{R}^2$ and $|\mathbf{a}_j| = (|a_{j0}|^2 + |a_{j1}|^2)^{1/2}$. It is not difficult to show that if

$$u(re^{i\theta}) = \sum_{j=0}^{\infty}(a_{j0}r^j\cos j\theta + a_{j1}r^j\sin j\theta) \in h_v^\infty(\mathbf{D}),$$

then

$$\sum_{j=0}^{n} |\mathbf{a}_j|^2 \le Bg(n)^2; \tag{1.5}$$

see for example [4]. On the other hand, the inequality

$$\sum_{j=0}^{n} |\mathbf{a}_j| \le Bg(n) \tag{1.6}$$

is sufficient to imply that $u \in h_v^\infty(\mathbf{D})$, but it is not necessary. In the special case of the Hadamard gap series, (1.6) is both necessary and sufficient; see [5], and this is also the case when all the coefficients are positive [4]. But it is not possible in general to characterize all functions in $h_v^\infty(\mathbf{D})$ by the absolute value of their coefficients. We will obtain conditions on the coefficients that imply that u defined by (1.1) is in $h_v^\infty(\mathbf{D})$ almost surely, and similarly in higher dimensions.

Let the partial sums of $u(re^{i\theta}) = \sum_{j=0}^{\infty}(a_{j0}r^j \cos j\theta + a_{j1}r^j \sin j\theta)$ be denoted as

$$(s_n u)(re^{i\theta}) = \sum_{j=0}^{n-1}(a_{j0}r^j \cos j\theta + a_{j1}r^j \sin j\theta)$$

and denote the corresponding Cesàro means by

$$(\sigma_n u)(re^{i\theta}) = \frac{1}{n}\sum_{j=1}^{n}(s_j u)(re^{i\theta}) = \sum_{j=0}^{n-1}\left(1 - \frac{j}{n}\right)(a_{j0}r^j \cos j\theta + a_{j1}r^j \sin j\theta).$$

By [17, Theorem 3.4, p. 89], the maximum of the Cesàro means is less than or equal to the maximum of the function

$$\max_\theta |u(re^{i\theta})| \ge \max_\theta |(\sigma_n u)(re^{i\theta})| \qquad \text{for every } n. \tag{1.7}$$

Although functions in $h_v^\infty(\mathbf{D})$ cannot be characterized by the coefficients alone, they can be characterized by their Cesàro means. The following is [4, Theorem 1.4].

Theorem A *Assume that v satisfies (1.3). If u is a harmonic function on the unit disk, then $u \in h_v^\infty(\mathbf{D})$ if and only if $\|\sigma_n u\|_\infty \le Cg(n)$ for all $n \ge 1$ and some constant $C \ge 0$.*

If we consider the partial sums instead, then $u \in h_v^\infty(\mathbf{D})$ only implies that

$$\|s_n u\|_\infty \le Cg(n)\log n,$$

and this result is sharp, see [4].

Random Taylor series is a fascinating subject in harmonic analysis; we refer the reader to [9] for an excellent introduction to the subject and further references. One of the central results that we use goes back to R. Salem and A. Zygmund [13]; it gives an estimate for the distribution function of a random polynomial. In [13] trigonometric polynomials of the form $\sum_{j=0}^{N} \xi_j a_j \cos j\phi$ are considered, where ξ_j is a Rademacher sequence (a sequence of independent random variables that take the values 1 and -1 with equal probability) or a Steinhaus sequence (a sequence $\{e^{i\varphi_j}\}$ where φ_j are independent and have uniform distribution in $[0, 2\pi]$). In [9] the corresponding result is generalized to other series and subnormal random sequences (which include both Rademacher and Gaussian sequences and the real part of Steinhaus sequences).

Conditions on the coefficients of random Taylor series of analytic functions in various functions spaces have been studied previously in [2] and [4]. In [2] Anderson, Clunie, and Pommerenke showed that if $c_j \geq 0$, $\{e^{i\varphi_j}\}$ is a Steinhaus sequence and

$$\Big(\sum_{j=0}^{n} j^2 c_j^2\Big)^{1/2} = O\Big(\frac{n}{\sqrt{\log n}}\Big), \tag{1.8}$$

then $f(z, \varphi) = \sum_{j=0}^{\infty} c_j e^{i\varphi_j} z^j$ belongs to the Bloch space almost surely.

Gao characterized Bloch functions for the case where the random sequence is a Rademacher sequence; the results give necessary and sufficient conditions for a function to be a Bloch function almost surely; see [7]. The conditions are given in terms of non-decreasing rearrangements.

Let A_ν^∞ denote the space of analytic functions that fulfill $|u(z)| \leq K\nu(|z|)$ for some K. In [4] G. Bennett, D. A. Stegenga, and R. M. Timoney proved the following theorem.

Theorem B *If $\{c_j\}_{j=0}^{\infty}$ is a sequence satisfying*

$$\Big(\sum_{j=0}^{n} |c_j|^2\Big)^{1/2} \leq C\frac{g(n)}{\sqrt{\log n}}$$

and $\{e^{i\varphi_j}\}_{j=0}^{\infty}$ is a Steinhaus sequence, then $\sum_{j=0}^{\infty} c_j e^{i\varphi_j} z^j \in A_\nu^\infty$ almost surely.

1.3 Contents and Organization of this Paper

In this paper we consider random functions given by (1.1) or more generally by (1.2) with a random subnormal sequence ξ_{ml}. The reason for considering subnormal sequences is that they include both Rademacher and normalized Gaussian sequences, and the proofs are based only on the fundamental inequality $\mathcal{E}(e^{\lambda\xi}) \leq e^{\lambda^2/2}$ that is used to define subnormal sequences.

The main result of the paper is a sufficient condition on the coefficients $\{a_{ml}\}$ under which the random series (1.2) belongs to $h_\nu^\infty(\mathbf{B})$ almost surely. As a consequence of this result we obtain a generalization of Theorem B to harmonic functions of several variables. In dimension 2 our main result is similar to Theorem B, but instead of summing all coefficients from 0 to n, we sum coefficients between n_{k-1} and n_k for some sequence n_k that depends on g. In this way we obtain results also in the case when g grows more slowly than $\sqrt{\log x}$.

Usually we start with a weight v and ask for conditions on the coefficients a_{ml} that guarantee that the function defined by (1.2) is in h_v^∞ almost surely. Another way to look at the result is by starting with a sequence of coefficients $\{a_{ml}\}$ and asking for the correct order of growth of typical functions given by (1.2) . We give some examples and show that in some cases our main result gives a better (more slowly growing) estimate than Theorem B.

In Section 2 we collect necessary definitions and preliminary results, and we also formulate a statement that illustrates how adding randomness to the coefficients influences the growth of the function. The main result and some corollaries are given in Section 3. In Section 4 we show that the main result is sharp (in some sense). We also prove some necessary conditions on the coefficients of functions in $h_v^\infty(\mathbf{D})$ in Section 5. Our results can be applied to random functions in Bloch-type spaces and analytic growth spaces, and we obtain similar results for such functions in Section 6.

2 Motivation and Preliminaries

2.1 Subnormal Variables

We will now consider random functions given by (1.1) and (1.2), where $\xi = \{\xi_{ji}\}$ is a sequence of random variables. We will restrict ourselves to subnormal variables.

Definition 2.1 A real-valued random variable ω is called *subnormal* if

$$\mathcal{E}(e^{\lambda\omega}) \leq e^{\lambda^2/2} \qquad \text{for all} \quad -\infty < \lambda < \infty.$$

A sequence of independent subnormal variables is called a *subnormal sequence*.

The random variable that takes the values 1 and -1 with equal probability is subnormal, since $\mathcal{E}(e^{\lambda\omega}) = \frac{1}{2}(e^{\lambda} + e^{-\lambda}) \leq e^{\frac{1}{2}\lambda^2}$. A Rademacher sequence is the sequence of independent variables with such a probability distribution; thus it is a subnormal sequence. Any real random variable ω with $\mathcal{E}(\omega) = 0$ and $|\omega| \leq 1$ a.s. is subnormal. A Gaussian normal variable is subnormal if $\mathcal{E}(\omega) = 0$ and $Var(\omega) \leq 1$; see [9, p. 67] and [16, p. 292] for more on subnormal variables.

Unlike Rademacher and Steinhaus variables, subnormal variables are not necessarily symmetric.

2.2 Deterministic and Random Series in Growth Spaces

The result below illustrates that the random sequence influences the growth of the function. If the growth restriction on the coefficients is strong enough, we can get a result that implies that the function is in $h_v^\infty(\mathbf{D})$. Another assumption implies that the function is in $h_v^\infty(\mathbf{D})$ almost surely. The last point of the proposition concerns a function with large (carefully chosen) coefficients for which the choice of signs still makes the function belong to $h_v^\infty(\mathbf{D})$. The coefficients are large in the sense that $\sum_{j=0}^{n} a_j^2 \geq Cg(n)^2$ for some C, and this is as large as they can be according to (1.5).

Let $n_0 = 1$ and for some $A > 1$ define n_k by induction as

$$n_{k+1} = \min\big\{ l \in \mathbf{N} \mid g(l) \ge Ag(n_k)\big\}. \tag{2.1}$$

Choose A large enough to make $n_k \ge 2n_{k-1}$. This way of defining a sequence $\{n_k\}$ will be used several times. In particular, if $\nu(r) = (\frac{1}{1-r})^\alpha$ or $\max\{1, (\log\frac{1}{1-r})^\alpha\}$, we can choose $n_k = 2^k$ and $n_k = 2^{2^k}$, respectively.

Proposition 2.2 *Let*

$$u(re^{i\theta}, \xi) = \sum_{j=0}^{\infty}(a_{j0}\xi_{j0}r^j \cos j\theta + a_{j1}\xi_{j1}r^j \sin j\theta)\xi_j r^j \cos j\theta.$$

(i) *If* $|\mathbf{a}_j| \le \frac{g(n_k)}{n_k}$ *for* $n_{k-1} < j \le n_k$, *then* $u(re^{i\theta}, \xi) \in h_\nu^\infty(\mathbf{D})$ *for all sequences* $\{\xi_{ji}\}$ *with* $\xi_{ji} \in \{-1, 1\}$.

(ii) *If* $|\mathbf{a}_j| \le \frac{g(n_k)}{\sqrt{n_k \log n_k}}$ *for* $n_{k-1} < j \le n_k$ *and* $\{\xi_{ji}\}$ *is a subnormal sequence, then* $u(re^{i\theta}, \xi) \in h_\nu^\infty(\mathbf{D})$ *almost surely.*

(iii) *If* $a_j = \frac{g(n_k)}{\sqrt{n_k}}$ *for* $n_{k-1} < j \le n_k$, *then there exists a sequence* $\{\xi_j\}$ *with* $\xi_j \in \{-1, 1\}$ *such that* $u(re^{i\theta}, \xi) = \sum_{j=0}^{\infty} a_j\xi_j r^j \cos j\theta \in h_\nu^\infty(\mathbf{D})$.

Proof (i) follows from (1.6), and (ii) will follow from Corollary 3.3. The function in (iii) is constructed as in the proof of [4, Theorem 1.12(b)]; we will use this function in the proof of Proposition 5.1. ■

In Proposition 4.2 we will see that (ii) is sharp.

2.3 Preliminaries on Higher-dimensional Functions

We consider real-valued functions of $d + 1$ real variables, $d \ge 1$. Let F_n be the space of restrictions of polynomials on $\mathbf{R}^{d+1}$ of degree less than or equal to n to the unit sphere S^d. Then the Bernstein inequality

$$\|\nabla P\|_\infty \le n\|P\|_\infty \tag{2.2}$$

holds for all n and all $P \in F_n$, where the gradient is evaluated tangentially to the sphere; see, for example, [10, Theorem V]. For trigonometric polynomials this is a well-known inequality by Bernstein.

The next lemma will be used to prove our main result.

Lemma 2.3 *Let* $P_n \in F_n$, $M_n = \max_{S^d} |P_n|$ *and* $\alpha \in (0, 1)$. *Then there exists a spherical cap of measure* $C((1 - \alpha)/n)^d$ *in which* $|P_n| \ge \alpha M_n$, *and* C *depends on* d.

Proof Let $\delta(y, \zeta)$ be the geodesic distance between two points y and ζ on S^d. Then let $B(y, \phi) = \{\zeta \in S^d \mid \delta(y, \zeta) < \phi\}$ be the spherical cap of radius ϕ with center in y. It can be shown that for the d-dimensional surface measure of the cap

$$|B(y, \phi)| \ge C\phi^d, \tag{2.3}$$

where the constant depends on d.

Let y_0 be a point at which $|P_n| = M_n$, and let y_1 be the closest point where $|P_n| = \alpha M_n$; there is nothing to prove if such a point does not exist. Just as in the proof of [13, Lemma 4.2.3], we have

$$M_n(1-\alpha) = |P_n(y_0)| - |P_n(y_1)| \le |P_n(y_0) - P_n(y_1)| \le \delta(y_0, y_1) \max |\nabla P_n|,$$

and by (2.2), $\delta(y_0, y_1) \ge (1-\alpha)/n$. Therefore, by (2.3), there exists a spherical cap of measure at least $C((1-\alpha)/n)^d$ in which $|P_n| \ge \alpha M_n$. ∎

The next result is [9, Theorem 1, p. 68], which we will need to prove our main result.

Theorem C *Let E be a measure space with a positive measure μ, and $\mu(E) < \infty$. Let F be a linear space of measurable bounded functions on E, closed under complex conjugation, and suppose there exists $\rho > 0$ with the following property: if $f \in F$ and f is real, there exists a measurable set $I = I(f) \subset E$ such that $\mu(I) \ge \mu(E)/\rho$ and $|f(t)| \ge \frac{1}{2}\|f\|_\infty$ for $t \in I$. Let us consider a random finite sum*

$$P = \sum \xi_j f_j$$

where ξ_j is a subnormal sequence and $f_j \in F$. Then, for all $\kappa > 2$,

$$\mathcal{P}\Big(\|P\|_\infty \ge 3\Big(\sum \|f_j\|_\infty^2 \log(2\rho\kappa)\Big)^{1/2}\Big) \le \frac{2}{\kappa}.$$

3 Main Results

3.1 Sufficient Conditions on the Coefficients

We consider harmonic functions defined by (1.2), where Y_{ml} are spherical harmonics of degree m on the sphere S^d, and we use the notation $\mathbf{a}_m = (a_{m0}, \dots, a_{mL_m})$, so $|\mathbf{a}_m|^2 = \sum_{l=0}^{L_m} |a_{ml}|^2$. We are now ready to prove the following theorem.

Theorem 3.1 *Let $\xi = \{\xi_{ml}\}$ be a subnormal sequence. If there exists an increasing sequence $\{n_k\}$ of positive integers such that for all k we have $g(n_{k+1}) \le C_1 g(n_k)$ and*

$$\sum_{j=1}^{k} \sqrt{\Big(\sum_{m=n_{j-1}+1}^{n_j} |\mathbf{a}_m|^2\Big) \log n_j} \le C_2 g(n_k),$$

then $u(x,\xi) = \sum_{m=0}^{\infty} \sum_{l=0}^{L_m} a_{ml}\xi_{ml} r^m Y_{ml}(\frac{x}{r}) \in h_v^\infty(\mathbf{B})$ almost surely.

In two dimensions $|\mathbf{a}_m|^2$ is just $|a_{m0}|^2 + |a_{m1}|^2$, so the same assumptions imply that

$$u(re^{i\theta}, \xi) = \sum_{m=0}^{\infty} (a_{m0}\xi_{m0} r^m \cos m\theta + a_{m1}\xi_{m1} r^m \sin m\theta) \in h_v^\infty(\mathbf{D})$$

almost surely.

Proof Let $S_n(y,\xi) = \sum_{m=0}^{n} \sum_{l=0}^{L_m} a_{ml}\xi_{ml}Y_{ml}(y)$, where $y \in S^d$ and denote $M_n(\xi) = \max_{y\in S^d} |S_n(y,\xi)|$. Let $j = j(N)$ be such that $n_{j-1} < N \le n_j$ and define $Q_N(y,\xi) = S_N(y,\xi) - S_{n_{j-1}}(y,\xi)$ and $\mathfrak{M}_N(\xi) = \max_{y\in S^d} |Q_N(y,\xi)|$. Since harmonic polynomials on the sphere fulfill (2.2), by Lemma 2.3 there exists a spherical cap of measure $C(\frac{1}{2N})^d$ in which $|Q_N| \ge \frac{1}{2}\mathfrak{M}_N$, where C depends on d. Then we can apply Theorem C to Q_N with $E = S^d$, μ the surface measure on S^d, F the set of functions of the form $\sum_{m=0}^{N} \sum_{l=0}^{L_m} a_{ml}\xi_{ml}Y_{ml}(y)$, $n \le N$, $\kappa = 2N^2$, and ρ a constant that depends on d. Define

$$E_N = \left\{ \xi \;\middle|\; \mathfrak{M}_N(\xi) \ge K_1 \sqrt{\sum_{m=n_{j-1}+1}^{N} |\mathbf{a}_m|^2 \log N} \right\},$$

where K_1 is a constant that is chosen large enough to make $3\sqrt{\log 2\rho\kappa} \le K_1\sqrt{\log N}$. Then since $\sum_{N=1}^{\infty} \mathcal{P}(E_N) = \sum_{N=1}^{\infty} 1/N^2 < \infty$, we have by the Borel–Cantelli lemma (see for example [9, p. 7]) that for almost all ξ there is a $J = J(\xi)$ such that

$$\mathfrak{M}_N(\xi) \le K_1 \sqrt{\sum_{m=n_{j-1}+1}^{N} |\mathbf{a}_m|^2 \log N}$$

for $N \ge n_J$. Fix L and let $n_{k-1} < L \le n_k$. Then for $L > n_J$,

$$\begin{aligned} M_L(\xi) &\le M_{n_{J-1}}(\xi) + \sum_{j=J}^{k-1} \mathfrak{M}_{n_j}(\xi) + \mathfrak{M}_L(\xi) \\ &\le B_\xi + K_1 \sum_{j=J}^{k} \sqrt{\sum_{m=n_{j-1}+1}^{n_j} |\mathbf{a}_m|^2 \log n_j} \le B_\xi + K_1C_2 g(n_k) \\ &\le B_\xi + C_3 g(n_{k-1}) \le B_\xi + C_3 g(L) \qquad \text{for a.e. } \xi. \end{aligned}$$

Let B_ξ be large enough to make the inequality $M_L(\xi) \le B_\xi + C_3 g(L)$ also hold for $0 < L \le n_J$, and also let $M_0(\xi) \le B_\xi$. Let $r = |x|$ and $y = x/|x|$. By summation by parts,

$$\begin{aligned} &\Big| \sum_{m=0}^{n} \sum_{l=0}^{L_m} a_{ml}\xi_{ml} r^m Y_{ml}\Big(\frac{x}{r}\Big) \Big| \\ &\quad = \Big| r^n S_n(y,\xi) - (1-r) \sum_{k=0}^{n-1} S_k(y,\xi) r^k \Big| \\ &\quad \le r^n (C_3 g(n) + B_\xi) + (1-r)\Big(B_\xi + \sum_{k=1}^{n-1} (C_3 g(k) + B_\xi) r^k \Big). \end{aligned}$$

Then because of the doubling condition we get

$$\Big| \sum_{m=0}^{\infty} \sum_{l=0}^{L_m} a_{ml}\xi_{ml} r^m Y_{ml}\Big(\frac{x}{r}\Big) \Big| \le C_3(1-r) \sum_{k=1}^{\infty} g(k) r^k + B_\xi \qquad \text{for a.e. } \xi. \tag{3.1}$$

Pick N such that $1-\frac{1}{N-1}<r\leq 1-\frac{1}{N}$. Then

$$(3.2)\qquad (1-r)\sum_{k=1}^{N} g(k)r^k \leq (1-r)g(N)\sum_{k=1}^{N} r^k \leq g(N)$$

and

$$(3.3)\qquad \begin{aligned}(1-r)\sum_{k=N+1}^{\infty} g(k)r^k &= (1-r)\sum_{j=0}^{\infty} r^{2^jN}\sum_{i=1}^{2^jN} g(2^jN+i)r^i\\ &\leq (1-r)\sum_{j=0}^{\infty} g(2^{j+1}N)r^{2^jN}\sum_{i=1}^{2^jN} r^i\\ &\leq g(N)\sum_{j=0}^{\infty} D^{j+1}\Big[\Big(1-\frac{1}{N}\Big)^N\Big]^{2^j} \leq C_4 g(N).\end{aligned}$$

Here C_4 depends only on D. Then by (3.1), (3.2), and (3.3), $u \in h_v^\infty(\mathbf{B})$ almost surely. ■

Remark 3.2 If we had applied Theorem C to S_n instead of Q_n, we could have obtained

$$\max_{y\in S}|S_n(y,\xi)| \leq C\sqrt{\sum\nolimits_{m=0}^{n}|\mathbf{a}_m|^2\log n} + C_\xi \qquad \text{for a.e. } \xi.$$

Then if

$$(3.4)\qquad \Big(\sum_{m=0}^{n}|\mathbf{a}_m|^2\Big)^{1/2} \leq C\frac{g(n)}{\sqrt{\log n}},$$

we would get by partial summation as above that $u \in h_v^\infty(\mathbf{B})$ almost surely, and this generalizes Theorem B. But the approach in Theorem 3.1 is better for two reasons. First of all it makes sense even if g grows more slowly than $\sqrt{\log n}$. For some examples it also gives a better estimate, in the sense that when the coefficients are given and we want to estimate the correct order of growth of a function, Theorem 3.1 may give a more slowly growing estimate for g than we get by using (3.4). Let $n_k = 2^{2^k}$ for $k = 0, 1, \dots$ and define $a_0 = a_1 = a_2 = 0$ and

$$a_j = \frac{1}{\sqrt{n_k}}, \qquad n_{k-1} < j \leq n_k.$$

For $u(z,\xi) = \sum_{j=0}^{\infty} a_j\xi_j r^j \cos j\theta$, (3.4) gives $g(x) = (\log x \log\log x)^{1/2}$, since

$$\sum_{j=0}^{n_N} a_j^2 = \sum_{k=0}^{N}\frac{n_k - n_{k-1}}{n_k} \simeq N+1 \simeq \log\log n_N,$$

but Theorem 3.1 gives $g(x) = (\log x)^{1/2}$, since

$$\sum_{k=1}^{N} \sqrt{\Big(\sum\nolimits_{j=n_{k-1}+1}^{n_k} a_j^2\Big) \log n_k} \simeq C\sqrt{\log n_N}.$$

We will see in Proposition 4.2 that $g(x) = (\log x)^{1/2}$ is the optimal estimate for this function.

Corollary 3.3 *Let $\xi = \{\xi_{ml}\}$ be a subnormal sequence and define $\{n_k\}$ as in (2.1). If*

$$\Big(\sum_{m=n_{k-1}+1}^{n_k} |\mathbf{a}_m|^2\Big)^{1/2} \le C\frac{g(n_k)}{\sqrt{\log n_k}},$$

then $u(x,\xi) = \sum_{m=0}^{\infty}\sum_{l=0}^{L_m} a_{ml}\xi_{ml} r^m Y_{ml}(\frac{x}{r}) \in h_v^\infty(\mathbf{B})$ almost surely.

Proof By the doubling condition $g(n_k) \le Dg(n_k/2) \le DAg(n_{k-1})$, and since

$$\sum_{j=1}^{k} \sqrt{\sum\nolimits_{m=n_{j-1}+1}^{n_j} |\mathbf{a}_m|^2 \log n_j} \le C_1 g(n_k) \sum_{j=1}^{k} \frac{1}{A^{k-j}} \le C_2 g(n_k),$$

the result follows from Theorem 3.1. ∎

Remark 3.4 Now it follows easily that Proposition 2.2(ii) is true. Functions with coefficients

$$|\mathbf{a}_j| \le \frac{g(n_k)}{\sqrt{n_k \log n_k}}, \qquad n_{k-1} < j \le n_k,$$

are in $h_v^\infty(\mathbf{D})$ almost surely by Corollary 3.3.

Remark 3.5 It is not necessary to assume that $\{Y_{ml}\}$ is a basis in the proof of Theorem 3.1; we can use any combination of spherical harmonics. We will need this fact when we apply our results to Bloch-type functions.

4 Sharpness of Results

4.1 Sharpness of Corollary 3.3

We will now prove that Corollary 3.3 is sharp by giving an example. We will first prove it in the two-dimensional case and then indicate how it can be generalized to any dimension. The example is similar to the one given in the proof of [4, Theorem 1.18(b)]. We will use that

$$\Big\| \sum_{j=1}^{n} c_j \cos(N + 4^j)\theta \Big\|_\infty \ge c\sum_{j=1}^{n} |c_j| \tag{4.1}$$

for any N and some absolute constant $c > 0$. This can be shown by using Riesz products. Let A be a constant such that

$$\frac{1}{A-1} \le \frac{c}{8}, \tag{4.2}$$

where c is the constant in (4.1). Let $n_0 = 2$, and for some A that fulfills (4.2) define n_k by induction as in (2.1). We choose A big enough to make $n_k \ge 4n_{k-1}$.

Proposition 4.1 *Let $\{\nu_k\}$ be any sequence of positive numbers increasing to infinity and define $\{n_k\}$ as in (2.1). Then for the sequence $\{a_j\}$, where*

$$a_j = \nu_k \frac{g(n_k)}{\log n_k}, \qquad \text{when } j = n_{k-1} + 4^m, \quad 0 \le m \le \log_4 \frac{n_k}{2},$$

and $a_j = 0$ otherwise, we have

$$\Big(\sum_{j=n_{k-1}+1}^{n_k} a_j^2\Big)^{1/2} \le C\nu_k \frac{g(n_k)}{\sqrt{\log n_k}},$$

but $u(z,\xi) = \sum_{j=0}^{\infty} a_j \xi_j r^j \cos j\theta \notin h_\nu^\infty(\mathbf{D})$ for any choice of sequence $\{\xi_j\}$ where $\xi_j = \pm 1$.

Proof Inequality (4.2) implies

$$\sum_{k=1}^{N-1} \nu_k g(n_k) \le \frac{c}{8} \nu_N g(n_N).$$

Let σ_n be the Cesàro mean; then by (4.1) we have for $n = n_N$,

$$\begin{aligned}
\|\sigma_n u\|_\infty &= \Big\| \sum_{k=1}^{N} \nu_k \frac{g(n_k)}{\log n_k} \sum_{m=0}^{\lfloor \log_4 (n_k/2) \rfloor} \Big(1 - \frac{n_{k-1}+4^m}{n_N}\Big) \xi_{n_{k-1}+4^m} \cos(n_{k-1}+4^m)\theta \Big\|_\infty \\
&\ge \nu_N \frac{g(n_N)}{\log n_N} \Big\| \sum_{m=0}^{\lfloor \log_4 (n_N/2) \rfloor} \Big(1 - \frac{n_{N-1}+4^m}{n_N}\Big) \xi_{n_{N-1}+4^m} \cos(n_{N-1}+4^m)\theta \Big\|_\infty \\
&\quad - \Big\| \sum_{k=1}^{N-1} \nu_k \frac{g(n_k)}{\log n_k} \sum_{m=0}^{\lfloor \log_4 (n_k/2) \rfloor} \Big(1 - \frac{n_{k-1}+4^m}{n_N}\Big) \xi_{n_{k-1}+4^m} \cos(n_{k-1}+4^m)\theta \Big\|_\infty \\
&\ge c \frac{1}{4 \log 4} \nu_N g(n_N) - \frac{1}{\log 4} \sum_{k=1}^{N-1} \nu_k g(n_k) \\
&\ge \frac{1}{\log 4} \Big(\frac{c}{4} - \frac{c}{8}\Big) \nu_N g(n_N) = C \nu_N g(n_N)
\end{aligned}$$

Hence by Theorem A we get that $u(z,\xi) \notin h_\nu^\infty(\mathbf{D})$. ∎

To prove the same in $\mathbf{R}^{d+1}$, let $Y_{j0}(y) = \Re(y_1 + iy_2)^j = \cos j\theta$, where $y = (y_1, \dots, y_{d+1})$ and $\theta = \arctan \frac{y_2}{y_1}$. Also let $a_{j0} = a_j$, where a_j is as above, and $a_{ji} = 0$ otherwise. Then $u(x,\xi) = \sum_{j=0}^{\infty} a_{j0} \xi_{j0} r^j Y_{j0}(\frac{x}{r}) \notin h_\nu^\infty(\mathbf{B})$.

4.2 Sharpness of Proposition 2.2(ii)

The next example serves two purposes. One is to prove in another way that Corollary 3.3 is sharp; the other is to show that the estimate in Proposition 2.2(ii) cannot be improved.

To construct this example we need a result that is based on [13, Lemma 4.5.1]. This lemma is used in a similar way in [2, Theorem 3.7] to prove a result on the coefficients of Bloch functions.

Lemma A *Let $\xi = \{\xi_k\}_{k=0}^{\infty}$ be a Rademacher sequence. Let*

$$H_n(\theta,\xi) = \sum_{j=0}^{n} b_j \xi_j \cos j\theta, \quad R_n = \sum_{j=0}^{n} b_j^2, \qquad T_n = \sum_{j=0}^{n} b_j^4 \le c\,\frac{R_n^2}{n}.$$

Then

$$\max_{\theta} |H_n(\theta,\xi)| > C\sqrt{R_n \log n_n} \qquad (C > 0)$$

except for $(\xi_0, \xi_1, \dots, \xi_n) \in E_n$, where $\mathcal{P}(E_n) < B(c)\, n^{-1/10}$. The constant C is absolute and B depends on c.

Then we have the following proposition.

Proposition 4.2 *Let $\xi = \{\xi_j\}_{j=0}^{\infty}$ be a Rademacher sequence, let $n_0 = 1$ and for some A large enough define $\{n_k\}$ by induction as in (2.1). Let $\{\nu_k\}$ be any sequence of positive numbers increasing to ∞. Then for the sequence $\{a_j\}$ where $a_0 = a_1 = a_2 = 0$ and*

$$a_j = \nu_k \frac{g(n_k)}{\sqrt{n_k \log n_k}}, \qquad n_{k-1} < j \le n_k,$$

we have

$$\Big(\sum_{j=n_{k-1}+1}^{n_k} a_j^2\Big)^{1/2} \le \nu_k \frac{g(n_k)}{\sqrt{\log n_k}},$$

but almost surely $u(z,\xi) = \sum_{j=0}^{\infty} a_j \xi_j r^j \cos j\theta \notin h_v^{\infty}(\mathbf{D})$.

The main difference between the proof of [2, Theorem 3.7] and the proof of this result lies in the fact that we need to make it hold for slow growing weights as well, and we split the function u in two parts, which are estimated separately. Lemma A is applied to only a part of the function.

Proof The constants C_j, $j = 1, 2, \dots$ in this proof will be absolute constants. Define the sequence $\{n_k\}$ by induction as stated, where we choose $A \ge 2$ and such that the following condition is satisfied:

$$n_k > 2n_{k-1}. \tag{4.3}$$

One more condition on A will be specified later.

Fix $r_N = 1 - 1/n_N$ and split u into two parts

$$u(r_N e^{i\theta}, \xi) = \sum_{j=0}^{\infty} a_j \xi_j r_N^j \cos j\theta = \sum_{j=0}^{n_{N-1}} a_j \xi_j r_N^j \cos j\theta + \sum_{j=n_{N-1}+1}^{\infty} a_j \xi_j r_N^j \cos j\theta$$
$$= b_N(r_N e^{i\theta}, \xi) + d_N(r_N e^{i\theta}, \xi).$$

Then

$$|u(r_N e^{i\theta}, \xi)| = \Big| \sum_{j=0}^{\infty} a_j \xi_j r_N^j \cos j\theta \Big| \geq |d_N(r_N e^{i\theta}, \xi)| - |b_N(r_N e^{i\theta}, \xi)|. \tag{4.4}$$

We will estimate $|d_N(r_N e^{i\theta}, \xi)|$ from below and $|b_N(r_N e^{i\theta}, \xi)|$ from above. Let

$$h_N(\theta, \xi) = \sum_{j=n_{N-1}+1}^{n_N} \Big(1 - \frac{j}{n_N}\Big) a_j \xi_j r_N^j \cos j\theta.$$

This is the Cesàro mean of the partial sum of $d(r_N e^{i\theta}, \xi)$. By (1.7),

$$\max_\theta |d(r_N e^{i\theta}, \xi)| \geq \max_\theta |h_N(\theta, \xi)|. \tag{4.5}$$

We will apply Lemma A to h_N. Using (4.3), we get

$$R_{n_N} = \sum_{j=n_{N-1}+1}^{n_N} \Big(1 - \frac{j}{n_N}\Big)^2 a_j^2 r_N^{2j} \geq C_1 \sum_{j=n_N/2+1}^{3n_N/4} \Big(1 - \frac{j}{n_N}\Big)^2 a_j^2$$
$$\geq C_1 \frac{n_N}{4} \Big(\frac{1}{4}\Big)^2 \frac{\nu_N^2 g(n_N)^2}{n_N \log n_N} \geq C_2 \frac{\nu_N^2 g(n_N)^2}{\log n_N}.$$

Furthermore,

$$T_{n_N} = \sum_{j=n_{N-1}+1}^{n_N} \Big(1 - \frac{j}{n_N}\Big)^4 a_j^4 r_N^{4j} \leq \frac{(n_N - n_{N-1}) \nu_N^4 g(n_N)^4}{(n_N \log n_N)^2} \leq C_3 \frac{R_{n_N}^2}{n_N}.$$

Then by Lemma A,

$$\max_\theta |h_N(\theta, \xi)| > C_4 \sqrt{R_{n_N} \log n_N} \geq C_5 \nu_N g(n_N) \tag{4.6}$$

except for $\xi \in E_{n_N}$. Since $\sum_{k=1}^{\infty} \mathcal{P}(E_{n_k}) < \sum_{k=1}^{\infty} B(C_3)\, n_k^{-1/10}$, and this is finite by (4.3), we have by the Borel–Cantelli lemma that for almost all ξ there exists a $N_0 = N_0(\xi)$ such that (4.6) holds for all $N \geq N_0$. Hence by (4.5), for almost all ξ we have for $N \geq N_0(\xi)$ that

$$\max_\theta |d(r_N e^{i\theta}, \xi)| \geq C_5 \nu_N g(n_N). \tag{4.7}$$

Let $S_n(\theta,\xi) = \sum_{k=0}^{n} a_k\xi_k \cos k\theta$ and $M_n(\xi) = \max_{0\le\theta\le 2\pi} |S_n(\theta,\xi)|$. Let $j = j(n)$ be such that $n_{j-1} < n \le n_j$ and define $Q_n(\theta,\xi) = S_n(\theta,\xi) - S_{n_{j-1}}(\theta,\xi)$ and $\mathfrak{M}_n(\xi) = \max_{0\le\theta\le 2\pi} |Q_n(\theta,\xi)|$. Just as in the proof of Theorem 3.1, it can be shown that for almost all ξ there is $J = J(\xi)$ such that

$$\mathfrak{M}_n(\xi) \le K_1 \sqrt{\Big(\sum\nolimits_{l=n_{j-1}+1}^{n} a_l^2\Big) \log n_j} \le K_1 \nu_j g(n_j)$$

for $n \ge n_J$. Fix L and let $n_{k-1} < L \le n_k$. Then for a.e. ξ and $L \ge n_J(\xi)$,

$$M_L(\xi) \le M_{n_{J-1}}(\xi) + \sum_{j=J}^{k-1} \mathfrak{M}_{n_j}(\xi) + \mathfrak{M}_L(\xi) \le B_\xi + K_1 \sum_{j=J}^{k} \nu_j g(n_j) \tag{4.8}$$

$$\le B_\xi + K_1 \nu_k g(n_k) \sum_{l=0}^{k-J} \frac{1}{A^l} \le B_\xi + 2K_1 \nu_k g(n_k).$$

Let B_ξ be large enough to make the inequality $M_L(\xi) \le B_\xi + 2K_1 g(L)$ also hold for $0 < L \le n_J$, and also let $M_0(\xi) \le B_\xi$.

We will now estimate $b_N(r_N e^{i\theta},\xi)$. By summation by parts and (4.8),

$$\begin{aligned}
\big|\, b_N(r_N e^{i\theta},\xi)\big| &= \Big|\sum_{l=0}^{n_{N-1}} a_l \xi_l r_N^l \cos l\theta\Big| \\
&= \Big| r_N^{n_{N-1}} S_{n_{N-1}}(\theta,\xi) - (1-r_N) \sum_{l=0}^{n_{N-1}-1} S_l(\theta,\xi) r_N^l \Big| \\
&\le r_N^{n_{N-1}} M_{n_{N-1}}(\xi) + (1-r_N)\Big(B_\xi + \sum_{j=0}^{N-2} \big(B_\xi + 2K_1\nu_j g(n_j)\big) \sum_{l=n_j}^{n_{j+1}-1} r_N^l\Big) \\
&\le \big(2K_1\nu_{N-1} g(n_{N-1}) + B_\xi\big) + B_\xi + 2K_1 \nu_{N-1} g(n_{N-1}) \sum_{j=0}^{N-2} \frac{1}{A^j}
\end{aligned}$$

Then

$$\max_\theta |b_N(r_N e^{i\theta},\xi)| \le 2B_\xi + 6K_1 \nu_{N-1} g(n_{N-1}) \qquad \text{for a.e. } \xi. \tag{4.9}$$

For almost every ξ and $N \ge J(\xi)$ we get, by letting $A \ge 12K_1/C_5$ and using (4.4), (4.7), and (4.9), that

$$\begin{aligned}
\max_\theta |u(r_N e^{i\theta},\xi)| &> C_5 \nu_N g(n_N) - 6K_1 \nu_{N-1} g(n_{N-1}) - 2B_\xi \\
&\ge C_5 \nu_N g(n_N) - \frac{6K_1}{A} \nu_{N-1} g(n_N) - 2B_\xi \\
&\ge \frac{C_5}{2} \nu_N g(n_N) - 2B_\xi = \frac{C_5}{2} \nu_N v(r_N) - 2B_\xi.
\end{aligned}$$

Then almost surely $u(z,\xi) = \sum_{j=0}^{\infty} a_j \xi_j r^j \cos j\theta \notin h_v^\infty(\mathbf{D})$. ■

https://doi.org/10.4153/CJM-2013-029-7 Published online by Cambridge University Press

5 Some Results for Deterministic Functions

5.1 Necessary Conditions on a General Function in $h_v^\infty(\mathbf{D})$

We will now prove some estimates for the growth of the coefficients of functions in $h_v^\infty(\mathbf{D})$. We know that $|\mathbf{a}_j| \le Cg(j)$ from, for example, (1.5). For Hadamard gap series there exist examples of functions in $h_v^\infty(\mathbf{D})$ for which

$$\limsup_{j\to\infty} \frac{|\mathbf{a}_j|}{g(j)} > 0,$$

for example $u(z) = \sum_{k=0}^\infty g(n_k) r^{n_k} \cos n_k\theta$, where $\{n_k\}$ is defined by (2.1); see [5]. But all the coefficients cannot grow this fast if $u \in h_v^\infty(\mathbf{D})$:

Proposition 5.1 *Let*

$$u(re^{i\theta}) = \sum_{j=0}^\infty (a_{j0} r^j \cos j\theta + a_{j1} r^j \sin j\theta) \in h_v^\infty(\mathbf{D})$$

and define a sequence $\{n_k\}$ as before. Let $k = k(j)$ be such that $n_{k-1} < j \le n_k$. Then

$$\liminf_{j\to\infty} \frac{|\mathbf{a}_j|\sqrt{n_k}}{g(j)} < \infty. \tag{5.1}$$

Moreover, there exists a function in $h_v^\infty(\mathbf{D})$ for which $\liminf_{j\to\infty} |\mathbf{a}_j|\sqrt{n_k}/g(j) > 0$, so the result is sharp.

A related result is given in [4, Theorem 1.16(a)]. There it is proven that if $u(z) = \sum_{j=0}^\infty b_j z^j \in A_v^\infty$ and $|b_n|$ increases with j, then $|b_j| = O(g(j)/\sqrt{j})$.

When g grows like x^α it would be equivalent to replace n_k in (5.1) by j, but for slow-growing functions like $\log x$, that would give a weaker statement, since n_k in that case grows very fast.

Proof In [4, Theorem 1.12(b)] it is proven that

$$\sum_{j=0}^n |\mathbf{a}_j| \le Cg(n)\sqrt{n} \tag{5.2}$$

whenever $u \in h_v^\infty(\mathbf{D})$. Then since $n_k \ge 2n_{k-1}$,

$$\frac{n_k}{2} \min_{j\in(n_{k-1},n_k]} |\mathbf{a}_j| \le \sum_{j=n_{k-1}+1}^{n_k} |\mathbf{a}_j| \le Cg(n_k)\sqrt{n_k},$$

thus

$$\min_{j\in(n_{k-1},n_k]} |\mathbf{a}_j| \le 2Cg(n_k)/\sqrt{n_k} \le 2ADCg(j)/\sqrt{n_k},$$

where D and A are as in (1.4) and (2.1), respectively, and the result follows.

https://doi.org/10.4153/CJM-2013-029-7 Published online by Cambridge University Press

The function used in [4] to prove that Theorem 1.12(b) is sharp can also be used here. To construct this function, it is used that there exists a sequence $\{\xi_j\}$ in $\{-1, 1\}$ such that the polynomials

$$P_m(z) = \frac{\sum_{j=1}^{m} \xi_j z^j}{\sqrt{m}}$$

satisfy $\|P_m\|_\infty \leq 5$; see [12]. These are called Rudin–Shapiro polynomials. Now define

$$u(z) = \Re \sum_{k=1}^{\infty} g(n_k) z^{n_{k-1}} P_{n_k - n_{k-1}}(z) = \sum_{k=1}^{\infty} \frac{g(n_k) r^{n_{k-1}}}{\sqrt{n_k - n_{k-1}}} \sum_{j=1}^{n_k - n_{k-1}} \xi_j r^j \cos(n_{k-1} + j)\theta.$$

By (1.7) we have $\|\sigma_n u\|_\infty \leq \|s_n u\|_\infty$, so $u \in h_v^\infty(\mathbf{D})$ by Theorem A, since $\|s_n u\|_\infty \leq Cg(n)$. The coefficients have the desired growth, since $n_k \geq 2n_{k-1}$. ■

The function constructed in the above proof also proves Proposition 2.2(iii).

The estimate $|\mathbf{a}_j| \leq p_j g(j)/\sqrt{j}$, where $\{p_j\}$ is a sequence going to infinity, holds for most of the coefficients. More precisely, we have the following proposition.

Proposition 5.2 *Assume that $u(re^{i\theta}) = \sum_{j=0}^{\infty} (a_{j0} r^j \cos j\theta + a_{j1} r^j \sin j\theta) \in h_v^\infty(\mathbf{D})$ and let p_j be an increasing sequence of positive numbers such that $\lim_{j\to\infty} p_j = \infty$. Define $N(n)$ as the number of $\mathbf{a}_j$ satisfying $j \leq n$ and $|\mathbf{a}_j| \leq p_j g(j)/\sqrt{j}$. Then*

$$\lim_{n\to\infty} N(n)/n = 1.$$

A similar result was proved by F. G. Avhadiev and I. R. Kayumov in [3] for Bloch functions using a different argument.

Proof Let $I_k = |\{j \mid 2^{k-1} < j \leq 2^k, |\mathbf{a}_j| > p_j g(j)/\sqrt{j}\}|$. Since by (5.2) we have

$$I_k p_{2^{k-1}} g(2^{k-1})/\sqrt{2^k} < \sum_{j=2^{k-1}+1}^{2^k} |\mathbf{a}_j| \leq Cg(2^k)\sqrt{2^k},$$

it follows that $I_k < DC2^k/p_{2^{k-1}}$. If $2^{m-1} < n \leq 2^m$, then

$$N(n) \geq n - \sum_{k=1}^{m} I_k = n - o(n). \qquad ■$$

6 Application to Other Spaces

6.1 Bloch-type Spaces

We will now see that our results for growth spaces can easily be applied to Bloch-type spaces $\mathcal{B}_w$. We will consider these spaces in several dimensions, and they are defined as the spaces of functions that fulfill

$$\|u\|_{\mathcal{B}_w} = |u(0)| + \sup_{z\in\mathbf{B}} w(|z|)|\nabla u(z)| < \infty,$$

where w is as described in the introduction. We always assume that w fullfills a condition equivalent to (1.3):

$$w\Big(1-\frac{d}{2}\Big) \geq Bw(1-d). \tag{6.1}$$

Examples are $w(r)=(1-r)^\alpha$ and $(\log\frac{1}{1-r})^{-\alpha}$ for $\alpha>0$.

The function $u(x,\xi)=\sum_{m=0}^\infty\sum_{l=0}^{L_m} a_{ml}\xi_{ml}r^m Y_{ml}(\frac{x}{r})$ is in $\mathcal{B}_w$ if and only if all partial derivatives of u are in $h_v^\infty(\mathbf{B})$ for $v(r)=1/w(r)$. We can write $Y_{ml}(x)$ instead of $r^m Y_{ml}(\frac{x}{r})$, and $Y_{ml}(x)$ is a homogeneous harmonic polynomial. By [10, Theorem III] we have $|\frac{\partial}{\partial x_i}Y_{ml}(x)|\leq m$. Then

$$\frac{\partial}{\partial x_i}u(x,\xi)=\sum_{m=1}^\infty\sum_{l=0}^{L_m} a_{ml}\xi_{ml}\frac{\partial}{\partial x_i}Y_{ml}(x)=\sum_{m=1}^\infty\sum_{l=0}^{L_m} m a_{ml}\xi_{ml}\frac{\partial}{\partial x_i}\frac{Y_{ml}(x)}{m}$$

By Remark 3.5 and since $\frac{\partial}{\partial x_i}\frac{Y_{ml}(x)}{m}$ is a homogeneous harmonic polynomial bounded by 1 on the sphere, we can apply Theorem 3.1 with $w(r)=1/v(r)$. Then the next result generalizes (1.8) to all weights that satisfy (6.1). It also generalizes [8, Theorem 1] by Guo and Liu, which is proved for α-Bloch functions.

Corollary 6.1 *Let $u(x,\xi)=\sum_{m=0}^\infty\sum_{l=0}^{L_m} a_{ml}\xi_{ml}r^m Y_{ml}(\frac{x}{r})$, where $\xi=\{\xi_{ml}\}$ is a subnormal sequence. If there exists an increasing sequence $\{n_k\}$ of positive integers such that for all k we have $g(n_{k+1})\leq C_1 g(n_k)$ and*

$$\sum_{i=1}^k\sqrt{\Big(\sum\nolimits_{m=n_{i-1}+1}^{n_i} m^2|\mathbf{a}_m|^2\Big)\log n_i}\leq\frac{C_2}{w(1-1/n_k)},$$

then $u\in\mathcal{B}_w$ almost surely.

Similarly, Corollary 3.3 gives:

Corollary 6.2 *Let $\xi=\{\xi_{ml}\}$ be a subnormal sequence, let $A>1$, $n_0=1$ and define n_k by induction as $n_{k+1}=\min\{l\in\mathbf{N}\mid w(1-1/l)A\leq w(1-1/n_k)\}$. If*

$$\Big(\sum_{m=n_{k-1}+1}^{n_k} m^2|\mathbf{a}_m|^2\Big)^{1/2}\leq\frac{C}{w(1-1/n_k)\sqrt{\log n_k}},$$

then $u(x,\xi)=\sum_{m=0}^\infty\sum_{l=0}^{L_m} a_{ml}\xi_{ml}r^m Y_{ml}\left(\frac{x}{r}\right)\in\mathcal{B}_w$ almost surely.

The same results hold for analytic Bloch-type spaces as well; see the next section.

We will give examples of what the last corollary means for $w(r)=(1-r)^\alpha$ and $(\log\frac{1}{1-r})^{-\alpha}$ for $\alpha>0$. The sequence n_k can be chosen as $n_k=2^k$ and $n_k=2^{2^k}$, respectively, and a sufficient condition to be in $\mathcal{B}_w$ almost surely when $w(r)=(1-r)^\alpha$ is

$$\Big(\sum_{m=2^{k-1}+1}^{2^k} m^2|\mathbf{a}_m|^2\Big)^{1/2}\leq C\frac{2^{\alpha k}}{\sqrt{k}},$$

and for $w(r) = (\log \frac{1}{1-r})^{-\alpha}$ it is

$$\Big(\sum_{m=2^{2^{k-1}}+1}^{2^{2^k}} m^2 |\mathbf{a}_m|^2 \Big)^{1/2} \le C 2^{\alpha 2^k - k/2}.$$

In the same way as in Proposition 4.1 and 4.2 it can be shown that Corollary 6.2 is sharp; just replace a_j by $j a_j$ when defining the coefficients.

Proposition 5.1 and 5.2 can also be applied to Bloch-type functions in the disk:

Proposition 6.3 *Let $u(re^{i\theta}) = \sum_{j=0}^{\infty} (a_{j0} r^j \cos j\theta + a_{j1} r^j \sin j\theta) \in \mathcal{B}_w$ and define a sequence $\{n_k\}$ as before. Let $k = k(j)$ be such that $n_{k-1} < j \le n_k$. Then*

$$\liminf_{j\to\infty} |\mathbf{a}_j| j w(1 - 1/j) \sqrt{n_k} < \infty.$$

Moreover, there exists a function in $\mathcal{B}_w$ for which $\liminf_{j\to\infty} |\mathbf{a}_j| \sqrt{n_k}/g(j) > 0$, so the result is sharp.

For $w(r) = (1-r)^\alpha$ this is $\liminf_{j\to\infty} |\mathbf{a}_j| j^{1-\alpha} \sqrt{2^k} < \infty$, and since n_k does not grow very fast in this case, this is equivalent to

$$\liminf_{j\to\infty} |\mathbf{a}_j| j^{1-\alpha} \sqrt{j} < \infty.$$

For the usual Bloch functions we have $\liminf_{j\to\infty} |\mathbf{a}_j| \sqrt{j} < \infty$.

Proposition 6.4 *Assume that $u(re^{i\theta}) = \sum_{j=0}^{\infty} (a_{j0} r^j \cos j\theta + a_{j1} r^j \sin j\theta) \in \mathcal{B}_w$ and let p_j be an increasing sequence of positive numbers such that $\lim_{j\to\infty} p_j = \infty$. Define $N(n)$ as the number of $\mathbf{a}_j$ satisfying $j \le n$ and $|\mathbf{a}_j| \le \frac{p_j}{j w(1-1/j)\sqrt{j}}$. Then*

$$\lim_{n\to\infty} N(n)/n = 1.$$

This generalizes [3, Corollary 2], which is proved for Bloch functions.

6.2 Analytic Growth Spaces and Bloch-type Spaces

Let A_v^∞ denote the space of analytic functions on $\mathbf{D}$ that fulfill $|u(z)| \le K v(|z|)$ for some K, as mentioned in the introduction. We can prove a result similar to Theorem 3.1 in this case as well, and this generalizes Theorem B. The proof is similar to the proof of Theorem 3.1; we apply Theorem C with F equal to the set of complex trigonometric polynomials.

Theorem 6.5 *Let $\xi = \{\xi_m\}$ be a subnormal sequence. If there exists an increasing sequence $\{n_k\}$ of positive integers such that for all k we have $g(n_{k+1}) \le C_1 g(n_k)$ and*

$$\sum_{j=1}^{k} \sqrt{\Big(\sum_{m=n_{j-1}+1}^{n_j} |a_m|^2 \Big) \log n_j} \le C_2 g(n_k),$$

then $u(z, \xi) = \sum_{m=0}^{\infty} a_m \xi_m z^m \in A_v^\infty$ almost surely.

https://doi.org/10.4153/CJM-2013-029-7 Published online by Cambridge University Press

A result similar to Corollary 3.3 follows easily. We can also apply Theorem 6.5 to get results similar to Corollary 6.1 and Corollary 6.2 for analytic Bloch-type spaces.

Acknowledgments I wish to thank my Ph.D. advisor Eugenia Malinnikova for suggesting this problem and for valuable discussions. I also thank the referee for useful suggestions. This paper was finished during my stay at the Centre of Advanced Study at the Norwegian Academy of Science and Letters, and I am grateful to the centre for its hospitality and support.

References

[1] E. Abakumov and E. Doubtsov, *Reverse estimates in growth spaces.* Math. Z. **271**(2012), no. 1, 399–413. http://dx.doi.org/10.1007/s00209-011-0869-8
[2] J. M. Anderson, J. Clunie, and Ch. Pommerenke, *On Bloch functions and normal functions.* J. Reine Angew. Math. **270**(1974), 12–37.
[3] F. G. Avhadiev and I. R. Kayumov, *Estimates for Bloch functions and their generalization.* Complex Variables Theory Appl. **29**(1996), no. 3, 193–201. http://dx.doi.org/10.1080/17476939608814887
[4] G. Bennett, D. A. Stegenga, and R. M. Timoney, *Coefficients of Bloch and Lipschitz functions.* Illinois J. Math. **25**(1981), no. 3, 520–531.
[5] K. S. Eikrem, *Hadamard gap series in growth spaces.* Collect. Math. **64**(2013), no. 1, 1–15. http://dx.doi.org/10.1007/s13348-012-0065-0
[6] K. S. Eikrem and E. Malinnikova, *Radial growth of harmonic functions in the unit ball.* Math. Scand. **110**(2012), no. 2, 273–296.
[7] F. Gao, *A characterization of random Bloch functions.* J. Math. Anal. Appl. **252**(2000), no. 2, 959–966. http://dx.doi.org/10.1006/jmaa.2000.7192
[8] J. Guo and P. Liu, *Random α-Bloch function.* Chinese Quart. J. Math. **16**(2001), no. 4, 100–103.
[9] J.-P. Kahane, *Some random series of functions.* Second ed., Cambridge Studies in Advanced Mathematics, 5, Cambridge University Press, Cambridge, 1985.
[10] O. D. Kellogg, *On bounded polynomials in several variables.* Math. Z. **27**(1928), no. 1, 55–64. http://dx.doi.org/10.1007/BF01171085
[11] L. A. Rubel and A. L. Shields, *The second duals of certain spaces of analytic functions.* J. Aust. Math. Soc. **11**(1970), no. 3, 276–280. http://dx.doi.org/10.1017/S1446788700006649
[12] W. Rudin, *Some theorems on Fourier coefficients.* Proc. Amer. Math. Soc. **10**(1959), 855–859. http://dx.doi.org/10.1090/S0002-9939-1959-0116184-5
[13] R. Salem and A. Zygmund, *Some properties of trigonometric series whose terms have random signs.* Acta Math. **91**(1954), 245–301. http://dx.doi.org/10.1007/BF02393433
[14] A. L. Shields and D. L. Williams, *Bounded projections, duality, and multipliers in spaces of analytic functions.* Trans. Amer. Math. Soc. **162**(1971), 287–302.
[15] ———, *Bounded projections, duality, and multipliers in spaces of harmonic functions.* J. Reine Angew. Math. **299/300**(1978), 256–279.
[16] K. R. Stromberg, *Probability for analysts.* Chapman and Hall Probability Series, Chapman & Hall, New York, 1994.
[17] A. Zygmund, *Trigonometric series.* Second ed., Cambridge University Press, London-New York, 1968.

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway
e-mail: kjersti.eikrem@gmail.com