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WEAK UNIQUENESS FOR ELLIPTIC OPERATORS IN $\mathbb{R}^3$ WITH TIME-INDEPENDENT COEFFICIENTS

CRISTINA GIANNOTTI

The author gives a proof with analytic means of weak uniqueness for the Dirichlet problem associated to a second order uniformly elliptic operator in $\mathbb{R}^3$ with coefficients independent of the coordinate x_3 and continuous in $\mathbb{R}^2 \setminus \{0\}$.

1. INTRODUCTION

Let $\mathcal{L}$ denote the class of second order, uniformly elliptic operators in $\mathbb{R}^3$ of the form

$$L = \sum_{i,j=1}^{3} \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \tag{1.1}$$

with bounded, measurable coefficients $\alpha_{ij} = \alpha_{ji}$ such that

$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^{3} \alpha_{ij}(x)\xi_i\xi_j \leqslant \lambda^{-1}|\xi|^2 \qquad \forall\, x, \xi \in \mathbb{R}^3 \tag{1.2}$$

for some $\lambda \in (0, 1)$.

We recall the definition of good solution to the Dirichlet problem associated with an operator $L \in \mathcal{L}$ (see for example [4]).

DEFINITION 1.1: Let $L \in \mathcal{L}$ and let Ω be a smooth, bounded domain in $\mathbb{R}^3$, $f \in L^3(\Omega)$ and $\phi \in C^0(\partial\Omega)$. A function $u \in C^0(\overline{\Omega})$ is called a *good solution* to the Dirichlet problem

$$Lu = f \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega \tag{1.3}$$

if there exist a sequence of operators

$$L_n = \sum_{i,j=1}^{3} \alpha_{ij}^{(n)}(x) \frac{\partial^2}{\partial x_i \partial x_j} \in \mathcal{L}$$

with the same ellipticity constant λ as L and a sequence of functions u_n with the following properties:

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(a) for each n, the operator L_n has continuous coefficients in $\overline{\Omega}$ and $\alpha_{ij}^{(n)}$ tends to α_{ij} almost everywhere in Ω as $n \to \infty$;

(b) u_n is the unique solution in $W^{2,3}_{\mathrm{loc}}(\Omega) \cap C^0(\overline{\Omega})$ to the Dirichlet problem

$$L_n u = f \text{ in } \Omega \quad u = \phi \text{ on } \partial\Omega$$

and u_n tends to u uniformly in $\overline{\Omega}$.

Notice that by the results of Jensen in [**8**], the good solutions coincide with the L^p-viscosity solutions to the problem (1.3) (see also [**6**]). Furthermore, in [**9**] the definition of good solution is generalised to the case of fully nonlinear elliptic equations and it is showed that also in this case the good solutions coincide with the viscosity solutions. For reviews of the theory of viscosity solutions see for example [**2, 3**].

It is well known that a good solution to the problem (1.3) always exists. In fact, if $\{L_n\}$ is any sequence of operators approximating L, then Aleksandrov-Pucci and Krylov-Safonov estimates imply that the corresponding solutions u_n are uniformly bounded and equicontinuous in $\overline{\Omega}$ and hence the sequence $\{u_n\}$ admits a subsequence which converges uniformly in $\overline{\Omega}$. (see [**4**])

On the other hand, one cannot claim that all convergent subsequences have to converge to the same limit, that is, that there is uniqueness of good solutions. Indeed, for operators with bounded, measurable coefficients there may be non-uniqueness. In fact, Nadirashvili in [**12**] constructed an example of an operator defined on the unit ball in $\mathbb{R}^3$, such that the Dirichlet problem with suitable data admits at least two different good solutions (see also [**14**]).

However, the existence of operators, for which there is no uniqueness of good solutions, brings to the following natural question: for which classes of operators we have uniqueness? This question seems quite hard and up to now no complete answer is known, but several results have been obtained assuming special hypotheses on the discontinuity set of the coefficients α_{ij}. In [**4**] uniqueness has been proved if the coefficients are continuous except at a countable set of points with at most a finite number of accumulation points. In [**10**] Krylov generalises this result to include the case of a set of discontinuity with countable closure. A further generalisation was obtained by Safonov in [**13**], where uniqueness is proved if the set E of discontinuity points is so that for any subdomain $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$, the intersection $E \cap \Omega_0$ has zero Hausdorff measure of dimension α for a suitable $\alpha = \alpha(\lambda)$.

Notice that Safonov's result does not give control on such dimension α. In particular, his result is not able to imply the weak uniqueness in the case in which the set of discontinuity points contains a straight line. Partial results on weak uniqueness for discontinuities along a line were first obtained in [**7, 5**]. A recently published result of Krylov ([**11**, Theorem 2.18]) implies that uniqueness holds for all operators $L \in \mathcal{L}$ of the form

$$L = \sum_{i,j=1}^{3} \alpha_{ij}(x_1, x_2) \frac{\partial^2}{\partial x_i \partial x_j} \tag{1.4}$$

where the coefficients α_{ij} are bounded, measurable and independent of the variable x_3, without any further assumption of continuity. Krylov's theorem has been obtained using probabilistic tecniques. We give here a proof of the following fact.

THEOREM 1.2. *Let Ω be a smooth, bounded domain in $\mathbb{R}^3$, $f \in L^3(\Omega)$ and $\phi \in C^0(\partial\Omega)$. Then, for any $L \in \mathcal{L}$ with coefficients α_{ij} independent of the variable x_3 and continuous in $\mathbb{R}^2 \setminus \{0\}$, the Dirichlet problem*

$$Lu = f \ \text{ in } \Omega, \quad u = \phi \ \text{ on } \partial\Omega$$

has a unique good solution.

This statement can be obtained also as consequence of Krylov's result, but the proof we present here is based on purely analytic means and, for this reason, we consider it of certain interest.

2. Preliminaries and notations

Let us introduce the following notations: for $x \in \mathbb{R}^3$ we shall always denote the third coordinate x_3 by t and we indicate the x_3-axis by τ

$$\tau = \{x \in \mathbb{R}^3 : \ x_1 = x_2 = 0\} .$$

By $T \sim (-\pi, \pi]$ we denote the 1-dimensional torus and for any $\rho > 0$,

$$D_\rho := \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 < \rho\}, \quad \text{and} \quad D := D_1 .$$

Finally,

$$C_\rho := D_\rho \times T, \quad C := D \times T, \quad \partial C_\rho := \partial D_\rho \times T \quad \partial C := \partial D \times T .$$

Let $\mathcal{L}$ be the class of elliptic operators defined in the Introduction. Denote by $\widetilde{\mathcal{L}} \subset \mathcal{L}$ the subclass of the operators of the form (1.4) with coefficients α_{ij} independent of the variable t and continuous in $\mathbb{R}^2 \setminus \{0\}$. Let us also consider the class $\mathcal{L}^C \subset \mathcal{L}$ given by the operators L whose coefficients are defined in C (that is, with coefficients periodic in t with period 2π). Obviously, $\widetilde{\mathcal{L}} \subset \mathcal{L}^C$.

Note that the periodic cylinder C can be naturally mapped onto a solid torus and hence that the following Aleksandrov–Bakelman–Pucci estimate holds: for any $L \in \mathcal{L}^C$ and $u \in W^{2,3}_{\text{loc}}(C) \cap C^0(C)$

$$\sup_C u \leqslant \sup_{\partial C} u + N \big| (Lu) \big|_{L^3(C)} , \tag{2.1}$$

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where N is a positive constant depending only on the ellipticity constant λ.

The estimate (2.1) and the Krylov–Safonov inequality guarantee the existence of at least one good solution to the Dirichlet problem in C for any operator $L \in \mathcal{L}^C$ and the estimate (2.1) remains valid if u is a good solution.

If u is a good solution to the Dirichlet problem (1.3) and $\{L_n\}$, $\{u_n\}$ are sequences which determine u by Definition 1.1, we shall refer to u as an $\{L_n, u_n\}$*-good solution* or, shortly, an $\{L_n\}$*-good solution.*

Finally, in all what follows, if the problem (1.3) has a unique good solution for *any* $f \in L^3(\Omega)$ and *any* $\phi \in C^0(\partial\Omega)$ we shall say that *uniqueness holds for L in Ω.*

Now, let us state some properties of good solutions, which we shall use in the proofs of section 3.

PROPOSITION 2.1. *Let $L \in \mathcal{L}$ and let Ω be a smooth, bounded domain in $\mathbb{R}^3$. Then, uniqueness holds for L in Ω if and only if local uniqueness holds that is, if and only if for any $x \in \Omega$ there exists a ball $B_r(x) \subset \Omega$ such that uniqueness holds for L in $B_r(x)$.*

The proof of the "if" part in Proposition 2.1 is given in [**4**] while the "only if" part is due to Krylov (see [**10**]). Notice that, by the natural identification between the periodic cylinder C and a solid torus, the previous proposition holds also for $L \in \widetilde{\mathcal{L}}$ and $\Omega = C$.

LEMMA 2.2. (See [**5**].) *Assume that for all $f \in L^3(C)$, the problem*

$$Lu = f \ \text{ in } C, \quad u = 0 \ \text{ on } \partial C$$

with homogeneous boundary data has a unique good solution. Then uniqueness holds for L in C.

LEMMA 2.3. (See [**13**].) *Let $L \in \mathcal{L}^C$, $f \in L^3(C)$ and $\phi \in C^0(\partial C)$. If the problem*

$$Lu = f \ \text{ in } C, \quad u = \phi \ \text{ on } \partial C$$

has a solution $u \in W^{2,3}(C) \cap C^0(\overline{C})$ then u is also the unique good solution to the problem.

LEMMA 2.4. (Strong maximum principle. See [**4**].) *Let u be a good solution to $Lu = 0$ in C. If u achieves its maximum (minimum) in C, then it is constant in C.*

LEMMA 2.5. *Assume that the problem*

$$Lu = f \ \text{ in } C, \quad u = \phi \ \text{ on } \partial C \tag{2.2}$$

has a unique good solution $u \in C^0(C)$. Then, for any $g \in L^3(C)$, $\psi \in C^0(\partial C)$ and for any good solution v to

$$Lv = g \text{ in } C, \quad v = \psi \text{ on } \partial C \,,$$

the difference $v - u$ is a good solution to the problem

$$Lw = f - g \text{ in } C, \quad w = \phi - \psi \text{ on } \partial C \,.$$

PROOF: It follows by the fact that u is an $\{L_n\}$-good solution to (2.2) for all sequences $\{L_n\}$ converging to L. ∎

LEMMA 2.6. *Let u be a good solution to*

$$Lu = f \text{ in } C, \quad u = \phi \text{ on } \partial C \ . \tag{2.3}$$

Then, for any $0 < \rho < 1$, u is also a good solution to the problem

$$Lv = f \text{ in } C_\rho, \quad v = u \text{ on } \partial C_\rho \ .$$

PROOF: Suppose that u is an $\{L_n, u_n\}$-good solution to (2.3) and, for each n, let v_n be the unique solution to the problem

$$L_n v_n = f \text{ in } C_\rho, \quad v_n = u \text{ on } \partial C_\rho \ .$$

Then

$$L_n(u_n - v_n) = 0 \text{ in } C_\rho, \quad u_n - v_n = u_n - u \text{ on } \partial C_\rho$$

and, applying the estimate (2.1),

$$\sup_{\overline{C}_\rho}(u_n - v_n) \leqslant \sup_{\partial C_\rho}(u_n - u) \to 0 \text{ as } n \to \infty \ ,$$

that is, v_n tends to u uniformly in $\overline{C}_\rho$. ∎

3. PROOF OF THEOREM 1.2

In this section, we shall prove that uniqueness holds for operators $L \in \widetilde{\mathcal{L}}$ in C and then we shall show that this implies Theorem 1.2.

The first step consists in showing that the Dirichlet problem

$$Lu = f \text{ in } C, \quad u = 0 \text{ on } \partial C \tag{3.1}$$

for an operator $L \in \widetilde{\mathcal{L}}$ and with periodic f of a special simple form, always admits a good solution of class $C^{1,\alpha}(C)$.

THEOREM 3.1. *Let $L \in \widetilde{\mathcal{L}}$ and $f : C \to \mathbb{R}$ of the form*

$$f(x_1, x_2, t) = \sum_{\nu=-k}^{k} f_\nu(x_1, x_2) e^{i\nu t} \tag{3.2}$$

with $k \in \mathbb{N}$ and $f_\nu \in C_0^\infty(D)$. Then, the Dirichlet problem (3.1) has a (unique) solution

$$u \in W^{2,2}_{\gamma_0}(C) \cap C^0(\overline{C}) \cap C^{1,\alpha}(C), \ \alpha \in (0,1).$$

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PROOF: For this proof it is convenient to decompose the operator L as follows

$$\begin{aligned} Lu &= \sum_{i,j=1}^{2} a_{ij}(x_1,x_2)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{2} b_j(x_1,x_2)\frac{\partial^2 u}{\partial x_j \partial t} + c(x_1,x_2)\frac{\partial^2 u}{\partial t^2} \\ &= Su + L_1 u_t + c u_{tt}\ , \end{aligned}$$

with coefficients a_{ij}, b_j, $c \in L^\infty(\mathbb{R}^2) \cap C^0(\mathbb{R}^2 \setminus \{0\})$. Notice that the uniform ellipticity of L implies that also the operator S is a uniformly elliptic operator on $\mathbb{R}^2$.

Now, let $u \in W^{2,2}_{\gamma_0}(C)$ and let us expand u in Fourier series with respect to t

$$u(x_1,x_2,t) = \sum_{\nu=-\infty}^{+\infty} u_\nu(x_1,x_2)e^{i\nu t}\ .$$

It follows from definitions that u_ν is in $W^{2,2}_{\gamma_0}(D)$ for any $\nu \in \mathbb{Z}$. Moreover, u is a (strong) solution to the problem (3.1) if and only if for all $\nu \in \mathbb{Z}$ the function u_ν solves the problem

$$\begin{cases} Su_\nu + i\nu L_1 u_\nu - c\nu^2 u_\nu = f_\nu & \text{in } D \\ u_\nu = 0 & \text{on } \partial D, \end{cases}$$

where $f_\nu \equiv 0$ for all $|\nu| \geqslant k+1$. Now, by the estimates in [**1**] and a standard continuity method, this problem admits a unique solution $u_\nu \in W^{2,2}_{\gamma_0}(D)$. Moreover, using the estimates in [**15**] and a standard interpolation technique, it can be shown that such a solution is also in $C^{1,\alpha}(D)$ for some $\alpha \in (0,1)$ independent of ν. In particular, we have that $u_\nu \equiv 0$ in D for all $|\nu| \geqslant k+1$. We conclude that the function

$$u(x_1,x_2,t) = \sum_{\nu=-k}^{k} u_\nu(x_1,x_2)e^{i\nu t}$$

is the unique solution to the problem (3.1) with the required regularity. ∎

REMARK 3.2. The solution u in Theorem 3.1 is also the unique convolution type good solution to the problem (3.1), according to the definition in [**7**].

The second step consists in constructing a suitable auxiliary function.

LEMMA 3.3. *Let f be of the form* (3.2) *and u the solution to* (3.1) *in Theorem 3.1. Then:*

(a) *the function*

$$\widetilde{u}(x_1,x_2,t) = u(0,0,t) + x_1 u_{x_1}(0,0,t) + x_2 u_{x_2}(0,0,t) \tag{3.3}$$

is in $C^\infty(\overline{C})$ and there exists a positive constant K_1 such that

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$$L\widetilde{u} \leqslant K_1(a_{11} + a_{22}) \text{ in } \overline{C}; \tag{3.4}$$

(b) *if* $\operatorname{supp} f \subset C \setminus \tau$ *and* $f \equiv 0$ *in* C_{r_0} *for some* $0 < r_0 < 1$, *then the function*

$$U(x,t) = u(x,t) - \widetilde{u}(x,t) + \frac{K_1}{2}(x_1^2 + x_2^2) \tag{3.5}$$

satisfies the following three conditions:

(i) $U \in W^{2,2}_{\gamma_0}(C) \cap C^0(\overline{C}) \cap C^{1,\alpha}(C)$;
(ii) $|U(x_1, x_2, t)| \leqslant K_2 r^{1+\alpha}$ *in* C *with* $r = (x_1^2 + x_2^2)^{1/2}$;
(iii) $LU = -L\widetilde{u} + K_1(a_{11} + a_{22}) \geqslant 0$ *in* C_{r_0}.

PROOF: Since $u(0,0,t) = \sum_{-k}^{k} u_\nu(0,0)e^{i\nu t}$ and $u_{x_i}(0,0,t) = \sum_{-k}^{k} u_{\nu x_i}(0,0)e^{i\nu t}$ $(i = 1,2)$ are in $C^\infty(T)$, we have $\widetilde{u} \in C^\infty(\overline{C})$. Moreover, $L\widetilde{u}$ is bounded in $\overline{C}$ and hence (3.4) is satisfied for suitable large values of K_1.

For what concerns the function U, (i) and (iii) follow immediately from the definition of U and (3.4). The inequality (ii) is a consequence of the fact that $U \in C^{1,\alpha}(C)$ and

$$U(0,0,t) = U_{x_1}(0,0,t) = U_{x_2}(0,0,t) \equiv 0.$$

∎

Now, in order to prove uniqueness for $L \in \widetilde{\mathcal{L}}$ in C, we need only another lemma.

LEMMA 3.4. *Let* $h \in C^0(\overline{C}) \cap W^{2,3}_{\text{loc}}(C \setminus \tau)$ *be a nontrivial solution to*

$$Lh = 0 \text{ in } C \setminus \tau, \quad h = 0 \text{ on } \partial C$$

and assume that h *has a positive maximum* M *in* $\overline{C}$. *Then* h *satisfies*

$$h \leqslant M(1 - r) \quad \text{in } \overline{C}\,, \tag{3.6}$$

where $r = (x_1^2 + x_2^2)^{1/2}$.

PROOF: Let us consider the cylindrical coordinates (r, θ, t), where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Using these coordinates, L has the following form:

$$L = a\frac{\partial^2}{\partial r^2} + \frac{b}{r}\frac{\partial}{\partial r} + \frac{b}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{2d}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{2d}{r^2}\frac{\partial}{\partial \theta} + 2k\frac{\partial^2}{\partial r \partial t} + \frac{2h}{r}\frac{\partial^2}{\partial \theta \partial t} + c\frac{\partial}{\partial t^2}\,,$$

where $b > 0$ by (1.2). Then, if $g(r) = M(1 - r)$, we have that $g - h$ satisfies

$$\begin{cases} L(g-h) = -\dfrac{Mb}{r} < 0 & \text{in } C \setminus \tau \\ g - h = M - h \geqslant 0 & \text{on } \tau \\ (g - h) = 0 & \text{on } \partial C \end{cases}$$

Thus, inequality (3.6) follows by maximum principle. ∎

THEOREM 3.5. *The Dirichlet problem*

$$Lu = f \ \text{ in } C, \quad u = \phi \ \text{ on } \partial C \tag{3.7}$$

has a unique good solution for any $f \in L^3(C)$ *and* $\phi \in C^0(\partial C)$.

PROOF: By Lemma 2.2, it is enough to consider $\phi \equiv 0$. Assume first that f is of the form (3.2) and that $f \equiv 0$ in C_{r_0}, for some $r_0 \in (0,1)$. Let u be the solution to the problem (3.7) given in Theorem 3.1. Suppose that uniqueness does not hold and assume that $v \in C^0(\overline{C}) \cap W^{2,3}_{\text{loc}}(C \setminus \tau)$ is a good solution to (3.7) different from u. Then the difference $h = v - u$ belongs to $C^0(\overline{C}) \cap W^{2,3}_{\text{loc}}(C \setminus \tau)$ and it solves

$$Lh = 0 \ \text{ in } C \setminus \tau, \quad h = 0 \ \text{ on } \partial C.$$

Since h is not identically zero in C, it has a positive maximum or a negative minimum in $\overline{C}$ and, by maximum principle, this maximum or minimum must be taken on τ. Without loss of generality, let us assume that h has a positive maximum M in $\overline{C}$ and let $P_0 \in \tau$ be a point such that $h(P_0) = M$. Then, by Lemma 3.4 the inequality (3.6) holds for h. Now, let us consider the function $V = v - \widetilde{u} + \big(K_1/2\big)(x_1^2 + x_2^2)$ where $\widetilde{u}$ is defined in (3.3) and K_1 is the constant in (3.4). If U is the function (3.5), we have that $h = v - u = V - U$. By the inequality (3.6) and Lemma 3.3, it follows that

$$V = h + U \leqslant M(1-r) + K_2 r^{1+\alpha} \ \text{ in } C.$$

Now, the function $\psi(r) : [0, +\infty) \to \mathbb{R}$, $\psi(r) = M(1-r) + K_2 r^{1+\alpha}$ has a strict local maximum equal to M in $r = 0$. Thus, there exists $0 < \overline{r} \leqslant r_0$ such that $V(x_1, x_2, t) \leqslant \psi(\overline{r}) < M$ on $\partial C_{\overline{r}}$. On the other hand, we have that $V \equiv h$ on τ and so $v(P_0) = M$. We claim that this is impossible. In fact, by Lemma 2.3 and Lemma 2.5, V is a good solution to the problem

$$\begin{cases} Lw = f - L\widetilde{u} + K_1(a_{11} + a_{22}) & \text{in } C \\ w = -\widetilde{u} + K_1/2 & \text{on } \partial C. \end{cases}$$

By Lemma 2.6, V is also a good solution in $C_{\overline{r}}$ to the problem

$$\begin{cases} Lw = -L\widetilde{u} + K_1(a_{11} + a_{22}) \geqslant 0 & \text{in } C_{\overline{r}}, \\ w = V & \text{on } \partial\overline{C}. \end{cases}$$

If V is an $\{L_n, V_n\}$-good solution, then we have $L_n V_n \geqslant 0$ in $C_{\overline{r}}$, $V_n \equiv V$ on $\partial\overline{C}$ and the sequence $\{V_n\}$ converges uniformly to V in $\overline{C}$. This implies $V_n(P_0) \leqslant \max_{\partial\overline{C}} V_n$ and going to the limit we get a contradiction. Then $h = v - u \equiv 0$ in $\overline{C}$.

Now, let us consider $f \in L^3(C)$. Then there exists a sequence $\{f^{(m)}\}$ where

$$f^{(m)} = \sum_{\nu=-k_m}^{k_m} f_\nu^{(m)} e^{i\nu t}$$

with $f_\nu^{(m)} \in C_0^\infty(D)$ and $\operatorname{supp} f^{(m)} \subset C \setminus \tau$ such that $\{f^{(m)}\}$ converges to f in $L^3(C)$. From the previous result, each problem $Lu = f^{(m)}$ in C, with $u = 0$ on ∂C, has a unique good solution u_m. Let us suppose that our problem has two different good solutions v and w. Then, by Lemma 2.5 we have that also $u_m - v$ and $u_m - w$ are good solutions to the Dirichlet problem

$$Lu = f^m - f \ \text{ in } C, \quad u = 0 \ \text{ on } \partial C\ .$$

Then applying the estimate (2.1) we have

$$|u_m - v|_{L^\infty(C)},\ |u_m - w|_{L^\infty(C)} \leqslant N|f^m - f|_{L^3(C)}\ .$$

This means that u_m tends both to v and w uniformly in C and hence $v \equiv w$. ∎

Now, we are able to prove Theorem 1.2.

PROOF OF THEOREM 1.2: Since an operator $L \in \widetilde{\mathcal{L}}$ has smooth coefficients in $\mathbb{R}^3$ except on τ, uniqueness holds for L in any ball $B \subset \Omega$ such that $\overline{B} \cap \tau = \emptyset$. Moreover, by Theorem 3.5 and Proposition 2.1 it follows that uniqueness holds for L also in any ball $B \subset \Omega$ which intersect τ. Using again Proposition 2.1, we conclude that uniqueness holds for L in Ω. ∎

As a consequence we have uniqueness also for any operator L whose coefficients are discontinuos on a countable set of parallel line accumulating to the x_3-axis.

COROLLARY 3.6. *Let $L \in \mathcal{L}$ be an elliptic operator satisfying* (1.2), *with coefficients independent of x_3 and continuous at all points of $\mathbb{R}^2$ except possibly at a countable set of points with at most one cluster point $x_0 \in \mathbb{R}^2$. Then, uniqueness holds for L in any smooth, bounded domain $\Omega \subset \mathbb{R}^3$.*

PROOF: Assume $x_0 = 0$. As before, it is enough to prove uniqueness for the problem

$$Lu = f \ \text{ in } C, \quad u = 0 \ \text{ on } \partial C\ , \tag{3.8}$$

where f is of the form (3.2) and it is supported in $C \setminus \tau$. Then Theorem 3.1 and Lemma 3.3 still hold and the problem (3.8) has a good solution $u \in W^{2,2}_{\gamma_0}(C) \cap C^0(\overline{C}) \cap C^{1,\alpha}(C)$. Suppose that there is a good solution $v \not\equiv u$. Then, by Theorem 1.1 and Proposition 2.1, we have that $h = v - u$ is a good solution to $Lh = 0$ in $C \setminus \tau$. Again, by Lemma 2.4, it follows that h must take a positive maximum or a negative minimum on τ and without loss of generality we may assume that h has a positive maximum M at the point $P_0 \in \tau$. Then, we still have Lemma 3.4 and we can conclude as in the proof of Theorem 3.5. ∎

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Dipartimento di Matematica e Informatica
Via Madonna delle Carceri
I- 62032 Camerino (Macerata)
Italy
e-mail: cristina.giannotti@unicam.it