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PRIMITIVE IDEALS OF THE QUANTISED ENVELOPING ALGEBRA OF A COMPLEX LIE ALGEBRA

SEI-QWON OH

In this paper, we characterise all primitive ideals of the quantised enveloping algebra $U_q[sl(2,\mathbb{C})]$ of the complex Lie algebra $sl(2,\mathbb{C})$ and show how they are similar to those of $U[sl(2,\mathbb{C})]$, the enveloping algebra of $sl(2,\mathbb{C})$.

The quantised enveloping algebra $U_q[sl(2,\mathbb{C})]$ of the complex Lie algebra $sl(2,\mathbb{C})$ is the algebra $\mathbb{C}[E,K,K^{-1},F]$ defined by the relations

$$KE = q^2EK,\ KF = q^{-2}FK,\ EF - FE = \frac{K^2 - K^{-2}}{q^2 - q^{-2}},$$

where q is not a root of unity. Therefore $U_q[sl(2,\mathbb{C})]$ is the iterated Ore extension $\mathbb{C}[E][K,K^{-1};\sigma][F;\tau,\delta]$, where $\sigma(E) = q^2E$, $\tau(E) = E$, $\tau(K) = q^2K$, $\delta(E) = -(K^2 - K^{-2})/(q^2 - q^{-2})$, $\delta(K) = 0$. Hence $U_q[sl(2,\mathbb{C})]$ is a noetherian domain and the sets $\Re_1 = \{E^iK^jF^n | (i,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+,\ j \in \mathbb{Z}\}$ and $\Re_2 = \{F^iK^jE^n | (i,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+,\ j \in \mathbb{Z}\}$ are its $\mathbb{C}$-bases and its Gelfand-Kirillov dimension is 3. The element $\Omega = EF + (q^{-2}K^2 + q^2K^{-2})/(q^2 - q^{-2})^2 = FE + (q^2K^2 + q^{-2}K^{-2})/(q^2 - q^{-2})^2$ is a central element of $U_q[sl(2,\mathbb{C})]$.

The quantised enveloping algebra $U_q(\mathfrak{g})$ of a semisimple Lie algebra appears in [2] (originally defined by Drinfield, Jimbo, and others). Moreover the structure of a finite dimensional irreducible module of $U_q(\mathfrak{g})$ is also shown in [2]. In particular, all finite dimensional irreducible modules of $U_q[sl(2,\mathbb{C})]$ are of the form $L(\omega q^m)$, where $\omega \in \{1,-1,i,-i\}$, $m \in \mathbb{Z}^+$, where $L(\lambda)$ is the irreducible highest weight module with highest weight $\lambda \in \mathbb{C}^*$. (See Lemma 5 for the definition of $L(\lambda)$.)

The structure of a finite dimensional irreducible module of $U_q[sl(2,\mathbb{C})]$ is a parallel property of the (classical) enveloping algebra $U[sl(2,\mathbb{C})]$. It is well known that primitive ideals of $U[sl(2,\mathbb{C})]$ with infinite codimensions are of the form $\langle \Omega' - c\rangle$, where $c \in \mathbb{C}$ and Ω' is Casimir element of the enveloping algebra $U[sl(2,\mathbb{C})]$. In particular, the ideal $\langle \Omega' - c\rangle$ is maximal if and only if $c \in \mathbb{C}$ is not of the form $n^2 + 2n$, $n \in \mathbb{N}$. In this paper, we show the following quantum version:

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THEOREM 1.

(1) *All primitive ideals of $U_q[sl(2,\mathbb{C})]$ with finite codimensions are the annihilators of $L(\omega q^m)$, where $\omega \in \{1,-1,i,-i\}$, $m \in \mathbb{Z}^+$.*

(2) *For every $c \in \mathbb{C}$, the ideal $\langle \Omega - c\rangle$ is a primitive ideal of $U_q[sl(2,\mathbb{C})]$ with infinite codimension. In fact the ideal $\langle \Omega - c\rangle$ is the annihilator of $L(\lambda)$ for some $\lambda \in \mathbb{C}^*$. Conversely, any primitive ideal with infinite codimension is $\langle \Omega - c\rangle$ for some $c \in \mathbb{C}$. Moreover, the ideal $\langle \Omega - c\rangle$ is maximal if and only if $c \in \mathbb{C}$ is not of the form $\pm\left(q^{2(m+1)} + q^{-2(m+1)}\right)/\left(q^2 - q^{-2}\right)^2$, $m \in \mathbb{Z}^+$.*

COROLLARY 2. *The centre of $U_q[sl(2,\mathbb{C})]$ is just the subalgebra $\mathbb{C}[\Omega]$.*

PROOF: It follows immediately from Theorem 1 and 8.4.18 of [3]. ∎

Since Ω is central in $U_q[sl(2,\mathbb{C})]$ all primitive ideals contain an element $\Omega - c$ for some $c \in \mathbb{C}$ by 8.4.18 of [3]. First we prove that the ideal $\langle \Omega - c\rangle$ is completely prime.

LEMMA 3. *Let R be a domain, $\sigma \in Aut(R)$ and let δ be a left σ-derivation. Put $S = R[X;\sigma,\delta]$ the skew polynomial ring over R. If $aX + b \in S$ is a normal element and a is invertible in R, then the ideal $S(aX+b)$ is completely prime in S. (An element $y \in S$ is said to be normal provided $yS = Sy$.)*

PROOF: Notice that $deg(fg) = deg(f) + deg(g)$ for nonzero $f, g \in S$, and given $f \in S$, there are $h \in S$, $r \in R$ such that $f = h(aX+b) + r$ because a is invertible. Given $f = h_1(aX+b) + r_1$ and $g = h_2(aX+b) + r_2$, if $fg \in S(aX+b)$ then $r_1 r_2 \in S(aX+b)$. Thus $r_1 r_2 = 0$. Since R is a domain, $r_1 = 0$ or $r_2 = 0$. Therefore the ideal $S(aX+b)$ is completely prime in S. ∎

PROPOSITION 4. *For any $c \in \mathbb{C}$, the ideal $\langle \Omega - c\rangle$ of $U_q[sl(2,\mathbb{C})]$ is completely prime.*

PROOF: The set $\mathcal{A} = \{E^i | i \in \mathbb{Z}^+\}$ is a left Ore set of the subalgebra $\mathbb{C}[E,K,K^{-1}]$. Let R be the localisation $\mathcal{A}^{-1}\mathbb{C}[E,K,K^{-1}]$. Thus $R = \mathbb{C}[E,E^{-1},K,K^{-1}]$ is an integral domain. By Lemma 3, the ideal $\langle \Omega - c\rangle$ is completely prime in the skew polynomial ring $R[F;\sigma,\delta]$ where $\sigma(E) = E$, $\sigma(K) = q^2 K$, $\delta(E) = -\left(K^2 - K^{-2}\right)/\left(q^2 - q^{-2}\right)$, $\delta(K) = 0$.

We prove that the ideal $\langle \Omega - c\rangle$ is completely prime in $U_q[sl(2,\mathbb{C})]$. Given $f, g \in U_q[sl(2,\mathbb{C})]$, if $fg \in \langle \Omega - c\rangle$ then $f \in R[F;\sigma,\delta]\langle \Omega - c\rangle$ or $g \in R[F;\sigma,\delta]\langle \Omega - c\rangle$, say $f \in R[F;\sigma,\delta]\langle \Omega - c\rangle$. Hence $E^t f = h(\Omega - c)$ for some $t \in \mathbb{Z}^+$ and $h \notin EU_q[sl(2,\mathbb{C})]$. We give an order relation in the basis $\mathfrak{R}_1$ as follows:

$$E^i K^j F^n < E^{i'} K^{j'} F^{n'} \text{ if } (i,j,n) < (i',j',n') \text{ in the lexicographic order.}$$

Express $E^t f$ and h as linear combinations of the basis $\mathfrak{R}_1$. Since $h \notin EU_q[sl(2,\mathbb{C})]$, the set $T = \{K^j F^n \mid \text{the coefficient of } K^j F^n \text{ in the expression of } h \text{ is nonzero}\}$ is nonempty. Let K^rF^s be the maximal monomial in the set T. Since $\Omega - c$ is central and $KE = q^2EK$, we have the following:

$$\begin{aligned} h(\Omega - c) &= \left(\alpha K^r F^s + \sum_{(i,j,n)\neq(0,r,s)} \alpha_{ijn} E^i K^j F^n\right)(\Omega - c) \\ &= \alpha K^r\left(EF + \frac{q^{-2}K^2 + q^2K^{-2}}{(q^2-q^{-2})^2} - c\right)F^s \\ &\quad + \sum_{(i,j,n)\neq(0,r,s)} \alpha_{ijn} E^i K^j\left(EF + \frac{q^{-2}K^2 + q^2K^{-2}}{(q^2-q^{-2})^2} - c\right)F^n \\ &= \alpha q^{-2}\left(q^2 - q^{-2}\right)^{-2} K^{r+2}F^s + \text{ other terms }, \end{aligned}$$

where α is nonzero. Therefore we have $t = 0$ and so f is in the ideal $\langle \Omega - c\rangle$. $\square$

DEFINITION:

(1) Let $\rho : U_q[sl(2,\mathbb{C})] \longrightarrow End_{\mathbb{C}}(M)$ be a representation and $\lambda \in \mathbb{C}^*$. A nonzero vector $v \in M$ is said to be a *highest weight vector* with weight λ if $\rho(K)v = \lambda v$, $\rho(E)v = 0$.

(2) If M is generated by a highest weight vector v with weight λ, M is said to be a *highest weight module* with highest weight λ.

LEMMA 5.

(1) *Every homomorphic image of a highest weight module is also a highest weight module with the same highest weight.*

(2) *For every $\lambda \in \mathbb{C}^*$, there exists an irreducible highest weight module with highest weight λ, denoted by $L(\lambda)$.*

PROOF: See Proposition 4 of [2] and its proof. $\square$

LEMMA 6. *Let $\rho : U_q[sl(2,\mathbb{C})] \longrightarrow End_{\mathbb{C}}(L(\lambda))$ be a representation and $\lambda \in \mathbb{C}^*$. The irreducible module $L(\lambda)$ is finite dimensional over $\mathbb{C}$ if and only if λ is of the form ωq^m, where $\omega \in \{1, -1, i, -i\}$, $m \in \mathbb{Z}^+$.*

PROOF: Theorem 1 of [2]. $\square$

LEMMA 7. *Let R be a $\mathbb{C}$-algebra and let M be an irreducible R-module with $ann_R M = P$. Then P is of finite codimension (that is, $dim_{\mathbb{C}} R/P < \infty$) if and only if $dim_{\mathbb{C}} M$ is finite.*

PROOF: If P is of finite codimension then $dim_{\mathbb{C}} M$ is finite because M is a finitely generated R/P-module. Conversely, let $dim_{\mathbb{C}} M$ be finite. The kernel of the representation $\rho : R \longrightarrow End_{\mathbb{C}} M$ is just P. Hence P is of finite codimension. $\square$

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Proof of Theorem 1: If J is a primitive ideal containing $\langle\Omega - c\rangle$ properly, the Gelfand-Kirillov dimension of $U_q[sl(2,\mathbb{C})]/J$ is less than or equal to 1 by [1, 8.3.6]. Since there is no nonartinian finitely generated noetherian primitive $\mathbb{C}$-algebra with Gelfand-Kirillov dimension 1 (see the proof of [4, 3.2]), the codimension of J is finite by [1, 8.1.17]. Hence every primitive ideal with infinite codimension is of the form $\langle\Omega - c\rangle$. Therefore it is enough to show that there exists $\lambda \in \mathbb{C}^*$ such that λ is not of the form ωq^m and such that Ω acts on $L(\lambda)$ by the scalar c by Lemma 5, 6 and 7.

Let v be a highest weight vector of $L(\lambda)$ on which Ω acts by the scalar c. Then we have

$$cv = \Omega v = \left(FE + \frac{q^2K^2 + q^{-2}K^{-2}}{(q^2 - q^{-2})^2}\right)v = \frac{q^2\lambda^2 + q^{-2}\lambda^{-2}}{(q^2 - q^{-2})^2}v.$$

Thus we have the equation

$$(*) \qquad \frac{q^2\lambda^2 + q^{-2}\lambda^{-2}}{(q^2 - q^{-2})^2} = c$$

For every $c \in \mathbb{C}$, it is enough to show that the equation (*) has a root which is not of the form ωq^m. If $c \neq \pm\left(q^{2(m+1)} + q^{-2(m+1)}\right)/\left(q^2 - q^{-2}\right)^2$, (*) has four roots all of which are not of the form ωq^m. If $c = \left(q^{2(m+1)} + q^{-2(m+1)}\right)/\left(q^2 - q^{-2}\right)^2$, (*) has roots $\pm q^{-m-2}$. If $c = -\left(q^{2(m+1)} + q^{-2(m+1)}\right)/\left(q^2 - q^{-2}\right)^2$, (*) has roots $\pm i q^{-m-2}$. ∎

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Department of Mathematics
Chungnam National University
Taejon 305-764
South Korea