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THE FOCAL LOCUS OF A RIEMANNIAN 4-SYMMETRIC SPACE

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ABSTRACT. A compact Riemannian 4-symmetric space M can be regarded as a fibre bundle over a Riemannian 2-symmetric space with totally geodesic fibres isometric to a 2-symmetric space. Here the result of R. Crittenden for conjugate and cut points in a 2-symmetric space is extended to the focal points of the fibres of M. Also the restriction of the exponential map of M up to the first focal locus in the normal bundle of a fibre is proved to yield a covering map onto its image. It is shown that for the noncompact dual M^*, the fibres have no focal points and hence the exponential map of M^* restricted to the normal bundle of a fibre is a covering map. The classification of the compact simply connected 4-symmetric spaces G/L with G classical simple provides a large class of examples of these fibrations.

1. **Statement of Results.** Riemannian 4-symmetric spaces have two interesting features, on one hand, they can be represented as coset spaces G/L of a well defined type, see Proposition 4 below. On the other hand, at least in the compact case, they fibre over ordinary symmetric spaces with totally geodesic fibres isometric to another ordinary symmetric space. See Theorem A.

In this paper both features are combined to study the focal locus of the fibres. The motivation arises from the equivalent classical results for conjugate points for ordinary symmetric spaces. Therefore, in this first part we have included both the statements of the classical results and their extensions to 4-symmetric spaces. Section 2 contains some preliminary results and section 3 contains the proofs of the Theorems. Finally, in Section 4 we draw a table with a detailed description of the fibrations for all the compact simply connected 4-symmetric spaces G/L with G classical simple.

THEOREM A. *Let $M = G/L$ be a 4-symmetric space with metric induced from the Killing form of G, where G is a compact simply connected semisimple Lie group and L is the fixed point set of an automorphism σ of order four of G. Let K be the fixed point set of σ^2, the square of σ, and $F = K/L$. Then*

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(1) *Geodesics perpendicular to F minimize the distance to F up to the first focal point in M.*

(2) *F is a connected complete totally geodesic submanifold of M which is also a 2-symmetric space.*

(3) *Endow G/K with the metric induced by the Killing form of G. Then the canonical projection* $\pi: G/L \to G/K$ *is a Riemannian submersion of M onto the 2-symmetric space* $B = G/K$ *with fibre* $F = K/L$.

REMARKS. If $K = L$, then M is in fact a Hermitian 2-symmetric space, F reduces to a point and the statement is a particular case of Theorem 1 below. Thus the interesting case in the theorem is that for which $\dim F \geqq 1$. See Section 4 for examples.

Part (3) can be regarded as a particular case of the fibrations of reflexion spaces as obtained by O. Loos in [10]. However, here this fibration is an immediate consequence of the homogeneous structure of M.

The following is the classical motivating result. A proof of it can be found in Cheeger, J. and Ebin, D. G., [2, pp. 102-103].

THEOREM 1. *Let* $B = G/K$ *be a symmetric space with metric induced from the Killing form of G, where G is a compact simply connected semisimple Lie group and K is the fixed point set of an involutive automorphism* σ *of G. Then*

(1) *Geodesics minimize up to the first conjugate point in M.*

(2) *B is simply connected.*

(3) *K is connected.*

Part (1) is originally due to R. Crittenden. Under the condition that G is simply connected, (2) and (3) are equivalent and go back to E. Cartan. (If G is not simply connected, then (3) is not necessarily true). (3) has been extended by R. Bott to all automorphisms of compact simply connected semisimple Lie groups G, see [4, p. 351].

In what follows $\nu(F)$ will denote the normal bundle of F. $\mathrm{Exp}: \nu(F) \to M$ will be the usual exponential map of $\nu(F)$ onto M. Let $\nu(F)_r$ be the set of $\mathbf{v} \in \nu(F)$ such that there are no focal points of F along the geodesic segment $t \to \mathrm{Exp}(t\mathbf{v})$, $0 < t < 1$. Let $\nu(F)_s$ be the boundary of $\nu(F)_r$ in $\nu(F)$. Thus, $\nu(F)_s$ may be considered as the "first focal locus" of F in $\nu(F)$.

Here we also prove the following

THEOREM B. *Let M and F be as in Theorem* A. *Then* $\mathrm{Exp}: \nu(F)_r \to \mathrm{Exp}(\nu(F)_r)$ *is a covering map.*

The idea of duality for ordinary symmetric spaces extends in a natural way to 4-symmetric spaces, see section 2. Then Theorem A has the following extension to the noncompact case.

THEOREM C. *Let* $M = G/L$ *be a Riemannian* 4-*symmetric space as in Theorem*

A. *Let $M^* = G^*/L^*$ be its dual with metric induced from the Killing form of G^*, where G^* is a noncompact simply connected semisimple Lie group and L^* is the fixed point set of the induced automorphism σ of order four of G^*. Let K^* be the fixed point set of $(\sigma^*)^2$, the square of σ^*, and set $F^* = K^*/L^*$. Then*

(1) *F^* has no focal points in M^*.*

(2) *F^* is a connected complete totally geodesic submanifold of M^*.*

(3) $\mathrm{Exp}:\nu(F^*) \to M^*$ *is a diffeomorphism.*

(4) *Endow $B^* = G^*/K^*$ with the metric induced by the Killing form of G^*. Then the canonical projection $\pi: G^*/L^* \to G^*/K^*$ is a Riemannian submersion of M^* onto B^* with fibre F^*.*

The antecedent of Theorem C is the following classical result for 2-symmetric spaces. See [4, Ch. VI].

THEOREM 2. *Let $B = G/K$ be as in Theorem* 1. *Its dual $B^* = G^*/K^*$ is a* 2-*symmetric space with metric induced from the Killing form of G^*, where G^* is a noncompact simply connected semisimple Lie group and K^* is the* (*connected*) *subgroup of fixed points of an involution σ^* of G^*. Then, B^* has non positive curvature, B^* has no conjugate points and the exponential map is a diffeomorphism at each point of B^*.*

It is interesting to point out that the proofs of these theorems do not make any use of the theory of root systems. Root systems can be used to locate the focal points of the fibres, see [6].

2. **Preliminaries.** A Riemannian 4-symmetric space is a connected C^∞-Riemannian manifold (M, g) together with a family of isometries $(s_x)x$ in M, with the following properties:

(i) For each x in M, the isometry s_x is of order four and has x as an isolated fixed point. s_x is called the symmetry at x.

(ii) (Regularity condition) For any two points x and y in M, the symmetries s_x and s_y satisfy

$$s_x \circ s_y = s_p \circ s_x$$

here p is the point $s_x(y)$.

REMARK. n-symmetric spaces were first introduced by Ledger, A., [8] (see also [3]) without condition (ii), however, here we shall be exclusively concerned with the case when the spaces are regular, thus we have included the regularity condition as part of the definition. This condition is best explained in terms of Lie groups, see Proposition 4 below.

One of the central results in the theory of n-symmetric spaces is that they are homogeneous manifolds. Let $I(M, g)$ denote the group of isometries of the manifold M with Riemannian metric g.

THEOREM 3 [9]. *If (M, g) is a Riemannian n-symmetric space, then $I(M, g)$ acts transitively on M.*

The next proposition summarizes the basic features of the homogeneous structure of 4-symmetric spaces. For details see [6] or [7].

PROPOSITION 4. *Let (M, g) be a Riemannian 4-symmetric space. Let G be the identity component of the closed subgroup generated by the symmetries, $(s_x)x$ in M, in $I(M, g)$. Then*

(i) *G acts transitively on M, and for a fixed point $0 \in M$, M can be written as the homogeneous space G/L with L the isotropy group of G at 0.*

(ii) *Conjugation with respect to s_0, the symmetry at 0, induces an automorphism σ of order four on G such that the fixed point set G^σ satisfies*

$$(G^\sigma)_0 \subset L \subset G^\sigma.$$

Here $(G^\sigma)_0$ denotes the identity component of G^σ, (this is in fact the regularity condition in disguise). Also, if $\pi: G \to G/L$ is the canonical projection, then

$$s_0 \circ \pi = \pi \circ \sigma.$$

(iii) *Let $\mathfrak{g}$ be the Lie algebra of G, and let σ also denote the automorphism induced by σ on $\mathfrak{g}$. Then g splits as*

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{v} + \mathfrak{h} \text{ (vector space direct sum)}$$

where $\mathfrak{l}$ is the Lie algebra of L, $\mathfrak{v}$ is the eigenspace of σ for the eigenvalue -1 and $\mathfrak{h}$ is the eigenspace of σ^2 for the eigenvalue -1.

Furthermore, this decomposition is $Ad(L)$-variant and G/L is a reductive homogeneous space.

Finally we indicate how the ideal of duality for ordinary symmetric spaces extends to 4-symmetric spaces.

Let $\mathfrak{g}$ be a compact semisimple Lie algebra and let σ be an automorphism of order four of $\mathfrak{g}$. $\mathfrak{g}$ splits as in (iii) above into the vector space direct sum

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{v} + \mathfrak{h}$$

furthermore, if $\theta = \sigma^2$, then $(\mathfrak{g}, \theta)$ is an orthogonal-symmetric Lie algebra of the compact type – see e.g. Helgason [4] – If $\mathfrak{g}^{\mathbf{C}}$ denotes the complexification of $\mathfrak{g}$ and both σ and θ are extended to (complex) automorphisms of $\mathfrak{g}^{\mathbf{C}}$, (denote these extensions by the same letters), then both leave invariant the real form $\mathfrak{g}^*$ of $\mathfrak{g}$ defined as follows:

$$\mathfrak{g}^* = \mathfrak{l} + \mathfrak{v} + \sqrt{-1}\,\mathfrak{h}$$

$\mathfrak{g}^*$ is also semisimple and in fact $(\mathfrak{g}^*, \theta^*)$ with θ^* the restriction of θ to $\mathfrak{g}^*$, is the dual symmetric Lie algebra to $(\mathfrak{g}, \theta)$ in the sense of ordinary symmetric spaces. On the other hand if σ^* denotes the restriction of σ to $\mathfrak{g}^*$, then $\mathfrak{g}^*$ becomes

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an orthogonal 4-symmetric Lie algebra. The pair $(\mathfrak{g}^*, \sigma^*)$ is then called the 4-symmetric dual of $(\mathfrak{g}, \sigma)$.

3. **Proofs of the Theorems.** Proof of Theorem A. Part (3) is immediate and we shall use it to obtain part (1) as a special case of the following

THEOREM A′. *Let $\pi:M \to B$ be a Riemannian submersion with $B = G/K$ as in Theorem 1. Then geodesics perpendicular to a fibre F minimize the distance to the fibre up to the first focal point of F in M.*

PROOF OF THEOREM A′. Let $\gamma(t) = \operatorname{Exp} tX$, $t > 0$ be a perpendicular geodesic ray to a fibre F with $\|X\| = 1$. And let $p = \operatorname{Exp} t_0X$ be the first focal point in M along γ. We need to show that $d(p, F) = t_0$.

Let $\pi \circ \gamma$ be the geodesic ray in B issuing from $0 = \pi(F)$. O'Neill has shown [11, Theorem 4] that the order of $\gamma(t_0)$ as a focal point of the fibre F along γ is equal to the order of conjugacy of the end points of $\pi \circ \gamma$, $0 \leqq t \leqq t_0$, along $\pi \circ \gamma$. Thus $\pi(p)$ is the first conjugate point of 0 along $\pi \circ \gamma$.

Theorem 1 implies that $d(0, \pi(p)) = t_0$. If we now assume the $d(p, F) < t_0$ then a contradiction is obtained as follows: Let $\gamma_1 = \operatorname{Exp} tX_1$, $0 < t < \infty$ be a perpendicular geodesic ray to F with $\gamma_1(t_1) = p$ and $d(p, F) = t_1$, $t_1 < t_0$. As before $\pi \circ \gamma_1$ is a geodesic ray in B issuing from 0 with $\pi \circ \gamma_1(t_1) = \pi(p)$, but then $d(0, \pi(p)) = t_1 < t_0$ in contradiction with the previous observation that $d(0, \pi(p)) = t_0$.

Part (1) of Theorem A is now a direct consequence of this result and part (3). As for part (2), the fact that F is connected follows from the fact that K is connected. To see that it is complete and totally geodesic one only has to observe that it coincides with the connected component through 0 of the fixed pointset of s_0^2, a result that follows from the relation $\pi \circ \sigma = s_0 \circ \pi$. (See Proposition 4). Since s_0^2 is an isometry, its fixed point set is a totally geodesic submanifold.

PROOF OF THEOREM B. R. Hermann [5] has proved that if $\operatorname{Exp} \nu(F)_s$ is contained in the boundary of $\operatorname{Exp} \nu(F)_r$. Then $\operatorname{Exp}:\nu(F)_r \to \operatorname{Exp} \nu(F)_r$ is a covering map. Here we show that this condition is satisfied.

The idea is to show that if $X_s \in \nu(F)_s$ and $X_r \in \nu(F)_r$, the relation $\operatorname{Exp} X_s = \operatorname{Exp} X_r$ is impossible.

Once again we make use of the submersion π of M onto B. The geodesic rays $\gamma_s(t) = \operatorname{Exp} tX_s$, $\gamma_r(t) = \operatorname{Exp} tX_r$, $t > 0$, project down to the geodesic rays $\rho_r = \pi \circ \gamma_s$ and $\rho_s = \pi \circ \gamma_r$ issuing from $0 = \pi(F)$. As in the proof of Theorem A′ we have that $\rho_s(1)$ is conjugate to 0 along ρ_s and that $\rho_r(1)$ is a regular point along ρ_r. However $\rho_s(1) = \rho_r(1)$ and this is impossible for $B = G/K$, see [4, Lemma 8.1, p. 319].

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PROOF OF THEOREM C. Parts (2) and (4) are similarly proved to parts (2) and (3) of Theorem A. Part (1) is then an immediate consequence of (4), the fact that M^* has no conjugate points and the result by O'Neill mentioned in the proof of Theorem A′.

Then for (3) J. Bolton [1] has shown that for an immersion $F^* \to M^*$ which has no focal points, $\mathrm{Exp}:\nu(F^*) \to M^*$ is a covering map. Since M^* is simply connected the result follows.

4. **Examples.** The following table gives a complete classification (up to isometry) of all the compact simply connected Riemannian 4-symmetric spaces $M = G/L$ with G classical simple. At the same time, for each M, the base space B and the fibre F are described. Furthermore, the corresponding (noncompact) dual space M^* is also given along with the base space B^*. Here the fibre F^* coincides with F. Throughout, the notational conventions are the same as in [4, pp. 444-455]. For details the reader is referred to [6] where the classification for the exceptional simple Lie groups is also obtained.

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The compact simply connected Riemannian 4-symmetric spaces G/L with G classical simple

Compact			Noncompact Dual		
Total Space	Base Space	Fibre	Total Space	Base Space	
$\frac{SU(2p+q)}{Sp(p)\times SO(q)}$	$\frac{SU(2p+q)}{S(U_{2p}\times U_q)}$	$\frac{S(U_{2p}\times U_q)}{Sp(p)\times SO(q)}$	$\frac{SU(2p,q)}{Sp(p)\times SO(q)}$	$\frac{SU(2p,q)}{S(U_{2p}\times U_q)}$	$(P \geqq 1, q \geqq 1)$
$\frac{su(2n)^*}{su(n)+R}$	$\frac{SU(2n)}{S(U_n\times U_n)}$	$\frac{su(n)+su(n)}{\Delta su(n)}$	$\frac{su(n,n)}{su(n)+R}$	$\frac{SU(n,n)}{S(U_n\times U_n)}$	$(n > 2)$
$\frac{SU(p+q+r+s)}{S(U_p\times U_q\times U_r\times U_s)}$	$\frac{SU(p+q+r+s)}{S(U_{p+q}\times U_{r+s})}$	$\frac{S(U_{p+q}\times U_{r+s})}{S(U_p\times U_q\times U_r\times U_s)}$	$\frac{SU(p+q,r+s)}{S(U_p\times U_q\times U_r\times U_s)}$	$\frac{SU(p+q,r+s)}{S(U_{p+q}\times U_{r+s})}$	$(p \geqq 1, r \geqq 1,$ $p \geqq r \geqq q \geqq s \geqq 0)$
$\frac{SO(2p+q+r)}{U(p)\times SO(q)\times SO(r)}$	$\frac{SO(2p+q+r)}{SO(2p)\times SO(q+r)}$	$\frac{SO(2p)}{U(p)}\times\frac{SO(q+r)}{SO(q)\times SO(r)}$	$\frac{SO_0(2p,q+r)}{U(p)\times SO(q)\times SO(r)}$	$\frac{SO_0(2p,q+r)}{SO(2p)\times SO(q+r)}$	$(p \geqq 1, q \geqq 0, r \geqq 0)$
$\frac{SO(2p+2q)}{U(p)\times U(q)}$	$\frac{SO(2p+2q)}{U(p+q)}$	$\frac{U(p+q)}{U(p)\times U(q)}$	$\frac{SO^*(2p+2q)}{U(p)\times U(q)}$	$\frac{SO^*(2p+2q)}{U(p+q)}$	$(p \geqq q \geqq 1, p+q > 3)$
$\frac{Sp(p+q)}{U(p)\times U(q)}$	$\frac{Sp(p+q)}{U(p+q)}$	$\frac{U(p+q)}{U(p)\times U(q)}$	$\frac{Sp(p+q,R)}{U(p)\times U(q)}$	$\frac{Sp(p+q,R)}{U(p+q)}$	$(p \geqq q \geqq 0, p+q \geqq 3)$
$\frac{Sp(p+q+r)}{U(p)\times Sp(q)\times Sp(r)}$	$\frac{Sp(p+q+r)}{Sp(p)\times Sp(q+r)}$	$\frac{Sp(p)}{U(p)}\times\frac{Sp(q+r)}{Sp(q)\times Sp(r)}$	$\frac{Sp(p,q+r)}{U(p)\times Sp(q)\times Sp(r)}$	$\frac{Sp(p,q+r)}{Sp(p)\times Sp(q+r)}$	$(p+q+r \geqq 3, p \geqq 1,$ $q \geqq r \geqq 0, q \geqq 1)$

*For these spaces it is necessary to obtain a global formulation. Also, for the fibre, $su(n)$ is included diagonally.