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A COMMUTATIVITY CRITERION FOR PRESPECTRAL OPERATORS

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It is shown that if a bounded linear operator A commutes with a prespectral operator T of class Γ, then A commutes with the resolution of the identity of class Γ for T, say $P(\cdot)$, if and only if $A^*(\Gamma) \subseteq [P^c]^*\Gamma$. Here A^* is the dual operator of A and $[P^c]^*\Gamma$ is the linear span of the set $\{U^*\xi;\ U \in P(\cdot)^c,\ \xi \in \Gamma\}$ where $P(\cdot)^c$ denotes the commutant of the range of $P(\cdot)$.

One of the fundamental results in the theory of spectral operators is the commutativity theorem: a bounded operator commutes with a spectral operator if and only if it commutes with its resolution of the identity [1; Theorem 6.6]. This commutativity result is known to fail for prespectral operators. Indeed, U. Fixman showed that there exist on ℓ^∞ a prespectral operator T with a resolution of the identity $P(\cdot)$ of class $\Gamma = \ell^1$ and a bounded operator A which commutes with T but not with every value of $P(\cdot)$ [1; p.144]. The crucial point in this example is that Γ is not mapped into $[P^c]^*\Gamma$ by the dual operator A^* of A.

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Here $[P^c]^*\Gamma$ is the linear span of $\{U^*\xi;\ U \in P(\cdot)^c,\ \xi \in \Gamma\}$ and $P(\cdot)^c$ denotes the commutant of the range of $P(\cdot)$. The purpose of this note is to establish the fact that if a bounded operator A commutes with a prespectral operator T of class Γ , then A commutes with the resolution of the identity of class Γ for T , say $P(\cdot)$, if and only if $A^*(\Gamma) \subseteq [P^c]^*\Gamma$.

If X is a Banach space, then $L(X)$ denotes the space of all continuous linear operators of X into itself. The identity operator is denoted by I . The space of all continuous linear functionals on X is denoted by X^* . Let $\mathbb{C}$ denote the complex number field and $\mathcal{B}$ the σ-algebra of Borel subsets of $\mathbb{C}$.

Let Γ be a total subspace of X^* . A set function $P: \mathcal{B} \to L(X)$ is called a *spectral measure of class* Γ if and only if

(i) $P(\mathbb{C}) = I$,

(ii) $P(E \cap F) = P(E)P(F)$ for all $E,F \in \mathcal{B}$, and

(iii) for all $x \in X$ and all $\xi \in \Gamma$ the $\mathbb{C}$-valued set function $\langle P(\cdot)x,\xi\rangle$ is countably additive on $\mathcal{B}$.

It is usually assumed, in addition, that $\sup\{\|P(E)\|\ ;\ E \in \mathcal{B}\}$ is finite, but this already follows from the requirements (i)-(iii) [2; p.150]. Since I belongs to the range of $P(\cdot)$ it is clear that $\Gamma \subseteq [P^c]^*\Gamma \subseteq X^*$.

LEMMA 1. *Let* $P: \mathcal{B} \to L(X)$ *be a spectral measure of class* Γ . *Then* $P(\cdot)$ *is also a spectral measure of class* $\Lambda = [P^c]^*\Gamma$.

Proof. Let $x \in X$ and $\xi \in \Lambda$. Then $\xi = \sum_{r=1}^n U_r^*\xi_r$ for some $U_r \in P(\cdot)^c$ and $\xi_r \in \Gamma$, $r = 1,\dots,n$. It follows that

$$\langle P(E)x,\xi\rangle = \sum_{r=1}^n \langle P(E)x, U_r^*\xi_r\rangle = \sum_{r=1}^n \langle U_rP(E)x,\xi_r\rangle = \sum_{r=1}^n \langle P(E)U_rx,\xi_r\rangle\ ,$$

for each $E \in \mathcal{B}$. Accordingly, $\langle P(\cdot)x,\xi\rangle$ is countably additive. □

An operator $T \in L(X)$ is called a *prespectral operator of class* Γ

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if there is a spectral measure $P(\cdot)$ of class Γ, necessarily unique [1; Theorem 5.13], such that $T \in P(\cdot)^c$ and the spectrum of the restriction of T to each closed invariant subspace $P(E)X$, $E \in \mathcal{B}$, is contained in the closure of E in $\mathbb{C}$. The measure $P(\cdot)$ is called the *resolution of the identity of class* Γ *for* T. Spectral operators correspond to the case when $\Gamma = X^*$ [1; Theorem 6.5]. An example of a prespectral operator (of some class Γ) which is not a spectral operator is given by $Tf = g$, $f \in X = L^\infty([0,1])$, where $g(s) = sf(s)$, $s \in [0,1]$, and $\Gamma = L^1([0,1])$.

PROPOSITION 1. *Let* $T \in L(X)$ *be a prespectral operator of class* Γ *and* $P(\cdot)$ *be its resolution of the identity of class* Γ. *Then* T *is also a prespectral operator of class* $\Lambda = [P^c]^*\Gamma$ *with the same* $P(\cdot)$ *being its resolution of the identity of class* Λ.

Proof. It follows from Lemma 1 that $P(\cdot)$ is a spectral measure of class Λ which also satisfies, if considered as being a class Λ rather than class Γ, the properties $T \in P(\cdot)^c$ and the spectrum of the restriction of T to each closed invariant subspace $P(E)X$, $E \in \mathcal{B}$ is contained in the closure of E. Accordingly, $P(\cdot)$ is a resolution of the identity of class Λ for T and so is *the* resolution of the identity of class Λ for T [1; Theorem 5.13]. □

If $T \in L(X)$ is a prespectral operator of class Γ with resolution of the identity of class Γ, say $P(\cdot)$, and $A \in L(X)$ commutes with T, then it is known that

$$(1) \qquad A\left(\int_{\sigma(T)} f dP\right) = \left(\int_{\sigma(T)} f dP\right) A, \quad f \in C(\sigma(T)),$$

where $\sigma(T)$ is the spectrum of T [1; Theorem 5.12]. The 'integral' is defined via a process of continuous extension from the $\mathcal{B}$-simple functions [1; p.120].

The main result can now be published.

THEOREM 1. *Let* $T \in L(X)$ *be a prespectral operator of class* Γ *and* $A \in L(X)$ *commute with* T. *If* $P(\cdot)$ *is the resolution of the*

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identity of class Γ *for* T, *then* A *commutes with each value of* $P(\cdot)$ *if and only if* $A^*(\Gamma) \subseteq [P^c]^*\Gamma$.

Proof. If A commutes with each value of $P(\cdot)$, then $A \in P(\cdot)^c$ and hence, $A^*(\Gamma) \subseteq [P^c]^*\Gamma$ by definition of $[P^c]^*\Gamma$.

Conversely, suppose that $A^*(\Gamma) \subseteq [P^c]^*\Gamma$. Fix $x \in X$ and $\xi \in \Gamma$. Define $\mathbb{C}$-valued set functions μ and ν on $\mathcal{B}$ by $\nu(\cdot) = \langle P(\cdot)Ax, \xi\rangle$ and $\mu(\cdot) = \langle AP(\cdot)x, \xi\rangle = \langle P(\cdot)x, A^*\xi\rangle$. Then ν is σ-additive by definition of $P(\cdot)$ being a spectral measure of class Γ and μ is σ-additive by the hypothesis $A^*\xi \in [P^c]^*\Gamma$ and Lemma 1. Since μ and ν are regular it follows from (1) that

$$\int_{\sigma(T)} f d\nu = \int_{\sigma(T)} f d\mu , \quad f \in C(\sigma(T)) ,$$

and so the Riesz representation theorem implies that $\nu = \mu$. Since $x \in X$ and $\xi \in \Gamma$ are arbitrary it follows from the totality of Γ that $AP(E) = P(E)A$ for each $E \in \mathcal{B}$. □

COROLLARY 1.1. *Let* X *be a Banach space and* $T \in L(X^*)$ *be a prespectral operator of class* X . *If* $A \in L(X)$ *satisfies* $A^*T = TA^*$, *then* A^* *commutes with the resolution of the identity of class* X *for* T.

The difficulty with Theorem 1 is that to apply it in practice it is necessary to be able to identify the subspace $[P^c]^*\Gamma$ which, in turn, requires a specific knowledge of the resolution of the identity of class Γ for T , say $P(\cdot)$, and its commutant $P(\cdot)^c$. However, it is clear that if Γ itself happens to be an invariant subspace of A^* , then certainly A commutes with $P(\cdot)$. This sufficient condition, although more stringent than the hypothesis $A^*(\Gamma) \subseteq [P^c]^*\Gamma$ and hence less likely to be satisfied, nevertheless has the advantage that it is easier to verify. Actually, under some reasonable topological assumptions it turns out that the containment $A^*(\Gamma) \subseteq \Gamma$ is not too far from the condition $A^*(\Gamma) \subseteq [P^c]^*\Gamma$.

PROPOSITION 2. *Let* Γ *be a total subspace of* X^* *such that* Γ *is sequentially closed for the weak topology* $\sigma(\Gamma,X)$ *induced by the dual pairing* $\langle\Gamma,X\rangle$. *If* $P\colon \mathcal{B} \to L(X)$ *is a spectral measure of class* Γ *such that each operator* $P(E)$, $E \in \mathcal{B}$, *is continuous from* $(X, \sigma(X,\Gamma))$ *into* $(X, \sigma(X,\Gamma))$, *then* Γ *coincides with the linear span* $[\mathcal{U}(P)]^*\Gamma$, *of* $\{V^*\xi;\ V \in \mathcal{U}(P),\ \xi \in \Gamma\}$ *where* $\mathcal{U}(P)$ *denotes the closed algebra generated by* $\{P(E);\ E \in \mathcal{B}\}$ *with respect to the uniform operator topology in* $L(X)$.

Remark. An operator $S\colon X \to X$ is continuous from $(X, \sigma(X,\Gamma))$ into $(X, \sigma(X,\Gamma))$ if and only if $S^*(\Gamma) \subseteq \Gamma$.

Proof. The inclusion $\Gamma \subseteq [\mathcal{U}(P)]^*\Gamma$ always holds. To show the reverse inclusion it suffices to show, by definition of $[\mathcal{U}(P)]^*\Gamma$ and the fact that Γ is a subspace, that $V^*\xi \in \Gamma$ whenever $\xi \in \Gamma$ and $V \in \mathcal{U}(P)$. Noting that the range of $P(\cdot)$ is a Γ-σ-complete Boolean algebra in the sense of Definition 2 of [2] it follows from [2; Lemma 2] that if K is the maximal ideal space of $\mathcal{U}(P)$, then there exist a spectral measure $Q\colon \mathcal{B}_K \to L(X)$ of class Γ and a function $f \in C(K)$ such that $V = \int_K f\,dQ$, where $\mathcal{B}_K$ is the σ-algebra of Borel subsets of K . In addition, the range of Q coincides with $\{P(E);\ E \in \mathcal{B}\}$. Choose a sequence of $\mathcal{B}_K$-simple functions, say $\{f_n\}$, such that $f_n \to f$ uniformly on K . Then $V = \lim \int_K f_n\,dQ$, where the limit exists in the uniform operator topology of $L(X)$ [1; p.120]. Accordingly

(2) $$\langle x, V^*\xi\rangle = \langle Vx, \xi\rangle = \lim \left\langle \left(\int_K f_n\,dQ\right)x, \xi\right\rangle = \lim \left\langle x, \left(\int_K f_n\,dQ\right)^*\xi\right\rangle ,$$

for each $x \in X$. But, if $g = \sum_{r=1}^{n} \alpha_r \chi_{F(r)}$ is a $\mathcal{B}_K$-simple function then it follows from the identity $\left(\int_K g\,dQ\right)^*\xi = \sum_{r=1}^{n} \alpha_r Q(F(r))^*\xi$, the inclusion $\{Q(F(r))\}_{r=1}^{n} \subseteq \{P(E);\ E \in \mathcal{B}\}$ and the assumption that Γ is invariant for each operator $P(E)^*$, $E \in \mathcal{B}$, that $\left(\int_K g\,dQ\right)^*\xi \in \Gamma$. Accordingly, the sequence $\left\{\left(\int_K f_n\,dQ\right)^*\xi\right\}_{n=1}^{\infty}$ is contained in Γ and, by (2), it converges to $V^*\xi$ with respect to the topology $\sigma(\Gamma,X)$. Then the $\sigma(\Gamma, X)$-sequential closedness of Γ implies that $V^* \xi \in \Gamma$. □

Remark. It is always the case that $U(P) \subseteq P(\cdot)^c$ and hence, under the assumptions of Proposition 2, the subspace Γ can be a proper subspace of $[P^c]^*\Gamma$ only if the containment $U(P) \subseteq P(\cdot)^c$ is proper. Of course, if it is known for some reason that $U(P) = P(\cdot)^c$, then under the assumptions of Proposition 2 it follows that a bounded operator A commuting with a prespectral operator T of class Γ (having $P(\cdot)$ as its resolution of the identity of class Γ) commutes with $P(\cdot)$ if and only if $A^*(\Gamma) \subseteq \Gamma$.

Example. Let X be a weakly sequentially complete Banach space and $T \in L(X)$ be a spectral operator with a cyclic vector (that is if $Q: \mathcal{B} \to L(X)$ is the resolution of the identity for T, then there exists a vector x_o in X such that the linear span of $\{Q(E)x_o; E \in \mathcal{B}\}$ is dense in X). Then $T^* \in L(X^*)$ is a prespectral operator of class X with the property that if $AT^* = T^*A$ for some $A \in L(X^*)$, then A commutes with the resolution of the identity of class X for T^* if and only if $A^*(X) \subseteq X$. Indeed, with $\Gamma = X$ it follows from [2; Lemma 3] that $P(E) = Q(E)^*$, $E \in \mathcal{B}$, is the resolution of the identity of class Γ for T^*. Since Γ with the $\sigma(\Gamma, X^*)$ topology is simply X with its weak topology and $P(E)^* = Q(E)^{**}$ has $\Gamma = X \subseteq X^{**}$ as an invariant subspace for each $E \in \mathcal{B}$, it suffices to show that $U(P) = P(\cdot)^c$ (see Proposition 2 and the Remark following it). But, if $x_o^* \in X^*$ is any Bade functional for x_o, then x_o^* is a Γ=X-cyclic vector for $\{P(E)^*; E \in \mathcal{B}\}$ in the sense of Definition 3 of [2]; see the Remark on page 153 of [2]. Accordingly, the Corollary on page 155 of [2] implies that $U(P) = P(\cdot)^c$.

For a specific example, let $X = \ell^1(\mathbb{N})$ and $\{\lambda_n\}_{n=1}^\infty$ be a bounded sequence in $\mathbb{C}$. Then X is weakly sequentially complete and the operator $T \in L(X)$ defined by $Tx = y$, $x \in X$, where $y_n = \lambda_n x_n$, $n = 1, \ldots,$ is a spectral operator with a cyclic vector (for example $x_o = \{n^{-2}\}_{n=1}^\infty$).

References

[1] H.R. Dowson, *Spectral theory of linear operators*, London Math. Soc. Monograph No. 12 (Academic Press, London, 1978).

[2] B. Nagy, *On Boolean algebras of projections and prespectral operators*, Operator Theory: Advances and Applications Vol. 6 (Birkhauser Verlag, Basel, 1982) 145-162.

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