The Elliptic Cylinder Functions of The Second Kind.

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§ 1. *Introductory.*

The differential equation of Mathieu, or the equation of the elliptic cylinder functions

$$\frac{d^2y}{dz^2} + (a + k^2\cos^2 z)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is known by the theory of linear differential equations to have a general solution of the type

$$y = Ae^{\mu z}\phi(z) + Be^{-\mu z}\psi(z)$$

ϕ and ψ being periodic functions of z, with period 2π.

It is known that when a certain relation, or rather one of an infinite series of relations exists between a and k, μ vanishes, and ϕ and ψ cease to be distinct. Thus in these cases the general solution degenerates into a single solution, and it is the object of the present paper to discover and investigate the nature of the corresponding second solution.

These periodic solutions, or "elliptic cylinder functions,' denoted by

$$\begin{array}{llll} ce_0(z) & ce_1(z) & ce_2(z)\ldots\ldots\ldots\ldots ce_n(z)\ldots\ldots\ldots\ldots \\ & se_1(z) & se_2(z)\ldots\ldots\ldots\ldots se_n(z)\ldots\ldots\ldots\ldots^* \end{array}$$

reduce to

$$\begin{array}{llll} 1 & \cos z & \cos 2z\ldots\ldots\ldots\ldots \cos nz\ldots\ldots\ldots\ldots \\ & \sin z & \sin 2z\ldots\ldots\ldots\ldots \sin nz\ldots\ldots\ldots\ldots \end{array}$$

when the k in equation (1) reduces to zero, and hence the corre-

* Whittaker. International Congress, Cambridge, 1912.

sponding second solutions may be taken to be such that they reduce to

$$\begin{array}{llll} z & \sin z & \sin 2z \ldots\ldots\ldots\ldots \sin nz \ldots\ldots\ldots\ldots \\ & \cos z & \cos 2z \ldots\ldots\ldots\ldots \cos nz \ldots\ldots\ldots\ldots \end{array}$$

when k becomes zero.

For these second solutions, or "elliptic cylinder functions of the second kind," Professor Whittaker has suggested the notation

$$\begin{array}{llll} in_0(z) & in_1(z) & in_2(z) \ldots\ldots\ldots\ldots in_n(z) \ldots\ldots\ldots\ldots \\ & jn_1(z) & jn_2(z) \ldots\ldots\ldots\ldots jn_n(z) \ldots\ldots\ldots\ldots \end{array}$$

and this notation will be adopted throughout the present paper.

§2. *Method of procedure.*

It is known from general theory that if

$$\frac{d^2y}{dz^2} + f(z)y = 0$$

be a linear differential equation of the second order, and if one solution $y_1(z)$ of it be known, the second solution $y_2(z)$ can readily be obtained, for it is simply

$$y_1(z)\int^z \frac{dz}{\{y_1(z)\}^2}.$$

The lower limit of integration is quite arbitrary, and the terms arising from it may be discarded, as they merely give rise to a term in $y_1(z)$ which may be considered as absorbed into the first solution of the equation.

We shall now proceed to apply this process to finding the function $in_0(z)$ which corresponds to the first solution $ce_0(z)$.

It is very convenient, in the first place, slightly to modify Mathieu's equation, and to exhibit it in the form

$$\frac{d^2y}{dz^2} + (a + 16q + 16q\cos 2z)y = 0$$

where $32q = k^2$,

as we thereby avoid introducing high powers of 2 into the subsequent working.

Retaining terms up to q^4, the expression given for $ce_0(z)$ is

$$ce_0(z) = 1 + 4q\cos 2z + 2q^2\cos 4z + 4q^3(\tfrac{1}{9}\cos 6z - 7\cos 2z) + \tfrac{1}{18}q^4(\cos 8z - 320\cos 4z), \ldots\ldots\ldots\ldots (2)$$

whence we easily find that

$$\frac{1}{ce_0^2(z)} = 1 - 8q\cos 2z + (20\cos 4z + 24)q^2 - (\tfrac{368}{9}\cos 6z + 112\cos 2z)q^3$$
$$+ \tfrac{1}{9}(667\cos 8z + 1376\cos 4z + 486)q^4$$

$$\int\frac{dz}{ce_0^2(z)} = z - 4q\sin 2z + (5\sin 4z + 24z)q^2 - (\tfrac{194}{27}\sin 6z + 56\sin 2z)q^3$$
$$+ \tfrac{1}{9}(\tfrac{667}{8}\sin 8z + 344\sin 4z + 486z)q^4$$

and

$$ce_0(z)\int\frac{dz}{ce_0^2(z)} = z\,ce_0(z)[1 + 24q^2 + 54q^4 + \dots]$$
$$- 4q\sin 2z - 3q^2\sin 4z + q^3(-\tfrac{22}{27}\sin 6z - 42\sin 2z)$$
$$+ q^4(-\tfrac{25}{216}\sin 8z - \tfrac{824}{27}\sin 4z).$$

We may divide out the constant multiplier $(1 + 24q^2 + \dots)$ and thus obtain the second solution in the form

$$in_0(z) = z\,ce_0(z) - 4q\sin 2z - 3q^2\sin 4z + q^3(-\tfrac{22}{27}\sin 6z + 54\sin 2z)$$
$$+ q^4(-\tfrac{25}{216}\sin 8z + \tfrac{1120}{27}\sin 4z). \qquad\qquad (3)$$

This method is very simple and convenient in the present case, but its application becomes increasingly difficult when we proceed to higher and higher orders of the functions *ce* and *se*, and consequently we have to fall back on another method detailed in the next section of the paper.

§ 3. *Second method of finding the Second Solution.*

In this method we try to satisfy the differential equation by a series in ascending powers of q, whose coefficients are functions of z, and whose leading term is the corresponding second solution as given in § 1 of the differential equation (1) when k is put equal to zero.

As an example of this method we will investigate the function $in_1(z)$, which is the second solution corresponding to the elliptic cylinder function $ce_1(z)$. When k (or q) vanishes it reduces to $\sin z$, and hence we assume the expansion

$$in_1(z) = \sin z + qA(z) + q^2B(z) + q^3C(z) + \dots \qquad (4)$$

where A, B, C ... are functions of z alone and may be taken as not involving $\sin z$, which merely amounts to determining the arbitrary constant by which the solution can be multiplied.

The particular value of a which gives rise to the function $ce_1(z)$ is given by

$$a + 16q = 1 - 8q - 8q^2 + 8q^3 - \tfrac{8}{3}q^4 - \tfrac{88}{9}q^5 + \ldots$$

and hence the differential equation (1) becomes

$$\frac{d^2y}{dz^2} + [16q\cos 2z + 1 - 8q - 8q^2 + 8q^3 - \tfrac{8}{3}q^4 - \tfrac{88}{9}q^5 + \ldots]y = 0.$$

Substituting the expression (4) in this differential equation we obtain

$$\begin{aligned} &-\sin z + A''q + B''q^2 + C''q^3 + \ldots \\ &\quad + (1 - 8q - 8q^2 - 8q^3 - \ldots)(\sin z + Aq + Bq^2 + Cq^3 + \ldots) \\ &\qquad + 16q\cos 2z(\sin z + Aq + Bq^2 + \ldots) = 0. \end{aligned}$$

The terms in this equation independent of q are together identically equal to zero. If we equate to zero the term in the first power of q we obtain

$$A'' + A - 8\sin z + 16\cos 2z\sin z = 0,$$

or $\quad A'' + A - 16\sin z + 8\sin 3z = 0,$

whence we obtain $\quad A = -8z\cos z + \sin 3z.$

No terms in $\sin z$ and $\cos z$ are introduced since we have decided to normalise the function $in_1(z)$ by assuming that the coefficient of $\sin z$ in it is strictly unity, and we may further assume that no term in $\cos z$ appears, or in other words, that the function $in_1(z)$ is to be purely an odd function of z.

Equating now to zero coefficients involving the second power of q, we have

$$B'' + B - 8\sin z - 8A + 16A\cos 2z = 0,$$

or $\quad B'' + B + 8\sin 5z - 8\sin 3z - 64z\cos 3z = 0,$

i.e. $\quad B = \tfrac{1}{3}\sin 5z + 5\sin 3z - 8z\cos 3z.$

The coefficients of higher powers of q may be obtained in the same way, and thus we obtain an expression for $in_1(z)$, viz.:

$$\begin{aligned} in_1(z) = &-8qz(1 - 3q^2 + 6q^3\ldots)\{\cos z + q\cos 3z + q^2(\tfrac{1}{3}\cos 5z - \cos z)\ldots\} \\ &+ \sin z + q\sin 3z + q^2(\tfrac{1}{3}\sin 5z + 5\sin 3z) + q^3(\tfrac{1}{18}\sin 7z + \tfrac{8}{3}\sin 5z - \tfrac{35}{3}\sin 3z) \\ &+ \ldots\ldots \\ = &-8q(1 - 3q^2 + 6q^3\ldots)z\, ce_1(z) + \sin z + q\sin 3z + q^2(\tfrac{1}{3}\sin 5z + 5\sin 3z) \\ &+ \ldots\ldots \qquad (5) \end{aligned}$$

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This method has the great advantage over the former that, if we know a in terms of q, we can proceed to obtain as many new terms as we desire in the expansion, without beginning the whole process over again, as is necessary in the case of the first method.

§ 4. *Summary of results obtained.*

By the use of the above methods, both of which were employed in order that the one might be a check upon the other, the following expressions have been obtained, viz.:

Corresponding to $a + 16q = -32q^2 + 224q^4 - \frac{29696}{9}q^6 + \ldots$, the second solution is

$$\begin{aligned} in_0(z) = zce_0(z) - 4q\sin 2z - 3q^2\sin 4z + q^3(-\tfrac{22}{27}\sin 6z + 54\sin 2z) \\ + q^4(-\tfrac{25}{216}\sin 8z + \tfrac{1120}{27}\sin 4z) \\ + \tfrac{1}{27}q^5(-\tfrac{137}{500}\sin 10z + \tfrac{8627}{9}\sin 6z - 26068\sin 2z) \\ + \ldots\ldots(3) \end{aligned}$$

Corresponding to $a + 16q = 1 - 8q - 8q^2 + 8q^3 - \frac{8}{3}q^4 - \ldots$, we have

$$\begin{aligned} in_1(z) = -8q(1 - 3q^2 + 6q^3 + \tfrac{31}{9}q^4 + \ldots)zce_1(z) + \sin z + q\sin 3z \\ + q^2(\tfrac{1}{3}\sin 5z + 5\sin 3z) + q^3(\tfrac{1}{18}\sin 7z + \tfrac{8}{3}\sin 5z - \tfrac{35}{3}\sin 3z) \\ + q^4(\tfrac{1}{180}\sin 9z + \tfrac{61}{108}\sin 7z - \tfrac{343}{54}\sin 5z - \tfrac{17}{3}\sin 3z) + \ldots\ldots(5) \end{aligned}$$

Corresponding to $a + 16q = 1 + 8q - 8q^2 - 8q^3 - \frac{8}{3}q^4 - \ldots$, we have

$$\begin{aligned} jn_1(z) = -8q(1 - 3q^2 - 6q^3 + \tfrac{31}{9}q^4 \ldots)zse_1(z) + \cos z + q\cos 3z \\ + q^2(\tfrac{1}{3}\cos 5z - 5\cos 3z) + q^3(\tfrac{1}{18}\cos 7z - \tfrac{8}{3}\cos 5z - \tfrac{35}{3}\cos 3z) \\ + q^4(\tfrac{1}{150}\cos 9z - \tfrac{61}{108}\cos 7z - \tfrac{343}{54}\cos 5z + \tfrac{17}{3}\cos 3z) + \ldots\ldots\ldots\ldots(6) \end{aligned}$$

Corresponding to $a + 16q = 4 + \frac{80}{3}q^2 - \frac{6104}{27}q^4 + \ldots$, we have

$$\begin{aligned} in_2(z) = 8q^2(1 - \tfrac{68}{9}q^2 \ldots)zce_2(z) + \sin 2z + \tfrac{2}{3}q\sin 4z + \tfrac{1}{6}q^2\sin 6z \\ + q^3(\tfrac{1}{45}\sin 8z - \tfrac{53}{27}\sin 4z) + q^4(\tfrac{1}{540}\sin 10z - \tfrac{457}{540}\sin 6z) + \ldots\ldots\ldots(7) \end{aligned}$$

Corresponding to $a + 16q = 4 - \frac{16}{3}q^2 + \frac{40}{27}q^4 - \ldots$, we have

$$\begin{aligned} jn_2(z) = 8q^2(1 + \tfrac{106}{27}q^2 + \ldots)zse_2(z) + \cos 2z + 2q(\tfrac{1}{3}\cos 4z - 1) \\ + \tfrac{1}{6}q^2\cos 6z + q^3(\tfrac{1}{45}\cos 8z + \tfrac{91}{27}\cos 4z - \tfrac{8}{3}) + q^4(\tfrac{1}{540}\cos 10z + \tfrac{713}{540}\cos 6z) \\ + \ldots\ldots(8) \end{aligned}$$

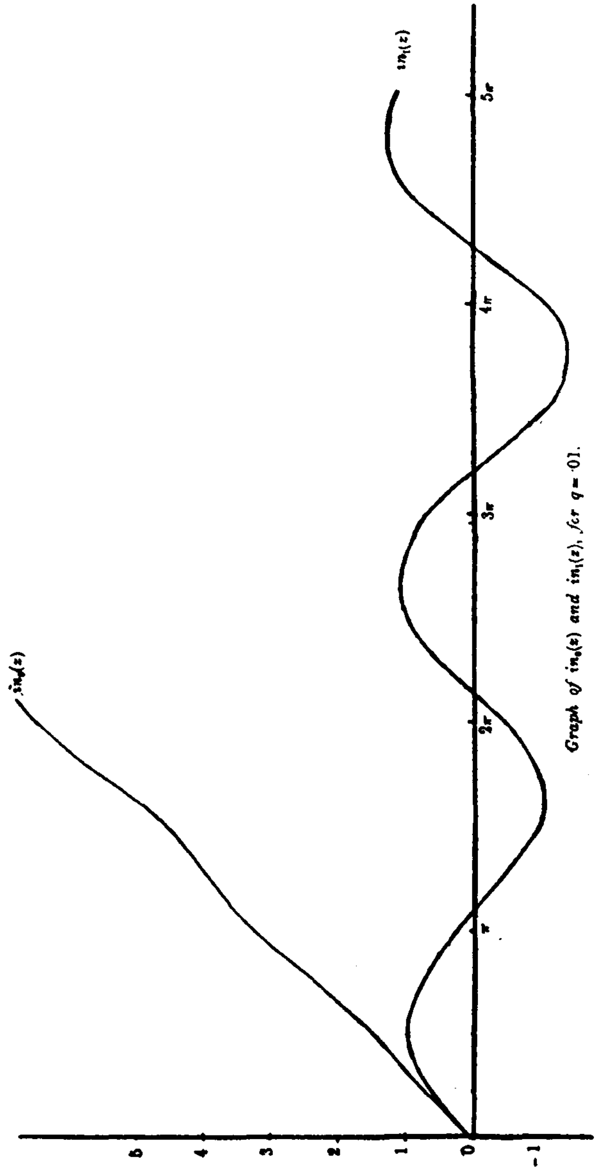
Corresponding to $a + 16q = 9 + 4q^2 - 8q^3 + \frac{13}{5}q^4 + \ldots$, we have

$$\begin{aligned} in_3(z) = -\tfrac{8}{3}q^3zce_3(z) + \sin 3z + q(\tfrac{1}{2}\sin 5z - \sin z) + q^2(\tfrac{1}{10}\sin 7z - \sin z) \\ + q^3(\tfrac{1}{90}\sin 9z + \tfrac{7}{40}\sin 5z - \tfrac{1}{2}\sin z) \\ + q^4(\tfrac{1}{1260}\sin 11z + \tfrac{17}{360}\sin 7z - \tfrac{7}{12}\sin 5z - \tfrac{1}{3}\sin z) + \ldots\ldots(9) \end{aligned}$$

and corresponding to $a + 16q = 9 + 4q^2 + 8q^3 + \frac{13}{5}q^4 + \ldots$, we have

$$\begin{aligned} jn_3(z) = -\tfrac{8}{3}q^3zse_3(z) + \cos 3z + q(\tfrac{1}{2}\cos 5z - \cos z) + q^2(\tfrac{1}{10}\cos 7z + \cos z) \\ + q^3(\tfrac{1}{90}\cos 9z + \tfrac{7}{40}\cos 5z - \tfrac{1}{2}\cos z) \\ + q^4(\tfrac{1}{1260}\cos 11z + \tfrac{17}{360}\cos 7z - \tfrac{7}{12}\cos 5z + \tfrac{1}{3}\cos z) + \ldots(10) \end{aligned}$$

Graphs of $in_0(z)$ and $in_1(z)$ for the particular case of $q = \cdot 01$ are appended.



Graph of $in_0(z)$ and $in_1(z)$, for $q = \cdot 01$.

§5. *Imaginary Values of the Argument.*

If in the differential equation

$$\frac{d^2y}{dz^2}+(a+16q+16q\cos 2z)y=0$$

we write iz for z, we obtain it in the form

$$\frac{d^2y}{dz^2}+(-a-16q-16q\cosh 2z)y=0. \quad \ldots\ldots\ldots(11)$$

The first solutions of this equation are obviously

$$ce_0(iz)\ \ ce_1(iz),\ \ ce_2(iz)\ \ldots\ldots\ldots$$
$$se_1(iz)\ \ se_2(iz),\ \ldots\ldots\ldots$$

or they may be expressed by the notation

$$ceh_0(z)\ \ ceh_1(z)\ \ ceh_2(z)\ \ldots\ldots\ldots$$
$$seh_1(z)\ \ seh_2(z)\ \ldots\ldots\ldots$$

where $\qquad ceh_n(z)\equiv ce_n(iz),\ \ seh_n(z)\equiv -i\ se_n(z).$

The corresponding second solutions are then

$$inh_0(z)=zceh_0(z)-4q\sinh 2z-3q^2\sinh 4z+q^3(-\tfrac{22}{27}\sinh 6z+54\sinh 2z)+\ldots\ldots\ldots$$

$$inh_1(z)=-8q(1-3q^2+6q^3\ldots)zceh_1(z)+\sinh z+q\sinh 3z+q^2(\tfrac{1}{3}\sinh 5z+5\sinh 3z)+\ldots\ldots\ldots$$

$$jnh_1(z)=-8q(1-3q^2-6q^3\ldots)zseh_1(z)+\cosh z+q\cosh 3z+q^2(\tfrac{1}{3}\cosh 5z-5\cosh 3z)+\ldots\ (12)$$

and so on, where

$$inh_n(z)\equiv -i\ in_n(iz)$$
$$jnh_n(z)\equiv jn_n(iz).$$

§6. *Nature of the functions.*

The elliptic cylinder functions of the second term are, of course, not periodic, but they exhibit a kind of quasi-periodicity, as is shown by the equations

$$in_n(z+2n\pi)=in_n(z)+2n\pi P_n ce_n(z)$$
$$jn_n(z+2n\pi)=jn_n(z)+2n\pi Q_n se_n(z),$$

where n is any integer and P_n and Q_n are power series in q, viz., the coefficients of $z\,ce_n(z)$ and $z\,se_n(z)$ in the expansions of $in_n(z)$ and $jn_n(z)$ respectively.

Thus $in_0(z+2n\pi) = in_0(z) + 2n\pi ce_0 z,$

$$in_1(z+2n\pi) = in_1(z) - 16n\pi q(1 - 3q^2 + 6q^3 + \ldots)ce_1(z),$$

$$jn_1(z+2n\pi) = jn_1(z) - 16n\pi q(1 - 3q^2 - 6q^3 + \ldots)se_1(z),$$

and so on.

It is also noteworthy that the terms of $in_n(z)$ coincide with those of $se_n(z)$ as far as the term in q^{n-1}, and thus up to this point are purely periodic. A term in $q^n z\, ce_n(z)$ is then introduced, and the sine terms in q^n and higher powers of q differ to a certain extent from the corresponding terms of $se_n(z)$. The function $jn_n(z)$ bears similar relations to the function $ce_n(z)$.

§ 7. *Genesis of the functions of the second kind.*

It has already been mentioned that Mathieu's differential equation admits of a general solution of the type

$$y = Ae^{\mu z}\phi(z) + Be^{-\mu z}\psi(z)$$

where ϕ and ψ are purely periodic functions of z, and that when one of a set of particular relations exists between the a and the k (or q) of the differential equation, μ vanishes and ϕ and ψ cease to be distinct, but reduce to one or other of the elliptic cylinder functions ce or se. It thus remains to attempt by a limiting process to obtain the corresponding second solution directly from the general solution.

In Professor Whittaker's notation,* the general solution which reduces to $ce_1(z)$ and $se_1(z)$ when μ vanishes is

$$y = A\Lambda(z, \sigma, q) + B\Lambda(z, -\sigma, q)$$

where $$\Lambda(z) = e^{\mu z}u(z),$$

$u(z)$ being purely periodic in z, σ is a parameter connected with a and q by the relation

$$a + 16q = 1 + 8q\cos 2\sigma + (-16 + 8\cos 4\sigma)q^2 - \ldots$$

and μ and $u(z)$ are given by the relations

$$\mu = 4q\sin 2\sigma - 12q^3\sin 2\sigma - 12q^4\sin 4\sigma + \ldots$$

$$u(z) = \sin(z-\sigma) + a_3\cos(3z-\sigma) + b_3\sin(3z-\sigma) + a_5\cos(5z-\sigma) +$$

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* Whittaker. *Proc. Edin. Math. Soc.*, XXXII. (1913–1914), p. 78.

where

$$b_3 = q + q^2\cos 2\sigma + (-\tfrac{14}{3} + 5\cos 4\sigma)q^3 + (-\tfrac{74}{9}\cos 2\sigma + 7\cos 6\sigma)q^4 + \ldots$$
$$a_3 = 3q^2\sin 2\sigma + 3q^3\sin 4\sigma + (-\tfrac{274}{9}\sin 2\sigma + 9\sin 6\sigma)q^4 + \ldots$$
$$b_5 = \tfrac{1}{3}q^3 + \tfrac{4}{9}q^2\cos 2\sigma + (-\tfrac{155}{54} + \tfrac{82}{27}\cos 4\sigma)q^4 + \ldots,$$

and so on,

$u(z)$ being a Fourier series whose coefficients are periodic functions of σ.

When σ vanishes, $\mu = 0$ and $\Lambda(z, \sigma, q)$ and $\Lambda(z, -\sigma, q)$ become identical and reduce to $se_1(z)$; when $\sigma = \frac{\pi}{2}$, μ vanishes and $\Lambda(z, \sigma, q)$ and $\Lambda(z, -\sigma, q)$ both reduce to $ce_1(z)$.

First of all consider the case when σ vanishes. In the first place, it is evident that if b_3', a_3', b_5', a_5'... are the coefficients $u(z, -\sigma)$ corresponding to the b_3, a_3, b_5, a_5 of $u(z, \sigma)$, then

$$b_3' = b_3 \quad b_5' = b_5 \quad b_7' = b_7 \ldots\ldots$$
$$a_3' = -a_3 \; a_5' = -a_5 \; a_7' = -a_7 \ldots$$

We now form the function

$$2F(z) = \Lambda(z, \sigma, q) - \Lambda(z, -\sigma, q)$$
$$= e^{\mu z}u(z, \sigma) - e^{-\mu z}u(z, -\sigma).$$

Next suppose that σ is very small, and that therefore we may write σ for $\sin\sigma$ and 1 for $\cos\sigma$. We thus have

$$2F(z) = (1 + 8q\sigma z - 24q^3\sigma z - 48q^4\sigma z + \ldots)\{\sin(z-\sigma) + a_3\cos(3z-\sigma)$$
$$+ b_3\sin(3z-\sigma) + a_5\cos(5z-\sigma) + b_5\sin(5z-\sigma) + \ldots\}$$
$$-(1 - 8q\sigma z + 24q^3\sigma z + 48q^4\sigma z + \ldots)\{\sin(z+\sigma) - a_3\cos(3z+\sigma)$$
$$+ b_3\sin(3z+\sigma) - a_5\cos(5z+\sigma) + b_5\sin(5z+\sigma) + \ldots\}$$
$$= 2\cos z\sin\sigma + 2b_3\cos 3z\sin\sigma + 2b_5\cos 5z\sin\sigma + 2b_7\cos 7z\sin\sigma + \ldots$$
$$- 2a_3\cos 3z\cos\sigma - 2b_5\cos 5z\cos\sigma - 2a_7\cos 7z\cos\sigma - \ldots$$
$$+ 8qz\sigma(1 - 3q^2 - 6q^3\ldots)(-2\sin z\cos\sigma - 2b_3\sin 3z\cos\sigma$$
$$- 2b_5\sin 5z\cos\sigma - 2b_7\sin 7z\cos\sigma - \ldots$$
$$+ 2a_3\sin 3z\sin\sigma + 2a_5\sin 5z\sin\sigma + 2a_7\sin 7z\sin\sigma + \ldots).$$

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Hence, to the fourth power of q

$$F(z)=\sigma\cos z+\sigma\cos 3z(q-5q^2-\tfrac{35}{3}q^3+\tfrac{17}{3}q^4\ldots)$$
$$+\sigma\cos 5z(\tfrac{1}{3}q^2-\tfrac{8}{3}q^3-\tfrac{343}{54}q^4+\ldots)+\sigma\cos 7z(\tfrac{1}{18}q^3-\tfrac{61}{108}q^4\ldots)$$
$$+\sigma\cos 9z(\tfrac{1}{180}q^4\ldots)+\ldots$$
$$-8qz\sigma(1-3q^2-6q^3\ldots)\{\sin z+(q+q^2+\tfrac{1}{3}q^3-\ldots)\sin 3z$$
$$+(\tfrac{1}{3}q^2+\tfrac{4}{9}q^3+\ldots)\sin 5z\}+\tfrac{1}{18}q^3\sin 7z+\ldots$$
$$=-8qz\sigma(1-3q^2-6q^3-\ldots)\{\sin z+q\sin 3z+q^2(\tfrac{1}{3}\sin 5z+\sin 3z)$$
$$+q^3(\tfrac{1}{18}\sin 7z+\tfrac{4}{9}\sin 5z+\tfrac{1}{3}\sin 3z)+\ldots\}$$
$$+\sigma\cos z+\sigma q\cos 3z+\sigma q^2(\tfrac{1}{3}\cos 5z-5\cos 3z)+$$
$$+\sigma q^3(\tfrac{1}{18}\cos 7z-\tfrac{8}{3}\cos 5z-\tfrac{35}{3}\cos 3z)$$
$$+\sigma q^4(\tfrac{1}{180}\cos 9z-\tfrac{61}{108}\cos 7z-\tfrac{343}{54}\cos 5z+\tfrac{17}{3}\cos 3z)+\ldots$$
$$=\sigma jn_1(z).$$

Hence we see that the function $jn_1(z)$ arises as the limit, when σ tends to zero, of the expression

$$\frac{\Lambda(z,\sigma,q)-\Lambda(z,-\sigma,q)}{2\sigma}.$$

We may now proceed in a similar manner to investigate the origin of the second solution which arises when σ tends towards the value $\frac{\pi}{2}$ and Λ merges into the particular function $ce_1(z)$.

For convenience we may employ instead of σ the parameter $\sigma'=\frac{\pi}{2}-\sigma$, so that when σ becomes $\frac{\pi}{2}$, σ' vanishes.

μ is now given by the expression

$$\mu=-(4q\sin 2\sigma'-12q^3\sin 2\sigma'-12q^4\sin 4\sigma'+\ldots)$$

and $u(z)$ by

$$u(z)=-\cos(z+\sigma')+a_3\sin(3z+\sigma')-b_3\cos(3z+\sigma')+\ldots$$

where

$$b_3=q-q^2\cos 2\sigma'+(-\tfrac{14}{3}+5\cos 4\sigma')q^3+(\tfrac{74}{9}\cos 2\sigma'-7\cos 6\sigma')q^4+\ldots$$
$$a_3=3q^2\sin 2\sigma'-3q^3\sin 4\sigma'+(-\tfrac{274}{9}\sin 2\sigma'+9\sin 6\sigma')q^4+\ldots$$
$$b_5=\tfrac{1}{3}q^2-\tfrac{4}{9}q^3\cos 2\sigma'+(-\tfrac{155}{54}+\tfrac{82}{27}\cos 4\sigma')q^4+\ldots$$
$$a_5=\tfrac{14}{9}q^3\sin 2\sigma'-\tfrac{44}{27}q^4\sin 4\sigma'+\ldots$$
$$b_7=\tfrac{1}{18}q^3-\tfrac{1}{12}q^4\cos 2\sigma'+\ldots$$
$$a_7=\tfrac{35}{108}q^4\sin 2\sigma'+\ldots$$
$$b_9=\tfrac{1}{180}q^4+\ldots$$

As before if a_3', b_3', a_5', b_5' are the coefficients of $u(z, -\sigma')$ corresponding to the a_3, b_3, a_5, b_5 of $u(z, \sigma')$ we have

$$b_3' = b_3 \qquad b_5' = b_5 \qquad b_7' = b_7 \;\ldots\ldots$$
$$a_3' = -a_3 \quad a_5' = -a_5 \quad a_7' = -a_7 \;\ldots$$

Also, as before, we form the function

$$2F(z) = \Lambda'(z, \sigma', q) - \Lambda'(z, -\sigma', q)$$
$$= e^{\mu z}u(z, \sigma') - e^{-\mu z}u(z, -\sigma'),$$

Λ' being what Λ becomes when for σ we write $\frac{\pi}{2} - \sigma'$.

We then obtain, using the condition that σ' is to be small,

$$2F(z) = (1 - 8q\sigma'z + 24q^3\sigma'z + 48q^4\sigma'z + \ldots)(-\cos(z+\sigma') + a_3\sin(3z+\sigma')$$
$$- b_3\cos(3z+\sigma') + a_5\sin(5z+\sigma') - b_5\cos(5z+\sigma') + \ldots)$$
$$- (1 + 8q\sigma'z - 24q^3\sigma'z - 48q^4\sigma'z + \ldots)(-\cos(z-\sigma') - a_3\sin(3z-\sigma')$$
$$- b_3(\cos 3z - \sigma') - a_5\sin(5z-\sigma') - b_5\cos(5z-\sigma') + \ldots).$$
$$= 2\sin z\sin\sigma' + 2b_3\sin 3z\sin\sigma' + 2b_5\sin 5z\sin\sigma' + \ldots$$
$$+ 2a_3\sin 3z\cos\sigma' + 2a_5\sin 5z\cos\sigma' + \ldots$$
$$+ 8q\sigma'z(1 - 3q^2 + 6q^3 + \ldots)(2\cos z\cos\sigma' + 2b_3\cos 3z\cos\sigma' + 2b_5\cos 5z\cos\sigma' + \ldots$$
$$- 2a_3\cos 3z\sin\sigma' - 2a_5\cos 5z\sin\sigma' - \ldots).$$

Hence to the fourth power of q.

$$F(z) = \sigma'\sin z + \sigma'\sin 3z(q + 5q^2 - \tfrac{35}{3}q^3 - \tfrac{17}{3}q^4 + \ldots)$$
$$+ \sigma'\sin 5z(\tfrac{1}{3}q^2 + \tfrac{8}{3}q^3 - \tfrac{343}{54}q^4 + \ldots) + \sigma'\sin 7z(\tfrac{1}{18}q^3 + \tfrac{61}{108}q^4 + \ldots)$$
$$+ \sigma'\sin 9z(\tfrac{1}{180}q^4 + \ldots)$$
$$+ 8q\sigma'z(1 - 3q^2 + 6q^3)\{\cos z + \cos 3z(q - q^2 - \tfrac{11}{3}q^3 + \tfrac{11}{9}q^4 + \ldots)$$
$$+ \cos 5z(\tfrac{1}{3}q^2 - \tfrac{4}{9}q^3 + \tfrac{1}{6}q^4 + \ldots) + \cos 7z(\tfrac{1}{18}q^3 - \tfrac{1}{12}q^4 + \ldots)$$
$$+ \cos 9z(\tfrac{1}{180}q^4 + \ldots)\}$$
$$= 8q\sigma'z(1 - 3q^2 + 6q^3)\{\cos z + q\cos 3z + q^2(\tfrac{1}{3}\cos 5z - \cos 3z)$$
$$+ q^3(\tfrac{1}{18}\cos 7z - \tfrac{4}{9}\cos 5z - \tfrac{11}{3}\cos 3z) + q^4(\tfrac{1}{180}\cos 9z - \tfrac{1}{12}\cos 7z$$
$$+ \tfrac{1}{6}\cos 5z + \tfrac{11}{9}\cos 3z) + \ldots\}$$
$$+ \sigma'\sin z + \sigma'q\sin 3z + \sigma'q^2(\tfrac{1}{3}\sin 5z + 5\sin 3z) + \sigma'q^3(\tfrac{1}{18}\sin 7z + \tfrac{8}{3}\sin 5z$$
$$- \tfrac{35}{3}\sin 3z)$$
$$+ \sigma'q^4(\tfrac{1}{180}\sin 9z + \tfrac{61}{108}\sin 7z - \tfrac{343}{54}\sin 5z - \tfrac{17}{3}\sin 3z) + \ldots$$
$$= \sigma' in_1(z).$$

Thus the function $in_1(z)$ is seen to be the limit when σ' tends to zero, of the function

$$\frac{\Lambda'(z, \sigma' q) - \Lambda'(z, -\sigma', q)}{2\sigma'}.$$

Hence the function $in_1(z)$ arises as the limit, when σ tends to the value $\frac{\pi}{2}$, of the expression

$$\frac{\Lambda(z, \sigma, q) - \Lambda(z, -\sigma, q)}{\pi - 2\sigma}.$$