Bull. Aust. Math. Soc. **84** (2011), 408–413
doi:10.1017/S000497271100267X

ON MINIMAL DEGREES OF PERMUTATION REPRESENTATIONS OF ABELIAN QUOTIENTS OF FINITE GROUPS

CLARA FRANCHI

(Received 2 December 2010)

Abstract

For a finite group G, we denote by $\mu(G)$ the minimum degree of a faithful permutation representation of G. We prove that if G is a finite p-group with an abelian maximal subgroup, then $\mu(G/G') \leq \mu(G)$.

2010 *Mathematics subject classification*: primary 20B05.
Keywords and phrases: permutation representation, finite p-group.

1. Introduction

For a finite group G, the *minimal faithful permutation degree* $\mu(G)$ is defined as the least positive integer n such that G is isomorphic to a subgroup of the symmetric group S_n. A faithful permutation representation of degree $\mu(G)$ is called a *minimal (faithful) permutation representation* of G. By Cayley's theorem $\mu(G) \leq |G|$, and it is easy to see that equality holds if and only if G is cyclic of prime power order, a generalized quaternion 2-group or the Klein 4-group [7].

If H is a subgroup of G, then $\mu(H) \leq \mu(G)$, but the situation for quotient groups can be quite different. For example, Neumann pointed out in [11] that the direct product of m copies of the dihedral group of order 8 has a natural faithful representation of degree $4m$ but it has an extraspecial quotient which has no faithful permutation representation of degree less than 2^{m+1}. On the other hand, particular classes of quotients behave just like the subgroups. For example, $\mu(G/N) \leq \mu(G)$ provided G/N has no nontrivial abelian normal subgroups (Kovács and Praeger [10]). Using this result, Holt and Walton [6] proved that there exists a constant c such that $\mu(G/N) \leq c^{\mu(G)-1}$ for all finite groups G and all normal subgroups N. (The constant is approximately 5.34.)

If $A = A_1 \times \cdots \times A_r$ is an abelian group, with each A_i cyclic of prime power order a_i, then $\mu(A) = a_1 + \cdots + a_r$ ([14] and [12, Ch. II, Theorem 4]; see also [7, 8]). Thus, in particular, $\mu(A/N) \leq \mu(A)$ for every subgroup N of A. According to [9], the question whether $\mu(G/N) > \mu(G)$ can happen with G/N abelian, goes back at least to Easdown and Praeger [3], the conjecture being that it cannot. In the last paragraph of Section 1 of [4], it was shown that a minimal counterexample G would have to have prime-power order and N would have to be the commutator subgroup G' (see also [2, 10]).

In this note, we carry on the analysis of a such a counterexample, showing that it cannot be a nonabelian finite p-group with an abelian maximal subgroup. Namely, we prove the following.

THEOREM. *Let G be a nonabelian finite p-group with an abelian maximal subgroup. Then $\mu(G/G') \leq \mu(G)$.*

Notation is standard. We refer to [1] for notation and terminology about permutation groups. If H is a subgroup of a group G we denote by ρ_H the standard representation of G on the right cosets of H. All groups considered are finite.

2. Proof of the theorem

Recall that if $AB = A \times B$ is a direct product of groups A and B, a subgroup G of AB is called a *subdirect product* of A and B if $AG = BG = AB$.

LEMMA 1. *Let G be a subdirect product of two groups, A and B, such that G/G' is not a subdirect product of A/A' and B/B', and set*

$$R = G'(B \cap G) \cap G'(A \cap G), \quad L = G'(A \cap G)(B \cap G).$$

Then R/G' is isomorphic to a section of A' which is a central section of A, and if A is nilpotent then G/L is not cyclic.

PROOF. Since G is a subdirect product of A and B, we have $A \times B = AG = BG$ and $A \cong G/(B \cap G)$, $B \cong G/(A \cap G)$. As G/G' is not a subdirect product of A/A' and B/B', it is easy to see that $R > G' > 1$. Observe that $A \cap G' = G'$ or $B \cap G' = G'$ would imply that $R = G'$, which is a contradiction. Hence $A \cap G' < G'$ and $B \cap G' < G'$ and no generality is lost by assuming that $A \cap G' = B \cap G' = 1$. Then $A \cap G$ and $B \cap G$ lie in the centre $Z(G)$ of G (because they are normal subgroups which avoid the derived group). Let $\alpha : G \to A$ be the restriction to G of the projection of $A \times B$ on the first component, that is $(ab)\alpha = a$ whenever $a \in A$, $b \in B$. Note that $G\alpha = A$, $\ker \alpha = B \cap G$, $R\alpha = G'\alpha = A'$, and of course $(Z(G))\alpha \leq Z(A)$. Now $A \cap R = (A \cap R)\alpha \leq R\alpha = A'$ and $A \cap R \leq A \cap G = (A \cap G)\alpha \leq (Z(G))\alpha \leq Z(A)$ show that $A \cap R$ is a subgroup of A' which is central in A. Since $G' \leq R \leq AG'$, by Dedekind's law, $R = (A \cap R)G'$. As $A \cap G' = 1$, this yields $R = (A \cap R) \times G'$, whence $R/G' \cong A \cap R$. The first statement of the lemma is proved.

Observe next that the complete inverse image of $A'(A \cap G)$ under α is L, so G/L is isomorphic to the largest abelian quotient of $A/(A \cap G)$. Suppose that A is nilpotent. If $A' \nleq A \cap G$, then $A/(A \cap G)$ is a nonabelian nilpotent group. As such, it must have a noncyclic abelian quotient, therefore in this case G/L cannot be cyclic. If $A' \leq A \cap G$, that is, if $G'\alpha \leq A \cap G$, then G' lies in the complete inverse image of $A \cap G$ under α, so $G' \leq (A \cap G)(B \cap G)$. In this case $L = (A \cap G)(B \cap G) \leq Z(G)$, and as a central

https://doi.org/10.1017/S000497271100267X Published online by Cambridge University Press

quotient of a nonabelian group can never be cyclic, the desired conclusion is once more at hand. □

We quote in the following lemma a consequence of [7, Theorem 2] that will be useful in what follows. We denote by C_{p^α} the cyclic group of order p^α.

Lemma 2. *Let U be an abelian group of exponent dividing p^n, $n > 1$. If V is a subgroup or a quotient of U of order $|U|/p$, then $\mu(U) \le \mu(V) + p^n - p^{n-1}$.*

Proof. If $U \cong V \times C_p$, the claim holds because $p^n - p^{n-1} \ge p$. Otherwise, an unrefinable direct decomposition of U has the same number of cyclic direct summands as V, the difference being that a C_{p^m} in V is replaced by a $C_{p^{m+1}}$ in U. (When V is a subgroup, this follows immediately from [9, Lemma 1]; when V is a factor group, it comes dually.) In this case, $\mu(U) = \mu(V) - p^m + p^{m+1}$ and the claim holds because $m + 1 \le n$ and so $p^{m+1} - p^m \le p^n - p^{n-1}$. □

Recall that a subgroup H of a group G is called *meet-irreducible* if it is not the intersection of two subgroups H_1, H_2, with $H_i > H$ for $i = 1, 2$.

Lemma 3. *Let P be a nonabelian p-group which is a transitive permutation group of degree p^n such that the stabilizer of a point is meet-irreducible. Suppose that P contains a nontransitive maximal abelian subgroup M. Then every section of P' which is central in P has order at most p and P/P' is isomorphic to one of the following groups, where $\alpha \le n - 2$:*

(i) $C_{p^\alpha} \times C_p \times C_p$;
(ii) $C_{p^{\alpha+1}} \times C_p$;
(iii) $C_{p^\alpha} \times C_{p^2}$.

In particular $\mu(P/P') \le p^{n-1} + p$.

Proof. Let S be the stabilizer of a point in P. Then $S \le M$, since M is not transitive, and $|M : S| = p^{n-1}$. It follows that $\{x^{p^{n-1}} \mid x \in M\}$ is a normal subgroup of P contained in S, so it must be 1 as S is core-free. Moreover, as S is meet-irreducible, M/S is a cyclic group. Thus, by a result of Ore on monomial representations [13, Ch. IV, Theorem 1], P embeds into the wreath product $C_{p^{n-1}} \mathrm{wr}\, C_p$ in such a way that M embeds into the base subgroup B. Observe that B has the structure of an $\mathcal{A}$-module isomorphic to $\mathcal{A}_\mathcal{A}$, where $\mathcal{A} = (\mathbb{Z}/p^{n-1}\mathbb{Z})C_p$, and subgroups of B which are normalized by P are precisely the $\mathcal{A}$-submodules. In what follows we identify M with its image in $\mathcal{A}_\mathcal{A}$ and denote by W the augmentation ideal of $\mathcal{A}_\mathcal{A}$.

Since P' is contained in $M \cap W$ and since every section of M which is central in P is a trivial $\mathcal{A}$-module, the last sentence of [5, Lemma 1.2.1] gives that every section of P' that is central in P has order dividing p. To prove the second part of the claim, note that, by [5, Lemma 1.2.1] and using the same notation, $P' = W_j$ for some $j > 0$. By [5, Proposition 1.2.2] (and using the same notation, except for replacing n by $n - 1$) the largest trivial submodule of $\mathcal{A}_\mathcal{A}/W_j$ is easily seen to be $A(n - 1, j + 1)/W_j$ if $W_j < W$ and $\mathcal{A}_\mathcal{A}/W$ otherwise. Hence M/P' is a subgroup either of $C_{p^{n-2}} \times C_p$ or of $C_{p^{n-1}}$.

https://doi.org/10.1017/S000497271100267X Published online by Cambridge University Press

Using that M/P' is a maximal subgroup of the noncyclic P/P' and by arguing as in the proof of Lemma 2, the second claim of the lemma follows. □

Recall that by [16], $\mu(G) = \mu(H) + \mu(K)$ whenever G is a nilpotent group with a nontrivial direct factorization $G = H \times K$. In particular, whenever G is a subdirect product of two nilpotent groups A and B, we have $\mu(G) \leq \mu(A) + \mu(B)$. We will use this fact in the remainder of the article without making reference to it.

Lemma 4. *Let G be a finite nilpotent group and suppose that $\mu(H/H') \leq \mu(H)$ for each homomorphic image H of G such that $\mu(H) < \mu(G)$. If G has a minimal faithful representation with an abelian transitive constituent, then $\mu(G/G') \leq \mu(G)$.*

Proof. Suppose that G has a minimal faithful representation on a set Ω with an abelian transitive constituent $A = G^{\Delta}$, and set $B = G^{\Omega \setminus \Delta}$. Then $\mu(G) = \mu(A) + \mu(B)$. As G is a subdirect product of A and B and A is abelian, $G' = 1 \times B'$, so G/G' is a subdirect product of A and B/B'. Now B is a homomorphic image of G with $\mu(B) < \mu(G)$; so by hypothesis $\mu(B/B') \leq \mu(B)$. Hence

$$\mu(G/G') \leq \mu(A) + \mu(B/B') \leq \mu(A) + \mu(B) = \mu(G),$$

as wanted. □

Proof of the Theorem. Let G be a finite p-group with an abelian maximal subgroup M and assume, for a proof by contradiction, that G is a counterexample of minimal degree. In particular, G is nonabelian. By [7, Lemma 1] there exists a faithful representation ρ of G on some set Ω which not only has minimal degree but is such that each point stabilizer is meet-irreducible. Let Δ be an orbit of maximal length p^n in such a representation ρ, and set $\Gamma = \Omega \setminus \Delta$, $A = G^{\Delta}$ and $B = G^{\Gamma}$. Then G is a subdirect product of A and B, and A is nonabelian by Lemma 4. As B has an abelian maximal subgroup as well, minimality of $\mu(G)$ implies that

$$\mu(B/B') \leq \mu(B) = \mu(G) - p^n. \tag{1}$$

Let S be the point stabilizer in G of a point of Δ. By our choice of ρ, this S is meet-irreducible. By Lemma 4, G has no abelian transitive constituent, and so $n \geq 2$. Finally note that the exponent of G, and hence of G/G', is at most p^n.

Assume first that M is not transitive on Δ. Then A satisfies the hypothesis of Lemma 3 and so each section of A' which is central in A has order at most p and

$$\mu(A/A') \leq p^{n-1} + p. \tag{2}$$

Thus if G/G' were a subdirect product of A/A' and B/B', using (1) and (2) we would get

$$\mu(G/G') \leq \mu(A/A') + \mu(B/B') \leq p^{n-1} + p + \mu(G) - p^n \leq \mu(G),$$

contradicting that G is a counterexample. Therefore Lemma 1 applies, yielding that R/G' is isomorphic to a section of A' that is central in A and that G/L is not cyclic.

https://doi.org/10.1017/S000497271100267X Published online by Cambridge University Press

In particular G/R, which is easily seen to be a subdirect product of A/A' and B/B', is not the whole direct product of these groups, so

$$\mu(G/R) \leq \mu(A/A') + \mu(B/B') - p. \tag{3}$$

Since sections of A' which are central in A have order dividing p, we have that $|R/G'| = p$. So, first by applying Lemma 2 with $U = G/G'$ and $V = G/R$ and then by using (3) and (2), we get

$$\begin{aligned} \mu(G/G') &\leq \mu(G/R) + p^n - p^{n-1} \leq \mu(A/A') + \mu(B/B') - p + p^n - p^{n-1} \\ &\leq p^{n-1} + p + \mu(G) - p^n - p + p^n - p^{n-1} = \mu(G), \end{aligned}$$

which is again a contradiction.

Hence M is transitive on Δ. Then S is not contained in M and $|M : M \cap S| = p^n$. Since M is an abelian maximal subgroup of G, we have that $G = SM$ and $S \cap M$ is a normal subgroup of G. Now if the kernel of the action of G on Δ, $\mathrm{core}_G(S)$, were bigger than $S \cap M$, then, by maximality of M, it would be $\mathrm{core}_G(S) = S$ and we would have that $A = G/S \cong M/M \cap S$ is abelian, contradicting Lemma 4. Hence $\mathrm{core}_G(S) = S \cap M$ and $A = G/S \cap M$.

Suppose first that $M/M \cap S$ is not cyclic. Then there exist two subgroups S_1, S_2 such that $S_1 \cap S_2 = M \cap S$ and $S_1 S_2 = M$. In particular, if $|M : S_1| = p^k$, then $1 \leq k \leq n-1$ and $|M : S_2| = p^{n-k}$. Consider the action of M on the set Ω via ρ and let $\{K_1, \ldots, K_r\}$ be a set of representatives of the point stabilizers of this action, one for each orbit, where we assume $K_1 = S \cap M$. Let σ be the representation of M defined by setting $\sigma = \rho_{S_1} + \rho_{S_2} + \sum_{i=2}^{r} \rho_{K_i}$. Then σ is a faithful representation of M of degree $\mu(G) - p^n + p^{n-k} + p^k$, whence

$$\mu(M) \leq \mu(G) - p^n + p^{n-k} + p^k. \tag{4}$$

By Lemma 2, applied with $U = G/G'$ and $V = M/G'$, we have that

$$\mu(G/G') \leq \mu(M/G') + p^n - p^{n-1}. \tag{5}$$

Observe that M abelian and $G' > 1$ imply that

$$\mu(M/G') \leq \mu(M) - p. \tag{6}$$

Hence, using (5), (6) and (4),

$$\begin{aligned} \mu(G/G') &\leq \mu(M/G') + p^n - p^{n-1} \leq \mu(M) - p + p^n - p^{n-1} \\ &\leq \mu(G) - p^n + p^k + p^{n-k} - p + p^n - p^{n-1} \\ &= \mu(G) - (p^{n-k-1} - 1)(p^k - p) \leq \mu(G), \end{aligned}$$

which is a contradiction.

Therefore, $M/M \cap S$ must be cyclic. Then, $A = G/M \cap S$ is a nonabelian group with a cyclic maximal subgroup. The structure of nonabelian p-groups with a cyclic maximal subgroup is well known (see for example [15, 5.3.4]) and shows that A/A' is either $C_{p^{n-1}} \times C_p$ or $C_2 \times C_2$. In either case $\mu(A/A') \leq p^{n-1} + p$, and one obtains a contradiction as in the case when M is not transitive on Δ. This proves the theorem. □

https://doi.org/10.1017/S000497271100267X Published online by Cambridge University Press

Acknowledgement

My interest in this problem dates from the summer of 2001, which I spent at the Australian National University studying with Professor László Kovács. After all these years, I wish to express again my deep gratitude for his help and kind assistance during my stay in Canberra.

References

[1] J. D. Dixon and B. Mortimer, *Permutation Groups* (Springer, New York, 1996).

[2] D. Easdown, 'Minimal faithful permutation and transformation representations of groups and semigroups', *Contemp. Math.* **131** (1992), 75–84.

[3] D. Easdown and C. E. Praeger, 'On the minimal faithful degree of a finite group', Research Report, School of Mathematics and Statistics, University of Western Australia, 1987.

[4] D. Easdown and C. E. Praeger, 'On minimal faithful permutation representations of finite groups', *Bull. Aust. Math. Soc.* **38** (1988), 207–220.

[5] D. L. Flannery, 'The finite irredicible linear 2-groups of degree 4', *Mem. Amer. Math. Soc.* **129**(613) (1997).

[6] D. F. Holt and J. Walton, 'Representing the quotient group of a finite permutation group', *J. Algebra* **248** (2002), 307–333.

[7] D. L. Johnson, 'Minimal permutation representations of finite groups', *Amer. J. Math.* **93** (1971), 857–866.

[8] G. I. Karpilovsky, 'The least degree of a faithful representation of abelian groups', *Vestn. Khar'k. Univ.* **53** (1970), 107–115 (in Russian).

[9] L. G. Kovács and C. E. Praeger, 'Finite permutation groups with large abelian quotients', *Pacific J. Math.* **136** (1989), 283–292.

[10] L. G. Kovács and C. E. Praeger, 'On minimal faithful permutation representations of finite groups', *Bull. Aust. Math. Soc.* **62**(2) (2000), 311–317.

[11] P. M. Neumann, 'Some algorithms for computing with finite permutation groups', in: *Proceedings of Groups–St. Andrews 1985*, London Mathematical Society Lecture Notes, 121 (eds. E. F. Robertson and C. M. Campbell) (Cambridge University Press, Cambridge, 1987), pp. 59–92.

[12] O. Ore, 'Contributions to the theory of groups of finite order', *Duke Math. J.* **5** (1939), 431–460.

[13] O. Ore, 'Theory of monomial groups', *Trans. Amer. Math. Soc.* **51** (1942), 15–64.

[14] A. Powsner, 'Über eine Substitutionsgruppe kleinster Grades, die einer gegebenen Abelschen Gruppe isomorph ist', *Commun. Inst. Sci. Math. et Mécan., Univ. Kharkoff et Soc. Math. Kharkoff, IV. s.* **14** (1937), 151–157 (translated from Russian); Autorreferat, *Zbl.* 0019.15506.

[15] D. J. S. Robinson, *A Course in the Theory of Groups* (Springer, New York, 1982).

[16] D. Wright, 'Degrees of minimal embeddings for some direct products', *Amer. J. Math.* **97** (1975), 897–903.

CLARA FRANCHI, Dipartimento di Matematica e Fisica 'Niccolò Tartaglia', Università Cattolica del Sacro Cuore, Via Musei 41, 25121 Brescia, Italy
e-mail: c.franchi@dmf.unicatt.it