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Positive Ulrich sheaves

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Abstract. We provide a criterion for a coherent sheaf to be an Ulrich sheaf in terms of a certain bilinear form on its global sections. When working over the real numbers, we call it a positive Ulrich sheaf if this bilinear form is symmetric or Hermitian and positive-definite. In that case, our result provides a common theoretical framework for several results in real algebraic geometry concerning the existence of algebraic certificates for certain geometric properties. For instance, it implies Hilbert's theorem on nonnegative ternary quartics, via the geometry of del Pezzo surfaces, and the solution of the Lax conjecture on plane hyperbolic curves due to Helton and Vinnikov.

1 Introduction

A widespread principle in real algebraic geometry is to find and use algebraic certificates for geometric statements. For example, a *sum of squares* representation of a homogeneous polynomial $p \in \mathbb{R}[x_1, \dots, x_n]_{2d}$ of degree $2d$ is a finite sequence of polynomials $g_1, \dots, g_r \in \mathbb{R}[x_1, \dots, x_n]_d$ such that

$$p = g_1^2 + \cdots + g_r^2$$

and serves as an algebraic certificate of the geometric property that p takes only nonnegative values at real points: $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. In an influential paper, Hilbert [Hil88] showed that the converse is true if (and only if) $2d = 2$, or $n = 1$, or $(2d, n) = (4, 3)$. While in the first two cases this can be seen quite easily via linear algebra and the fundamental theorem of algebra, respectively, the proof for the case $(2d, n) = (4, 3)$ of ternary quartics is nontrivial. There have been several different new proofs of this statement in the last 20 years: via the Jacobian of the plane curve defined by p [PRSS04] relying on results of Coble [Cob82], using elementary techniques [PS12], and as a special case of more general results on varieties of minimal degree [BSV16].

Another instance, that has attracted a lot of attention recently, is the following. Let $h \in \mathbb{R}[x_0, \dots, x_n]_d$ be a homogeneous polynomial for which there are real symmetric (or complex Hermitian) matrices $A_0, \dots, A_n$ and $r \in \mathbb{Z}_{>0}$ such that

$$h^r = \det(x_0 A_0 + \cdots + x_n A_n)$$

and $e_0 A_0 + \cdots + e_n A_n$ is positive definite for some $e \in \mathbb{R}^{n+1}$. In this case, we say that h^r has a *definite symmetric (or Hermitian) determinantal representation* and it

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is a certificate that h is *hyperbolic with respect to e* in the sense that the univariate polynomial $h(te-v) \in \mathbb{R}[t]$ has only real zeros for all $v \in \mathbb{R}^{n+1}$. The minimal polynomial of a Hermitian matrix has only real zeros. Lax [Lax58] conjectured that for $n=2$ and arbitrary $d \in \mathbb{Z}_{>0}$, the following strong converse is true: every hyperbolic polynomial $h \in \mathbb{R}[x_0, x_1, x_2]_d$ has a definite symmetric determinantal representation (up to multiplication with a nonzero scalar). This conjecture was solved to the affirmative by Helton and Vinnikov [HV07]. Furthermore, every hyperbolic $h \in \mathbb{R}[x_0, x_1, x_2, x_3]_3$ has a definite Hermitian determinantal representation [BK07], and for every quadratic hyperbolic polynomial $h \in \mathbb{R}[x_0, \dots, x_n]_2$, there is an $r \in \mathbb{Z}_{>0}$ such that h^r has definite symmetric (or Hermitian) determinantal representation [NT12]. On the other hand, if $d \geq 4$ and $n \geq 3$ or if $d \geq 3$ and $n \geq 5$, there are hyperbolic polynomials $h \in \mathbb{R}[x_0, \dots, x_n]_d$ such that no power h^r has a definite symmetric (or Hermitian) determinantal representation (see [Brä11, Kum16b]; resp. [BKS+ar, Sau19]). The case $(d,n) = (3,4)$ is open.

The condition of $h \in \mathbb{R}[x_0, \dots, x_n]_d$ being hyperbolic with respect to e can be phrased in geometric terms as follows: let $X = \mathcal{V}(h) \subset \mathbb{P}^n$ be the hypersurface defined by h. Then h is hyperbolic with respect to e if and only if the linear projection $\pi_e\colon X \to \mathbb{P}^{n-1}$ with center e is *real fibered* in the sense that $\pi_e^{-1}(\mathbb{P}^{n-1}(\mathbb{R})) \subset X(\mathbb{R})$. This leads to a natural generalization of hyperbolicity to arbitrary embedded varieties introduced in [SV18] and further studied in [KS20]. A subvariety $X \subset \mathbb{P}^n$ of dimension k is *hyperbolic* with respect to a linear subspace $E \subset \mathbb{P}^n$ of codimension $k+1$ if $X \cap E = \varnothing$ and the linear projection $\pi_E\colon X \to \mathbb{P}^k$ with center E is real fibered. Furthermore, a polynomial $p \in \mathbb{R}[x_1, \dots, x_n]_{2d}$ is nonnegative if and only if the double cover $X \to \mathbb{P}^n$ ramified along the zero set of p is real fibered, where X is defined as the zero set of $y^2 - p$ in a suitable weighted projective space. Thus, both abovementioned geometric properties of polynomials, being nonnegative and being hyperbolic, can be seen as special instances of real-fibered morphisms.

Let $f\colon X \to Y$ be a morphism of schemes. A coherent sheaf $\mathcal{F}$ on X is called *f-Ulrich* if there is a natural number $r > 0$ such that $f_*\mathcal{F} \cong \mathcal{O}_Y^r$. If $X \subset \mathbb{P}^n$ is a closed subscheme, then one defines a coherent sheaf $\mathcal{F}$ on X to be an *Ulrich sheaf* per se if it is π-Ulrich for any finite surjective linear projection $\pi\colon X \to \mathbb{P}^k$. This is equivalent to $H^i(X, \mathcal{F}(-j)) = 0$ for $1 \leq j \leq \dim(X)$ and all i. See [Bea18] for an introduction to Ulrich sheaves. The question of which subvarieties of $\mathbb{P}^n$ carry an Ulrich sheaf is of particular interest in the context of Boij–Söderberg theory [ES11].

For real-fibered morphisms $f\colon X \to Y$, *positive f*-Ulrich sheaves have been defined in [KS20] and it was shown that a hypersurface in $\mathbb{P}^n$ carries a positive Ulrich sheaf if and only if it is cut out by a polynomial with a definite determinantal representation. For subvarieties $X \subset \mathbb{P}^n$ of higher codimension supporting a positive Ulrich sheaf is equivalent to admitting some generalized type of determinantal representation, that was introduced in [SV18] and motivated by operator theory. The existence of such a determinantal representation for X implies that some power of the Chow form of X has a definite determinantal representation. On the other hand, we will show that a real-fibered double cover $f\colon X \to \mathbb{P}^n$, which is ramified along the zero set of a homogeneous polynomial p, admits a positive f-Ulrich sheaf if and only if p is a sum of squares. Therefore, the notion of positive Ulrich sheaves encapsulates both types of algebraic certificates for the abovementioned geometric properties of homogeneous polynomials, namely being a sum of squares and having a definite determinantal representation.

The main result of this article is a criterion for a coherent sheaf to be a positive Ulrich sheaf, which implies all the abovementioned existence results on representations as sums of squares and determinantal representations, and more. It only comprises a positivity criterion that can be checked locally as well as a condition on the dimension of the space of global sections (but surprisingly not of the higher cohomology groups) of the coherent sheaf at hand.

To this end, after some preparations in Section 3, we characterize Ulrich sheaves in terms of a certain bilinear mapping and its behavior on the level of global sections in Section 4 (see, in particular, Theorem 4.6). In this part, we work over an arbitrary ground field and we believe that these results can be of independent interest.

After we review some facts on the codifferent sheaf in Section 5, that will be important later on, we focus on varieties over $\mathbb{R}$. In Section 6, we recall some facts about real-fibered morphisms. We then define positive Ulrich sheaves in Section 7. Theorem 7.2 is the abovementioned convenient criterion for checking whether a sheaf is a positive Ulrich sheaf. In Section 8, we show that for a given polynomial having a determinantal or sum of squares representation is equivalent to the existence of a certain positive Ulrich sheaf. From this, the result on determinantal representations of quadratic hyperbolic polynomials from [NT12] follows directly. In order to make our general theory also applicable to other cases of interest, we specialize to Ulrich sheaves of rank one on irreducible varieties in Section 9. Theorem 9.3 gives a convenient criterion for a Weil divisor giving rise to a positive Ulrich sheaf. Namely, under some mild assumptions, if $f: X \to Y$ is a real-fibered morphism and D a Weil divisor on X such that $2D$ differs from the ramification divisor of f only by a principal divisor defined by a nonnegative rational function, then the sheaf associated with D is a positive f-Ulrich sheaf whenever its space of global sections has dimension $\deg(f)$. A particularly nice case is that of hyperbolic hypersurfaces: The existence of a definite determinantal representation is guaranteed by the existence of a certain *interlacer*, generalizing the construction for plane curves from [PV13] (see Corollary 9.9).

In Section 10, we apply our theory to the case of curves and show how it easily implies the Helton–Vinnikov theorem on plane hyperbolic curves [HV07] as well as its generalization to curves of higher codimension from [SV18] using the 2-divisibility of the Jacobian. Finally, in Section 11, we consider the anticanonical map on real del Pezzo surfaces in order to reprove Hilbert's theorem on ternary quartics [Hil88] and the existence of a Hermitian determinantal representation on cubic hyperbolic surfaces [BK07]. We further prove a new result on quartic del Pezzo surfaces in $\mathbb{P}^4$. Apart from our general theory, the only ingredients for this part are basic properties of (real) del Pezzo surfaces as well as the Riemann–Roch theorem.

2 Preliminaries and notation

For any scheme X and $p \in X$, we denote by $\kappa(p)$ the residue class field of X at p. If X is separated, reduced (but not necessarily irreducible), and of finite type over a field K, we say that X is a variety over K. For any coherent sheaf $\mathcal{F}$ on X, we denote by $\operatorname{rank}_p(\mathcal{F})$ the dimension of the fiber of $\mathcal{F}$ at p considered as $\kappa(p)$-vector space. If X is irreducible with generic point ξ, we simply denote $\operatorname{rank}(\mathcal{F}) = \operatorname{rank}_\xi(\mathcal{F})$. If X is a scheme (over a field K) and L a field (extension of K), then we denote by $X(L)$ the set of all morphisms $\operatorname{Spec}(L) \to X$ of schemes (over K). For a field K, we let $\mathbb{P}^n_K = \operatorname{Proj}(K[x_0, \dots, x_n])$, and if the field is clear from the context, we omit the index and just write $\mathbb{P}^n$. We say that a scheme is *Noetherian* if it can be covered by a finite number of open affine subsets that are spectra of Noetherian rings.

3 Bilinear mappings on coherent sheaves

Definition 3.1 Let X be a scheme, and let $\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{G}$ be coherent sheaves on X. A $\mathcal{G}$-*valued pairing* of $\mathcal{F}_1$ and $\mathcal{F}_2$ is a morphism of coherent sheaves $\varphi: \mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{G}$. Let K be a field and $\alpha \in X(K)$, i.e., a morphism $\alpha: \operatorname{Spec}(K) \to X$. Then we get a bilinear map $\alpha^*\varphi$ on $\alpha^*\mathcal{F}_1 \times \alpha^*\mathcal{F}_2$ with values in $\alpha^*\mathcal{G}$ which are just finite-dimensional K-vector spaces. We say that φ is *nondegenerate at* $\alpha \in X(K)$ if the map $\alpha^*\mathcal{F}_1 \to \operatorname{Hom}_K(\alpha^*\mathcal{F}_2, \alpha^*\mathcal{G})$ induced by $\alpha^*\varphi$ is an isomorphism.

For the rest of this section, unless stated otherwise, let X always be a geometrically integral scheme with generic point ξ which is proper over a field K.

Lemma 3.1 *Let $\mathcal{F}$ be a coherent sheaf on X which is generated by global sections. If there is a K-basis of $H^0(X, \mathcal{F})$ which is also a $\kappa(\xi)$-basis of $\mathcal{F}_\xi$, then $\mathcal{F} \cong \mathcal{O}_X^r$, where $r = \dim H^0(X, \mathcal{F})$.*

Proof Let $\mathcal{K}$ be the kernel of the map $\mathcal{O}_X^r \to \mathcal{F}$ that sends the unit vectors to the K-basis of $H^0(X, \mathcal{F})$. Since $\mathcal{F}$ is generated by global sections, this map is surjective. We thus obtain the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_X^r \to \mathcal{F} \to 0.$$

Passing to the stalk at ξ gives $\mathcal{K}_\xi = 0$ by our assumption. Since $\mathcal{K}$ is torsion-free as a subsheaf of $\mathcal{O}_X^r$, this implies that $\mathcal{K} = 0$ and therefore $\mathcal{O}_X^r \cong \mathcal{F}$. ∎

Remark 3.2 Let $\varphi: \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2 \to \mathcal{O}_X$ be a pairing of the coherent sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$. This induces a bilinear mapping

$$V_1 \times V_2 \to K,$$

where $V_i = H^0(X, \mathcal{F}_i)$ since $H^0(X, \mathcal{O}_X) = K$.

Lemma 3.3 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent sheaves on X, let $V_i = H^0(X, \mathcal{F}_i)$, and let $s_1, \dots, s_r$ be a basis of V_1. Let $\varphi: \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2 \to \mathcal{O}_X$ be a pairing such that the induced bilinear mapping $V_1 \times V_2 \to K$ is nondegenerate. Then the images of $s_1, \dots, s_r$ in the $\kappa(\xi)$-vector space $(\mathcal{F}_1)_\xi$ are linearly independent.*

Proof Since the bilinear mapping $V_1 \times V_2 \to K$ is nondegenerate, there is a basis $t_1, \dots, t_r \in V_2$ that is dual to $s_1, \dots, s_r$ with respect to this bilinear mapping. Suppose $f_1, \dots, f_r \in \kappa(\xi)$ such that

$$f_1 s_1 + \cdots + f_r s_r = 0.$$

Tensoring with t_j and applying φ yields $f_j \cdot \varphi(s_j \otimes t_j) = 0$ and therefore $f_j = 0$ since $\varphi(s_j \otimes t_j) = 1$ by assumption. ∎

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Proposition 3.4 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent sheaves on X, let $V_i = H^0(X, \mathcal{F}_i)$, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be generated by global sections. Let $\varphi\colon \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2 \to \mathcal{O}_X$ be a pairing such that the induced bilinear mapping $V_1 \times V_2 \to K$ is nondegenerate. Then:*

(a) *$\mathcal{F}_i \cong \mathcal{O}_X^r$, where $r = \dim V_i$.*
(b) *The morphism $\mathcal{F}_1 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{O}_X)$ corresponding to φ is an isomorphism.*

Proof Let $s_1, \dots, s_r \in V_1$ be a basis of V_1. By assumption, $s_1, \dots, s_r$ span the $\kappa(\xi)$-vector space $(\mathcal{F}_1)_\xi$ and, by Lemma 3.3, they are linearly independent. Thus, by Lemma 3.1, we have $\mathcal{F}_1 \cong \mathcal{O}_X^r$. The same argument applies to $\mathcal{F}_2$ and part (b) then follows immediately from (a). ■

Lemma 3.5 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent torsion-free sheaves on X. Assume that the image of $V_i = H^0(X, \mathcal{F}_i)$ spans the stalk $(\mathcal{F}_i)_\xi$ as $\kappa(\xi)$-vector space. Furthermore, let $\varphi\colon \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2 \to \mathcal{O}_X$ be a pairing such that the induced bilinear mapping $V_1 \times V_2 \to K$ is nondegenerate. Then the corresponding morphism $\mathcal{F}_1 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{O}_X)$ is injective.*

Proof Let $s_1, \dots, s_r \in V_1$ be a basis of V_1, and let $t_1, \dots, t_r \in V_2$ be the dual basis with respect to the bilinear mapping $V_1 \times V_2 \to K$. Let $U \subset X$ be some open affine subset, $A = \mathcal{O}_X(U)$ and $M_i = \mathcal{F}_i(U)$. For any $0 \neq g \in M_1$, there is $0 \neq a \in A$ such that $a \cdot g$ is in the submodule of M_1 that is spanned by the restrictions $s_i|_U$:

$$a \cdot g = f_1 \cdot s_1|_U + \cdots + f_r \cdot s_r|_U$$

for some $f_j \in A$ that are not all zero. Let, for instance, $f_i \neq 0$, then

$$\varphi(a \cdot g \otimes t_i|_U) = a \cdot f_i \cdot \varphi(s_i \otimes t_i) \neq 0.$$

This shows that the map $M_1 \to \mathrm{Hom}_A(M_2, A)$ induced by φ in injective. ■

Lemma 3.6 *Let $\mathcal{F}$ be a coherent sheaf on X and $\mathcal{G}$ a subsheaf with $\mathcal{G}_\xi = \mathcal{F}_\xi$. Then the natural map $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X)$ is injective.*

Proof Let $U \subset X$ be some open affine subset, $A = \mathcal{O}_X(U)$, $M = \mathcal{F}(U)$, and $N = \mathcal{G}(U)$. Consider a morphism $\varphi\colon M \to A$ such that $\varphi|_N = 0$. For every $g \in M$, there is a nonzero $t \in A$ such that $t \cdot g \in N$. Thus, $\varphi(t \cdot g) = t \cdot \varphi(g) = 0$, and therefore $\varphi(g) = 0$. This shows that $\varphi = 0$. ■

Theorem 3.7 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent torsion-free sheaves on X, and let $V_i = H^0(X, \mathcal{F}_i)$. Let $\varphi\colon \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2 \to \mathcal{O}_X$ be a pairing such that the induced bilinear mapping $V_1 \times V_2 \to K$ is nondegenerate. If $\dim V_1 \geq \operatorname{rank} \mathcal{F}_1$, then $\mathcal{F}_1 \cong \mathcal{O}_X^r$ and $r = \dim V_1 = \operatorname{rank} \mathcal{F}_1$. Furthermore, $\mathcal{F}_1 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{O}_X)$ is an isomorphism.*

Proof By Proposition 3.4(a), it suffices to show that $\mathcal{F}_i$ is generated by global sections. Let $\mathcal{G}_i$ be the subsheaf of $\mathcal{F}_i$ generated by its global sections V_i. We get the commutative diagram

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}_2, \mathcal{O}_X) \\ \mathcal{F}_1 \arrow[r] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{O}_X) \arrow[u] \end{tikzcd}.$$

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The homomorphism $\mathcal{G}_1 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}_2, \mathcal{O}_X)$ is an isomorphism by Proposition 3.4(b). By Lemma 3.3 and the condition on the dimension it follows that the image of $V_i = H^0(X, \mathcal{F}_i)$ spans the stalk $(\mathcal{F}_i)_\xi$ as $\kappa(\xi)$-vector space and that $(\mathcal{G}_i)_\xi = (\mathcal{F}_i)_\xi$. Thus, the bottom and right maps in the diagram are also injective by Lemmas 3.5 and 3.6, respectively. This implies that $\mathcal{G}_1 = \mathcal{F}_2$, and therefore $\mathcal{F}_1$ is generated by global sections. ∎

Example 3.8 This example is to illustrate that the assumption in Theorem 3.7 of being nondegenerate on global sections is crucial. Let $X = \mathbb{P}^1$ and ξ the generic point of $\mathbb{P}^1$. Consider the coherent torsion-free sheaf $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ on $\mathbb{P}^1$. We have $\dim H^0(\mathbb{P}^1, \mathcal{F}) = 2 = \operatorname{rank}_\xi \mathcal{F}$. On $\mathcal{F}$, we define the pairing $\varphi\colon \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{O}_X$ that sends $(a, b) \otimes (c, d)$ to $ad + bc$ which is nondegenerate at ξ. But the induced bilinear form on the global sections of $\mathcal{F}$ is identically zero and $\mathcal{F} \ncong \mathcal{O}_{\mathbb{P}^1}^2$.

4 Ulrich sheaves

Definition 4.1 Let $f\colon X \to Y$ be a morphism of schemes. A coherent sheaf $\mathcal{F}$ on X is *f-Ulrich* if $f_*\mathcal{F} \cong \mathcal{O}_Y^r$ for some natural number r.

Remark 4.1 Let $f_i\colon X_i \to Y$ be finitely many morphisms of schemes. Let X be the disjoint union of the X_i and $f\colon X \to Y$ the morphism induced by the f_i. A coherent sheaf $\mathcal{F}$ on X is f-Ulrich if and only if $\mathcal{F}|_{X_i}$ is f_i-Ulrich for all i. Thus, we can usually restrict to the case when X is connected.

A case of particular interest is when X is an embedded projective variety.

Proposition 4.2 [ESW03, Proposition 2.1] *Let $X \subset \mathbb{P}^n$ be a closed subscheme, and let $f\colon X \to \mathbb{P}^k$ be a finite surjective linear projection from a center that is disjoint from X. Let $\mathcal{F}$ be a coherent sheaf whose support is all of X. Then the following are equivalent:*

(i) *$\mathcal{F}$ is f-Ulrich.*
(ii) *$H^i(X, \mathcal{F}(-i)) = 0$ for $i > 0$ and $H^i(X, \mathcal{F}(-i-1)) = 0$ for $i < k$.*
(iii) *$H^i(X, \mathcal{F}(j)) = 0$ for all $1 \le i \le k-1$, $j \in \mathbb{Z}$; $H^0(X, \mathcal{F}(j)) = 0$ for $j < 0$ and $H^k(X, \mathcal{F}(j)) = 0$ for $j \ge -k$.*
(iv) *The module $M = \oplus_{i\in\mathbb{Z}} H^0(X, \mathcal{F}(i))$ of twisted global sections is a Cohen–Macaulay module over the polynomial ring $S = K[x_0, \dots, x_n]$ of dimension $k+1$ whose minimal S-free resolution is linear.*

If $\mathcal{F}$ as in Proposition 4.2 satisfies these equivalent conditions, then we say that $\mathcal{F}$ is an *Ulrich sheaf* on X without specifying the finite surjective linear projection f as conditions (ii)–(iv) do not depend on the choice of f. A major open question in this context is the following.

Problem 4.3 [ESW03, p. 543] *Is there an Ulrich sheaf on every closed subvariety $X \subset \mathbb{P}^n$?*

We now want to apply the results from Section 3 to give a criterion for a sheaf to be Ulrich. For this, we need a relative notion of nondegenerate bilinear mappings. Let $f\colon X \to Y$ be a finite morphism of Noetherian schemes. For any quasi-coherent sheaf $\mathcal{G}$ on Y, we consider the sheaf $\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G})$. Since this is a quasi-coherent $f_*\mathcal{O}_X$-module, it corresponds to a quasi-coherent $\mathcal{O}_X$-module which we will denote by $f^!\mathcal{G}$. We recall the following basic lemma (cf. [Har77, Chapter III Section 6, Exercise 6.10]).

Lemma 4.4 *Let $f: X \to Y$ be a finite morphism of Noetherian schemes. Let $\mathcal{F}$ be a coherent sheaf on X, and let $\mathcal{G}$ be a quasi-coherent sheaf on Y. There is a natural isomorphism*

$$f_* \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) \to \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G})$$

of quasi-coherent $f_*\mathcal{O}_X$-modules.

Let $f: X \to Y$ be a finite morphism of Noetherian schemes. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent sheaves on X and consider an $f^!\mathcal{O}_Y$-valued pairing, i.e., a morphism $\mathcal{F}_1 \otimes \mathcal{F}_2 \to f^!\mathcal{O}_Y$ of coherent $\mathcal{O}_X$-modules. This corresponds to a morphism $\mathcal{F}_1 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}_2, f^!\mathcal{O}_Y)$. Lemma 4.4 tells us that this gives us a morphism

$$f_*\mathcal{F}_1 \to \mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{F}_2, \mathcal{O}_Y),$$

which corresponds to an $\mathcal{O}_Y$-valued pairing on the pushforwards $f_*\mathcal{F}_1$ and $f_*\mathcal{F}_2$.

Remark 4.5 Let us explain here the affine case in more detail. To that end, let $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$, and $f: X \to Y$ be induced by the finite ring homomorphism $f^\#: A \to B$. Then $f^!\mathcal{O}_Y$ is the sheaf on X associated with the B-module $\mathrm{Hom}_A(B, A)$ whose B-module structure is given by $(b \cdot \varphi)(m) = \varphi(b \cdot m)$ for all $b, m \in B$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are the coherent sheaves associated with the B-modules M_1 and M_2, then a morphism $\mathcal{F}_1 \otimes \mathcal{F}_2 \to f^!\mathcal{O}_Y$ of coherent $\mathcal{O}_X$-modules corresponds to a homomorphism $\psi: M_1 \otimes_B M_2 \to \mathrm{Hom}_A(B, A)$ of B-modules. This gives rise to the following A-bilinear map on $M_1 \times M_2$:

$$(m_1, m_2) \mapsto (\psi(m_1 \otimes m_2))(1).$$

This gives the $\mathcal{O}_Y$-valued pairing of the pushforwards $f_*\mathcal{F}_1$ and $f_*\mathcal{F}_2$.

Definition 4.2 Let $f: X \to Y$ be a finite morphism of Noetherian schemes. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent sheaves on X and consider an $f^!\mathcal{O}_Y$-valued pairing $\varphi: \mathcal{F}_1 \otimes \mathcal{F}_2 \to f^!\mathcal{O}_Y$. For a field K, we say that φ is *nondegenerate at $\alpha \in Y(K)$* if the induced $\mathcal{O}_Y$-valued pairing of the pushforwards $f_*\mathcal{F}_1$ and $f_*\mathcal{F}_2$ is nondegenerate at α.

Now, let Y be a geometrically irreducible variety which is proper over a field K. Let further $f: X \to Y$ be a finite surjective morphism and $\mathcal{F}_1$, $\mathcal{F}_2$ coherent sheaves on X with a pairing $\mathcal{F}_1 \otimes \mathcal{F}_2 \to f^!\mathcal{O}_Y$. We have seen that this induces an $\mathcal{O}_Y$-valued pairing on the pushforwards which in turn induces a K-bilinear mapping

$$H^0(X, \mathcal{F}_1) \times H^0(X, \mathcal{F}_2) \to K.$$

Theorem 4.6 *Let X be an equidimensional variety over a field K with irreducible components $X_1, \ldots, X_s$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent torsion-free sheaves on X, and let $V_i = H^0(X, \mathcal{F}_i)$. Let Y be a geometrically irreducible variety which is proper over K and $f: X \to Y$ a finite surjective morphism. Assume that there is an $f^!\mathcal{O}_Y$-valued pairing* *of $\mathcal{F}_1$ and $\mathcal{F}_2$ such that the induced K-bilinear mapping $V_1 \times V_2 \to K$ is nondegenerate. Then the following are equivalent:*

(i) $\dim V_1 \geq \sum_{i=1}^{s} \deg(f|_{X_i}) \cdot \operatorname{rank}(\mathcal{F}_1|_{X_i})$.
(ii) *$\mathcal{F}_1$ is f-Ulrich.*

Proof We will apply Theorem 3.7 to the coherent sheaves $f_*\mathcal{F}_1$ and $f_*\mathcal{F}_2$. First, we need that the $f_*\mathcal{F}_i$ are torsion-free. This follows from the assumptions that f is finite and surjective, X is equidimensional and $\mathcal{F}_i$ is torsion-free. Further, by [Kol96, Chapter VI Section 2, Proposition 2.7], one has $\operatorname{rank}(f_*\mathcal{F}_1) = \sum_{i=1}^{s} \deg(f|_{X_i}) \cdot \operatorname{rank}(\mathcal{F}_1|_{X_i})$. ■

Remark 4.7 Let $f\colon X \to Y$ as in Theorem 4.6. Assume that $\mathcal{F}_1$ is an f-Ulrich sheaf on X, and let $\mathcal{F}_2 = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{O}_Y)$. Then we have a canonical map $\mathcal{F}_1 \otimes \mathcal{F}_2 \to f^!\mathcal{O}_Y$ which satisfies the assumptions of Theorem 4.6.

Example 4.8 Let $X = \mathbb{P}^1$ and consider the natural morphism $f\colon X \times X \to X^{(2)}$, where $X^{(2)}$ is the symmetric product. This corresponds to the inclusion $S^\sigma \subset S$, where S is the bigraded ring $K[x_0, x_1, y_0, y_1]$ and S^σ is the ring of invariants under the action of the automorphism $\sigma\colon S \to S$ that interchanges x_i and y_i for $i = 0, 1$. For every $f, g \in S$, we have that $g\sigma(f) - \sigma(g)f$ is divisible by $x_0y_1 - x_1y_0$. Thus,

$$B(f,g) = \frac{g\sigma(f) - \sigma(g)f}{x_0y_1 - x_1y_0} \in S^\sigma.$$

Since the denominator has bidegree $(1,1)$, this defines a morphism

$$\mathcal{O}_{X\times X}(1,1) \to \mathscr{H}om_{\mathcal{O}_{X^{(2)}}}(f_*\mathcal{O}_{X\times X}, \mathcal{O}_{X^{(2)}})$$

of $\mathcal{O}_{X\times X}$-modules that sends a section g to $B(f,g)$. Thus, we get an $f^!\mathcal{O}_{X^{(2)}}$-valued pairing of $\mathcal{F}_1 = \mathcal{O}_{X\times X}(1,0)$ and $\mathcal{F}_2 = \mathcal{O}_{X\times X}(0,1)$. The induced K-bilinear mapping is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so, in particular, nondegenerate. Thus, we can apply Theorem 4.6 to deduce that $\mathcal{F}_1$ and $\mathcal{F}_2$ are f-Ulrich.

5 The codifferent sheaf

In this section, we recall some properties of $f^!\mathcal{O}_Y$ and its relation to the codifferent sheaf. Most of the results should be well known, but for a lack of adequate references, we will include (at least partial) proofs here.

Lemma 5.1 *Let $f\colon X \to Y$ be a finite morphism of Noetherian schemes.*

(a) *If f is flat and both X and Y are Gorenstein, then $f^!\mathcal{O}_Y$ is a line bundle.*
(b) *If Y is a smooth variety over K and X is Gorenstein, then $f^!\mathcal{O}_Y$ is a line bundle.*

Proof We first note that (b) is a special case of part (a) because, in this situation, the morphism f is automatically flat as Gorenstein implies Cohen–Macaulay. In order to prove part (a), note that for every $y \in Y$, the canonical module of $\mathcal{O}_{Y,y}$ is $\mathcal{O}_{Y,y}$ itself

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by [HK71, Satz 5.9], and for every $x \in X$, the canonical module of $\mathcal{O}_{X,x}$ is $(f^!\mathcal{O}_Y)_x$ by [HK71, Satz 5.12]. Thus, again by [HK71, Satz 5.9], $f^!\mathcal{O}_Y$ is a line bundle on X if X is Gorenstein. ∎

Remark 5.2 The preceding lemma implies that the sheaf $f^!\mathcal{O}_Y$ is a line bundle whenever $f\colon X \to Y$ is a finite morphism of smooth varieties over a field K.

Definition 5.1 Let A be a Noetherian integral domain, and let $A \subset B$ be a finite ring extension such that for each minimal prime ideal $\mathfrak{p}$ of B, we have $\mathfrak{p} \cap A = (0)$. Let K be the quotient field of A, and let L be the total quotient ring of B. Then L is a finite-dimensional K-vector space and we have the K-linear map $\mathrm{tr}_{L/K}\colon L \to K$ that associates to every element $x \in L$ the trace of the K-linear map $L \to L$, $a \mapsto ax$. The *codifferent* of the ring extension $A \subset B$ is the B-module

$$\Delta(B/A) = \{g \in L \mid \mathrm{tr}_{L/K}(g \cdot B) \subset A\}.$$

Clearly, the map

$$\Delta(B/A) \to \mathrm{Hom}_A(B, A),\ g \mapsto (b \mapsto \mathrm{tr}_{L/K}(g \cdot b)$$

is a homomorphism of B-modules.

Now, let Y be an integral Noetherian scheme and $f\colon X \to Y$ a finite morphism such the generic point of each irreducible component of X is mapped to the generic point of Y. Let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$. By gluing the above, we define the quasi-coherent subsheaf $\Delta(X/Y)$ of $\mathcal{K}_X$ and we obtain a morphism of $\mathcal{O}_X$-modules $\Delta(X/Y) \to f^!\mathcal{O}_Y$. We call $\Delta(X/Y)$ the *codifferent sheaf* of *f*.

Example 5.3 Let A be an integral domain and $K = \mathrm{Quot}(A)$. Let $f \in A[t]$ be a monic polynomial over A which has only simple zeros in the algebraic closure of K, and let $B = A[t]/(f)$. Then the codifferent $\Delta(B/A)$ is the fractional ideal generated by $\frac{1}{f'}$ in the total quotient ring of B. Here, f' denotes the formal derivative of *f*. This follows from a lemma often attributed to Euler (see [Ser79, III, Section 6]).

Remark 5.4 In order to construct a pairing $\mathcal{F}_1 \otimes \mathcal{F}_2 \to f^!\mathcal{O}_Y$, it thus suffices by the discussion in Definition 5.1 to construct a pairing $\mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{K}_X$ whose image is contained in $\Delta(X/Y)$.

Lemma 5.5 *Let X and Y be varieties over a field of characteristic zero, X equidimensional and Y irreducible. Then, for any finite surjective morphism $f\colon X \to Y$, the map $\Delta(X/Y) \to f^!\mathcal{O}_Y$ is an isomorphism of $\mathcal{O}_X$-modules.*

Proof We can reduce to the affine case: let $A \subset B$ be a finite ring extension such that A is an integral domain and $\mathfrak{p} \cap A = (0)$ for all minimal prime ideals $\mathfrak{p}$ of B. Let E and F be the total quotient rings of A and B, respectively. Since $B \otimes_A E$ is finite-dimensional as an E-vector space and reduced, it is a ring of the form $F_1 \times \cdots \times F_r$ for some finite field extensions F_i of E. Since no element of A is a zero divisor in B, we actually have that $F = F_1 \times \cdots \times F_r$. Thus, we have an injective map $\mathrm{Hom}_A(B, A) \to \mathrm{Hom}_E(F, E)$ that is given by tensoring with E. By definition, the preimage of $\mathrm{Hom}_A(B, A)$ under the map

$$\psi\colon F \to \mathrm{Hom}_E(F, E),\ a \mapsto \mathrm{tr}_{F/E}(a \cdot -)$$

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is exactly $\Delta(B/A)$. It thus suffices to show that ψ is an isomorphism because then the restriction of ψ to $\Delta(B/A)$ is the desired isomorphism $\Delta(B/A) \to \mathrm{Hom}_A(B, A)$. The map ψ is the direct sum of the maps

$$\psi_i: F_i \to \mathrm{Hom}_E(F_i, E),\ a \mapsto \mathrm{tr}_{F_i/E}(a \cdot -),$$

and thus it suffices to show that each ψ_i is an isomorphism. We have that $\mathrm{tr}_{F_i/E}(1)$ is the dimension of F_i as an E-vector space. Since we work over a field of characteristic zero, this shows that each ψ_i is a nonzero map. But since $\mathrm{Hom}_E(F_i, E)$ is one-dimensional considered as a vector space over F_i, this implies that ψ_i is an isomorphism. ∎

Remark 5.6 Lemma 5.5 is no longer true over fields of positive characteristic because the trace $\mathrm{tr}_{L/K}$ is identically zero for inseparable field extensions $K \subset L$. This is one reason why we have not worked with the codifferent sheaf to begin with.

Proposition 5.7 *Let $f: X \to Y$ be a finite surjective morphism of varieties over a field of characteristic zero. Let X be equidimensional and Gorenstein. Let Y be smooth and irreducible. Then $\Delta(X/Y)$ is an invertible subsheaf of $\mathcal{K}_X$ and thus $\Delta(X/Y) = \mathcal{L}(R)$ for some Cartier divisor R on X. This Cartier divisor is effective and its support consists exactly of those points where f is ramified.*

Proof By Lemma 5.5, $\Delta(X/Y)$ is isomorphic to $f^!\mathcal{O}_Y$ which is an invertible sheaf by Lemma 5.1. Thus, $\Delta(X/Y)$ is an invertible subsheaf of $\mathcal{K}_X$, and we can write $\Delta(X/Y) = \mathcal{L}(R)$ for some Cartier divisor R on X. We first show that R is effective, which is equivalent to the constant 1 being a global section of $\Delta(X/Y)$. This can be checked locally. We thus consider a finite ring extension $A \subset B$, where A is an integral domain. Furthermore, this ring extension is flat by the assumptions on X and Y. Thus, without loss of generality, we can assume that B is free as A-module. Therefore, the A-linear map $B \to B$ given by multiplication with an element $b \in B$ can be represented by a matrix having entries in A. Using the notation of Definition 5.1, this shows that $\mathrm{tr}_{L/K}(1 \cdot B) \subset A$. Thus, the constant 1 is a global section of $\Delta(X/Y)$ and R is effective. The image of 1 under the map $H^0(X, \Delta(X/Y)) \to H^0(X, f^!\mathcal{O}_Y)$ is just the trace map and the subscheme associated with R is the zero set of this section. This consists exactly of the ramification points of f (see, for example, [Sta20, Tag 0BW9] or [Kum16a, Remark 2.2.19]). ∎

Definition 5.2 In the situation of Proposition 5.7, we call the Cartier divisor R on X that corresponds to the invertible subsheaf $\Delta(X/Y)$ of $\mathcal{K}_X$ the *ramification divisor* of f.

Lemma 5.8 *Let $f: X \to Y$ be a finite surjective morphism of varieties. Let X be equidimensional, and let Y be smooth and irreducible. Let $Z \subset X$ be of codimension at least two. Consider the open subset $V = X \backslash Z$ and its inclusion $\iota: V \to X$ to X. Then $\iota_*(\Delta|_V) = \Delta$, where $\Delta = \Delta(X/Y)$.*

Proof If we enlarge Z, then the statement becomes stronger, so we may assume that $Z = f^{-1}(Z')$ for some $Z' \subset Y$ of codimension at least two. We write $\mathcal{D} = \iota_*(\Delta|_V)$. Then

Δ is a subsheaf of $\mathcal{D}$ and we need to show equality. To that end, let $U \subset Y$ be an affine open subset and $W = f^{-1}(U)$. Then we have the following:

(1) $\mathcal{O}_X(W) \subset \mathcal{O}_X(W \setminus Z)$ and
(2) $\mathcal{O}_Y(U) = \mathcal{O}_Y(U \setminus Z')$.

Letting L be the total quotient ring of X and K the function field of Y, we get

$$\mathcal{D}(W) = \Delta(f^{-1}(U \setminus Z')) = \{a \in L \mid \mathrm{tr}_{L/K}(a\mathcal{O}_X(W \setminus Z)) \subset \mathcal{O}_Y(U \setminus Z')\}$$

and due to (1) and (2) the latter is contained in

$$\{a \in L \mid \mathrm{tr}_{L/K}(a\mathcal{O}_X(W)) \subset \mathcal{O}_Y(U)\} = \Delta(W).$$

Thus, we have $\mathcal{D}(W) = \Delta(W)$. Since $f: X \to Y$ is affine, the sets $W = f^{-1}(U)$ for $U \subset Y$ open and affine give an affine covering of X and thus $\mathcal{D} = \Delta$. ∎

6 Real-fibered morphisms

In this section, we recall the notion of real-fibered morphisms, basic examples and some of their properties.

Definition 6.1 Let $f: X \to Y$ be a morphism of varieties over $\mathbb{R}$. If $f^{-1}(Y(\mathbb{R})) = X(\mathbb{R})$, then we say that f is *real fibered*.

Example 6.1 Let $p \in \mathbb{R}[x_0, \ldots, x_n]_{2d}$ be a homogeneous polynomial of degree $2d$. Inside the weighted projective space $\mathbb{P}(d, 1, \ldots, 1)$, we consider the hypersurface X defined by $y^2 = p(x_0, \ldots, x_n)$ and the natural projection $\pi: X \to \mathbb{P}^n$ onto the x-coordinates. This is a double cover of $\mathbb{P}^n$ ramified at the hypersurface defined by $p = 0$. Clearly π is real fibered if and only if p is globally nonnegative, i.e., $p(x) \geq 0$ for all $x \in \mathbb{R}^{n+1}$.

Hyperbolic polynomials yield another class of examples.

Definition 6.2 Let $h \in \mathbb{R}[x_0, \ldots, x_n]_d$ be a homogeneous polynomial of degree d, and let $e \in \mathbb{R}^{n+1}$. We say that h is *hyperbolic with respect to e* if the univariate polynomial $h(te - v) \in \mathbb{R}[t]$ has only real zeros for all $v \in \mathbb{R}^{n+1}$. Note that this implies $h(e) \neq 0$. A hypersurface $X \subset \mathbb{P}^n$ is called *hyperbolic with respect to* $[e]$ if its defining polynomial is hyperbolic with respect to e.

Example 6.2 Let $X \subset \mathbb{P}^n$ be a hypersurface, and let $e \in \mathbb{P}^n$ be a point that does not lie on X. Then the linear projection $\pi_e: X \to \mathbb{P}^{n-1}$ with center e is real-fibered if and only if X is hyperbolic with respect to e.

One can generalize this notion naturally to varieties of higher codimension.

Definition 6.3 Let $X \subset \mathbb{P}^n$ be a variety of pure dimension d, and let $E \subset \mathbb{P}^n$ be a linear subspace of codimension $d+1$ which does not intersect X. We say that X is *hyperbolic with respect to E* if the linear projection $\pi_E: X \to \mathbb{P}^d$ with center E is real-fibered.

An important feature of real-fibered morphisms is the following.

Theorem 6.3 [KS20, Theorem 2.19] *Let $f: X \to Y$ be a real-fibered morphism of smooth varieties over $\mathbb{R}$. Then the ramification divisor of f has no real point.*

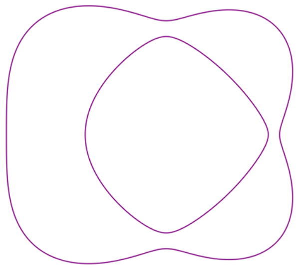

Figure 1: A plane quartic curve that is hyperbolic with respect to any point in the inner oval

The following property of real-fibered morphisms will come in handy later when we want to construct positive semidefinite bilinear forms.

Proposition 6.4 *Let Y be an irreducible smooth variety, and let X be an equidimensional variety over $\mathbb{R}$. Let $f\colon X \to Y$ be a finite surjective real-fibered morphism. Let K and L be the total quotient rings of Y and X and $\mathrm{tr}_{L/K}\colon L \to K$ the trace map. If $g \in L$ is nonnegative on $X(\mathbb{R})$ (wherever it is defined), then $\mathrm{tr}_{L/K}(g)$ is nonnegative on $Y(\mathbb{R})$ (wherever it is defined).*

Proof Assume that $g \in L$ is nonnegative on $X(\mathbb{R})$ (wherever it is defined). By generic freeness [Gro65, Lemma 6.9.2], there is a nonempty open affine subset $U \subset Y$ such that $B = \mathcal{O}_X(f^{-1}(U))$ is a free A-module, where $A = \mathcal{O}_Y(U)$ and such that $g \in B$. By a version of the Artin–Lang theorem [Bec82, Lemma 1.5], the real points of U are dense in $Y(\mathbb{R})$ with respect to the euclidean topology because Y is smooth. Thus, it suffices to show that $\mathrm{tr}_{L/K}(g) \in A$ is nonnegative on every real point p of U. Let $\mathfrak{m} \subset A$ be the corresponding maximal ideal. Then $C = B/\mathfrak{m}B$ is finite-dimensional as vector space over $\mathbb{R} = A/\mathfrak{m}$ and $\mathrm{Spec}(C)$ consists only of real points because f is real fibered. Thus, letting $\overline{g} \in C$ be the residue class of g and because g is nonnegative on $f^{-1}(p)$, the quadratic form

$$C \times C \to \mathbb{R},\ (a, b) \mapsto \mathrm{tr}_{C/\mathbb{R}}(\overline{g} \cdot a \cdot b)$$

is positive semidefinite by [PRS93, Theorem 2.1]. Thus, in particular, $\mathrm{tr}_{C/\mathbb{R}}(\overline{g}) \geq 0$. Finally, by flatness, we have that $\overline{\mathrm{tr}_{L/K}(g)} = \mathrm{tr}_{C/\mathbb{R}}(\overline{g}) \geq 0$. ∎

7 Positive semidefinite bilinear forms

The criterion for a coherent sheaf being Ulrich that we have seen in Section 4 very much relies on the induced bilinear form on global sections being nondegenerate. Verifying this condition might be hard in general. However, in this section, we show that when working over the real numbers, we have a more convenient criterion at hand, namely, positivity.

We first have to define Hermitian bilinear forms on sheaves. To this end, let X be a scheme, and let $\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{G}$ be coherent sheaves on X. A $\mathcal{G}$-valued pairing $\varphi\colon \mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{G}$ induces naturally the two morphisms $\varphi_1\colon \mathcal{F}_1 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G})$ and $\varphi_2\colon \mathcal{F}_2 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G})$.

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Definition 7.1 Let X be a scheme, and let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X. A $\mathcal{G}$-valued pairing $\varphi: \mathcal{F} \otimes \mathcal{F} \to \mathcal{G}$ is *symmetric* if $\varphi_1 = \varphi_2$.

Definition 7.2 Let X be a scheme over $\mathbb{R}$. Denote by $X_{\mathbb{C}} = X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ the base change to $\mathbb{C}$ and $\pi: X_{\mathbb{C}} \to X$ the natural projection. Further, let $\tau: X_{\mathbb{C}} \to X_{\mathbb{C}}$ be the automorphism induced by complex conjugation. Let $\mathcal{G}$ be a coherent sheaf on X, and let $\mathcal{F}$ be a coherent sheaf on $X_{\mathbb{C}}$. A pairing $\varphi: \mathcal{F} \otimes \tau^*\mathcal{F} \to \pi^*\mathcal{G}$ is *Hermitian* if φ_1 agrees with the pullback of φ_2 via τ.

From now on, in this section, X will always be a variety over $\mathbb{R}$.

Definition 7.3 Let $\mathcal{F}$ be a coherent sheaf on X. Consider a symmetric bilinear form $\varphi: \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{O}_X$. Let $\alpha \in X(\mathbb{R})$, i.e., a morphism $\alpha: \mathrm{Spec}(\mathbb{R}) \to X$. Then we get a symmetric bilinear form $\alpha^*\varphi$ on the pullback $\alpha^*\mathcal{F}$ which is just a finite-dimensional $\mathbb{R}$-vector space. We say that φ is *positive semidefinite at* $\alpha \in X(\mathbb{R})$ if $\alpha^*\varphi$ is positive semidefinite. We say that φ is *positive semidefinite* if it is positive semidefinite at every $\alpha \in X(\mathbb{R})$.

Analogously, let $\mathcal{F}$ be a coherent sheaf on $X_{\mathbb{C}}$ and consider a Hermitian bilinear form $\varphi: \mathcal{F} \otimes_{\mathcal{O}_X} \overline{\mathcal{F}} \to \mathcal{O}_X$. We can consider any $\alpha \in X(\mathbb{R})$ also as a point of $X_{\mathbb{C}}$ that is fixed by the involution. Like this, we obtain a Hermitian bilinear form $\alpha^*\varphi$ on the pullback $\alpha^*\mathcal{F}$ which is just a finite-dimensional $\mathbb{C}$-vector space. We say that φ is *positive semidefinite at* $\alpha \in X(\mathbb{R})$ if $\alpha^*\varphi$ is positive semidefinite. We say that φ is *positive semidefinite* if it is positive semidefinite at every $\alpha \in X(\mathbb{R})$.

A symmetric or Hermitian bilinear form on a coherent sheaf induces a symmetric or Hermitian bilinear form on the space of global sections. The next lemma shows how this behaves with respect to positivity.

Lemma 7.1 *Let X be irreducible and proper over $\mathbb{R}$ with generic point ξ. Let $\varphi: \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{O}_X$ be a positive semidefinite symmetric bilinear form on the coherent sheaf $\mathcal{F}$ and $V = H^0(X, \mathcal{F})$.*

(a) *If $X(\mathbb{R}) \neq \emptyset$, then the induced bilinear form $V \times V \to \mathbb{R}$ is positive semidefinite.*
(b) *If $X(\mathbb{R})$ is Zariski dense in X, φ is nondegenerate at ξ and $\mathcal{F}$ is torsion-free, then the induced bilinear form $V \times V \to \mathbb{R}$ is positive definite and thus in particular nondegenerate.*

The corresponding statements for Hermitian bilinear forms hold true as well.

Proof For part (a), we observe that if $\varphi(s,s) = -1$ for some $s \in V$, then φ is not positive semidefinite at any point from $X(\mathbb{R})$.

In order to show part (b), we first observe that since $X(\mathbb{R})$ is Zariski dense in X, the field $\kappa(\xi)$ has an ordering P. Consider a nonzero section $s \in V$. Since $\varphi(s,s) \geq 0$ at all points in $X(\mathbb{R})$, Tarski's principle implies that $\varphi(s,s)$ is also nonnegative with respect to the ordering P when considered as an element of $\kappa(\xi)$. Thus, the bilinear form induced by φ on the $\kappa(\xi)$-vector subspace of $\mathcal{F}_\xi$ spanned by V is also positive semidefinite (with respect to P). But since φ is nondegenerate at ξ, it is even positive definite (with respect to P). Finally, because $\mathcal{F}$ is torsion-free, the nonzero section s is mapped to a nonzero element in the stalk $\mathcal{F}_\xi$ and therefore by positive definiteness $\varphi(s,s) \neq 0$. This shows the claim. ∎

Thus, if we assume positive semidefiniteness, we only need to assure that our bilinear form is nondegenerate at the generic point rather than on global sections.

Theorem 7.2 *Let Y be a geometrically irreducible variety which is proper over $\mathbb{R}$. Let $f\colon X \to Y$ be a finite surjective morphism, where X is an equidimensional variety over $\mathbb{R}$ with $X(\mathbb{R})$ Zariski dense in X. Let $X_1, \dots, X_s$ be the irreducible components of X. Let $\mathcal{F}$ be a coherent torsion-free sheaf on X, and let $V = H^0(X, \mathcal{F})$. Let $\varphi\colon \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to f^!\mathcal{O}_Y$ be a symmetric bilinear map which is nondegenerate at the generic point of each X_i. If the induced $\mathcal{O}_Y$-valued bilinear form on $f_*\mathcal{F}$ is positive semidefinite, then the following are equivalent:*

(i) $\dim V \geq \sum_{i=1}^{s} \deg(f|_{X_i}) \cdot \operatorname{rank}(\mathcal{F}|_{X_i})$.
(ii) *$\mathcal{F}$ is an f-Ulrich sheaf.*

Proof Combining Lemma 7.1 and Theorem 3.7, it remains to show that the induced symmetric $\mathcal{O}_Y$-valued bilinear form on $f_*\mathcal{F}$ is nondegenerate at the generic point of Y. But this follows from the assumption that φ is nondegenerate at the generic point of each X_i by Lemma 7.3 whose easy proof we leave as an exercise. ∎

Lemma 7.3 *Let $A = K_1 \times \cdots \times K_r$ be the direct product of fields K_i each of which is a finite extension of the field K. Let M be a finitely generated A-module. Then $M \cong V_1 \times \cdots \times V_r$, where each V_i is a finite-dimensional K_i-vector space and the right-hand side is considered as an A-module in the obvious way. A homomorphism $\varphi\colon M \to \operatorname{Hom}_K(M, K)$ of A-modules such that all induced maps $V_i \to \operatorname{Hom}_K(V_i, K)$ are isomorphisms is an isomorphism itself. In particular, the induced K-bilinear form $M \otimes_K M \to K$ is nondegenerate.*

In the same manner, we obtain the Hermitian version.

Theorem 7.4 *Let Y be a geometrically irreducible variety which is proper over $\mathbb{R}$. Let $f\colon X \to Y$ be a finite surjective morphism, where X is an equidimensional variety over $\mathbb{R}$ with $X(\mathbb{R})$ Zariski dense in X. Let $X_1, \dots, X_s$ be the irreducible components of $X_{\mathbb{C}}$. Let $\mathcal{F}$ be a coherent torsion-free sheaf on $X_{\mathbb{C}}$, and let $V = H^0(X, \mathcal{F})$. Let $\varphi\colon \mathcal{F} \otimes_{\mathcal{O}_X} \overline{\mathcal{F}} \to f^!\mathcal{O}_{Y_{\mathbb{C}}}$ be a Hermitian bilinear map which is nondegenerate at the generic point of each X_i. If the induced $\mathcal{O}_{Y_{\mathbb{C}}}$-valued Hermitian bilinear form on $f_*\mathcal{F}$ is positive semidefinite, then the following are equivalent:*

(i) $\dim V \geq \sum_{i=1}^{s} \deg(f|_{X_i}) \cdot \operatorname{rank}(\mathcal{F}|_{X_i})$.
(ii) *$\mathcal{F}$ is an f-Ulrich sheaf.*

Definition 7.4 Let Y be a geometrically irreducible variety which is proper over $\mathbb{R}$. Let $f\colon X \to Y$ be a finite surjective morphism, where X is an equidimensional variety over $\mathbb{R}$ with $X(\mathbb{R})$ Zariski dense in X. Let $X_1, \dots, X_s$ be the irreducible components of $X_{\mathbb{C}}$.

(a) Let $\mathcal{F}$ be an f-Ulrich sheaf on X. If there exists a symmetric bilinear map $\varphi\colon \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to f^!\mathcal{O}_Y$ which is nondegenerate at the generic point of each X_i such that the induced $\mathcal{O}_Y$-valued bilinear form on $f_*\mathcal{F}$ is positive semidefinite, then we say that $\mathcal{F}$ is a *positive symmetric* f-Ulrich sheaf.
(b) Let $\mathcal{F}$ be an f-Ulrich sheaf on $X_{\mathbb{C}}$. If there exists a Hermitian bilinear map $\varphi\colon \mathcal{F} \otimes_{\mathcal{O}_X} \overline{\mathcal{F}} \to f^!\mathcal{O}_{Y_{\mathbb{C}}}$ which is nondegenerate at the generic point of each X_i such

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that the induced $\mathcal{O}_{Y_\mathbb{C}}$-valued bilinear form on $f_*\mathcal{F}$ is positive semidefinite, then we say that $\mathcal{F}$ is a *positive Hermitian* f-Ulrich sheaf.

In order to check the positivity condition, the following lemma will be useful.

Lemma 7.5 *Let Y be an irreducible smooth variety with $Y(\mathbb{R}) \neq \varnothing$, and let X be an equidimensional variety over $\mathbb{R}$. Let $f: X \to Y$ be a finite surjective real-fibered morphism. Let s be a global section of $\mathcal{K}_X$ which is nonnegative on $X(\mathbb{R})$, and let $\mathcal{D}$ be the subsheaf of $\mathcal{K}_X$ given by $s \cdot \Delta(X/Y)$. Let $\mathcal{L}$ be a subsheaf of $\mathcal{K}_X$ such that $\mathcal{L} \cdot \mathcal{L} \subset \mathcal{D}$. Then there is an $f^!\mathcal{O}_Y$-valued symmetric bilinear form on $\mathcal{L}$ such that the induced $\mathcal{O}_Y$-valued bilinear form on $f_*\mathcal{L}$ is positive semidefinite.*

Proof By assumption, we can define the symmetric $\Delta(X/Y)$-valued bilinear form on $\mathcal{L}$ that maps a pair of sections (g, h) to $\frac{g \cdot h}{s}$. Thus the induced $\mathcal{O}_Y$-valued bilinear form maps (g, g) to the trace of $\frac{g^2}{s}$ which is nonnegative by Proposition 6.4. ■

Remark 7.6 Let $\iota: X \hookrightarrow \mathbb{P}^n$ be an embedding of a k-dimensional projective variety, and let $\mathcal{F}$ be an Ulrich sheaf on X, i.e., $\mathcal{F}$ is π-Ulrich for some finite surjective linear projection $\pi: X \to \mathbb{P}^k$. One can show that

$$\iota_*(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \pi^!\mathcal{O}_{\mathbb{P}^k})) \cong \mathcal{E}xt^{n-k}(\iota_*\mathcal{F}, \omega_{\mathbb{P}^n})(k+1).$$

Thus, our notion of symmetry coincides with the one introduced in [ESW03, Section 3.1]. In particular, the property of being symmetric does not depend on the choice of the linear projection π but the positivity condition does, as the next example shows.

Example 7.7 Let $Y = \mathbb{P}^1$ and $X = \mathcal{V}(x_0^2 - x_1^2 - x_2^2) \subset \mathbb{P}^2$. We let $f_i: X \to \mathbb{P}^1$ be the linear projection with center e_i, where $e_1 = [1:0:0]$ and $e_2 = [0:1:0]$. Note that f_1 is real fibered but f_2 is not. Let $P = [1:1:0] \in X$ (Weil divisor on X) and $\mathcal{L} = \mathcal{L}(P)$ the corresponding invertible subsheaf of $\mathcal{K}_X$. A basis of the space of global sections V of $\mathcal{L}(P)$ is given by the two rational functions $g_1 = 1$ and $g_2 = \frac{x_0 + x_1}{x_2}$. The ramification divisor R_i of f_i is given by $R_1 = Q_0 + \overline{Q_0}$ and $R_2 = Q_1 + Q_2$ with $Q_0 = [0:1:\mathrm{i}]$, $Q_1 = [1:0:1]$ and $Q_2 = [1:0:-1]$. Denoting by Δ_i the codifferent sheaf associated with f_i, we thus get that

$$(a, b) \mapsto a \cdot b \cdot \frac{x_0 - x_1}{x_0} \tag{7.1}$$

defines a symmetric bilinear form $\mathcal{L} \otimes \mathcal{L} \to \Delta_1$. Via the isomorphism $\Delta_1 \cong f_1^!\mathcal{O}_{\mathbb{P}^1}$, this induces an $\mathcal{O}_{\mathbb{P}^1}$-valued bilinear form on $(f_1)_*\mathcal{L}$. With respect to the above basis of V, it is given by the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

which is positive definite. Thus, $\mathcal{L}$ is a positive symmetric f_1-Ulrich sheaf. On the other hand, via the isomorphism $\Delta_1 \to \Delta_2$ that is given by multiplication with $\frac{x_0}{x_1}$, we get

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from (7.1) and $\Delta_2 \cong f_2^! \mathcal{O}_{\mathbb{P}^1}$ an $\mathcal{O}_{\mathbb{P}^1}$-valued bilinear form on $(f_2)_*\mathcal{L}$. With respect to the above basis of V, it is given by the matrix

$$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$

which is not positive semidefinite. Thus, although $f_1^! \mathcal{O}_{\mathbb{P}^1} \cong f_2^! \mathcal{O}_{\mathbb{P}^1}$ as abstract line bundles, for checking the positivity condition, we need to specify the morphism. Moreover, Proposition 9.2 will show that actually no nonzero symmetric $f_2^! \mathcal{O}_{\mathbb{P}^1}$-valued bilinear form on $\mathcal{L}$ will induce a positive semidefinite bilinear form on $(f_2)_*\mathcal{L}$ since f_2 is not real-fibered.

Since, later on, we will focus on irreducible varieties, we close this section with an example for the reducible case. A systematic study of positive Ulrich sheaves on reducible hypersurfaces would be very interesting with regard to the so-called *generalized Lax conjecture* (see [Vin12, Conjecture 3.3]). We think it would be particularly beneficial to understand how the main result of [Kum17] fits into our context here.

Example 7.8 Let $l = x_0 - x_1$ and $h = x_0^2 - (x_1^2 + x_2^2 + x_3^2)$. Let $X = X_1 \cup X_2$, where $X_1 = \mathcal{V}(l) \subset \mathbb{P}^3$ and $X_2 = \mathcal{V}(h) \subset \mathbb{P}^3$. The linear projection

$$f\colon X \to \mathbb{P}^2,\ [x_0\colon x_1\colon x_2\colon x_3] \to [x_1\colon x_2\colon x_3]$$

with center $e = [1\colon 0\colon 0\colon 0]$ is real-fibered, and we want to construct a positive symmetric f-Ulrich sheaf on X. The function field of $\mathbb{P}^2$ is $K = \mathbb{R}(\frac{x_2}{x_1}, \frac{x_3}{x_1})$ and the total quotient ring of X is $L = L_1 \times L_2$, where L_i is the function field of X_i for $i = 1, 2$. We note that $\mathcal{O}_X$ can be identified with the following subsheaf of $\mathcal{K}_X$:

$$\mathcal{O}_X(U) = \{(a, b) \in \mathcal{O}_{X_1}(U \cap X_1) \times \mathcal{O}_{X_2}(U \cap X_2) \mid (a - b)|_{X_1 \cap X_2 \cap U} = 0\}$$

for $U \subset X$. We further define the subsheaf $\mathcal{P}$ of $\mathcal{K}_X$ via

$$\mathcal{P}(U) = \{(a, b) \in \mathcal{O}_{X_1}(U \cap X_1) \times \mathcal{O}_{X_2}(U \cap X_2) \mid (a + b)|_{X_1 \cap X_2 \cap U} = 0\}$$

for $U \subset X$. Note that $\mathcal{P}$ is a line bundle on X and $\mathcal{P} \cdot \mathcal{P} = \mathcal{O}_X$. Finally, let

$$\mathcal{E}(U) = \{(a, b) \in \mathcal{O}_{X_1}(U \cap X_1) \times \mathcal{O}_{X_2}(U \cap X_2) :$$

$$\left(a - \frac{x_3}{x_2} b\right)|_{X_1 \cap X_2 \cap U \cap U_2} = 0 \text{ and}$$

$$\left(\frac{x_2}{x_3} a - b\right)|_{X_1 \cap X_2 \cap U \cap U_3} = 0\}.$$

The coherent sheaf $\mathcal{E}$ is defined in such a way that $\mathcal{E} \cdot \mathcal{E} \subset \mathcal{P}$.

Now, let $V_i \subset \mathbb{P}^2$ be the open affine set, where $x_i \neq 0$ and $U_i = f^{-1}(V_i)$ for $i = 1, 2, 3$. By Example 5.3, the codifferent sheaf $\Delta(X/\mathbb{P}^2)$ is the subsheaf of $\mathcal{K}_X$ generated by $\frac{x_i^2}{\mathrm{D}_e(l \cdot h)}$ on U_i for $i = 1, 2, 3$. As an element of $L_1 \times L_2$, this is

$$\left(-\frac{x_i^2}{x_2^2 + x_3^2}, \frac{x_i^2}{2 \cdot (x_1^2 + x_2^2 + x_3^2 - x_0 x_1)}\right)$$

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on U_i for $i = 1, 2, 3$. Let $\mathcal{L}$ be the subsheaf of $\mathcal{K}_X$ generated by $(\frac{x_i}{x_1}, \frac{x_i}{x_1})$ on U_i. Consider

$$s = \left(\frac{x_2^2 + x_3^2}{x_1^2}, \frac{2 \cdot (x_1^2 + x_2^2 + x_3^2 - x_0 x_1)}{x_1^2} \right)$$

which is nonnegative on $X(\mathbb{R})$. We have $\mathcal{L} \cdot \mathcal{L} \subset s \cdot \mathcal{P} \cdot \Delta(X/\mathbb{P}^2)$. Thus, we have $\mathcal{F} \cdot \mathcal{F} \subset s \cdot \Delta(X/\mathbb{P}^2)$, where $\mathcal{F} = \mathcal{E} \cdot \mathcal{L}$. This gives us an $f^! \mathcal{O}_{\mathbb{P}^2}$-valued symmetric bilinear form on $\mathcal{L}$ such that the induced $\mathcal{O}_{\mathbb{P}^2}$-valued bilinear form on $f_* \mathcal{F}$ is positive semidefinite by Theorem 7.5. Here are linearly independent global sections of $\mathcal{F}$:

$$(0, x_0 - x_1),\ (x_2, -x_3),\ (x_3, x_2).$$

Thus, $\mathcal{F}$ is a positive symmetric f-Ulrich sheaf.

8 Ulrich sheaves, determinantal representations, and sums of squares

The main reason why we are interested in Ulrich sheaves is because they correspond to certain determinantal representations. For the applications we consider in this article, we are interested in the following situation. Let $S = \mathbb{R}[y, x_1, \dots, x_n]$ be the polynomial ring with the grading determined by letting $\deg(y) = e$ and $\deg(x_i) = 1$ for $i = 1, \dots, n$, let $h \in S_{de}$ be an irreducible homogeneous element of degree $d \cdot e$, and let $X = \mathcal{V}(h) \subset \mathbb{P}(e, 1, \dots, 1)$ be the hypersurface in the weighted projective space corresponding to S. Assume $h(1, 0, \dots, 0) = 1$. The following proposition and the subsequent remark are well known among experts. But since we are not aware of a reference for the precise statement that we need, we include a proof for the sake of completeness. The case $e = 1$ is treated, for example, in [Bea00, Theorem A] and the proof for the general case works essentially in the same way.

Proposition 8.1 *Let $f: X \to \mathbb{P}^{n-1}$ be the projection on the last coordinates, and let $\mathcal{F}$ be an f-Ulrich sheaf on X with* $\operatorname{rank}(\mathcal{F}) = r$. *Then there is a square matrix A of size $d \cdot r$ whose entries are homogeneous polynomials in the x_i of degree e such that $h^r =$* $\det(y \cdot I - A)$. *If $\mathcal{F}$ is a positive symmetric (resp. Hermitian) f-Ulrich sheaf, then A can be chosen to be symmetric (resp. Hermitian).*

Proof Let $\mathcal{O}_X(1)$ be the pullback of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ via f, and let $M = \oplus_{i \in \mathbb{Z}} H^0(X, \mathcal{F}(i))$ be the module of twisted global sections. Since $\mathcal{F}$ is an f-Ulrich sheaf, we have that M considered as a module over $R = \mathbb{R}[x_1, \dots, x_n] \subset S$ is isomorphic to $R^{d \cdot r}$. Multiplication with y is an R-linear map, homogeneous of degree e, that can thus be represented by a square matrix A of size $d \cdot r$ whose entries are real homogeneous polynomials in the x_i of degree e and whose minimal polynomial is h. Thus, $h^r = \det(y \cdot I - A)$. Further, a symmetric positive semidefinite bilinear form as in Theorem 7.2 yields a homomorphism of graded S-modules (of degree zero)

$$\psi: M \to \operatorname{Hom}_R(M, R),$$

which has the property that the induced R-bilinear form

$$\begin{aligned} R^{d \cdot r} \times R^{d \cdot r} &\to R \\ (a, b) &\mapsto (\psi(a))(b) \end{aligned}$$

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is symmetric, i.e., $(\psi(a))(b) = (\psi(b))(a)$. The degree zero part V of $M \cong_R R^{d\cdot r}$ is the space of global sections of $\mathcal{F}$ and the restriction of this symmetric bilinear form to V is thus positive definite by Lemma 7.1. We can therefore choose an orthonormal basis of V with respect to this bilinear form. Note that this will also be a basis of the R-module $R^{d\cdot r}$ that is orthonormal with respect to ψ. Since ψ is a homomorphism of S-modules, we have $(\psi(a))(y\cdot b) = (\psi(y\cdot a))(b)$, so multiplication with y is self-adjoint with respect to our above defined symmetric bilinear form. Thus, we can choose the representing matrix A of the R-linear map given by multiplication with y to be symmetric. The Hermitian case follows analogously. ∎

Remark 8.2 The converse of Proposition 8.1 is also true: if $h^r = \det(y\cdot I - A)$, then the cokernel M of the map $S^{d\cdot r} \to S^{d\cdot r}$ given by $y\cdot I - A$ is supported on X and M considered as R-module is just $R^{d\cdot r}$. If A is symmetric, then an isomorphism of M with $\mathrm{Hom}_R(M,R)$ as S-modules is given by sending the standard basis of $R^{d\cdot r}$ to its dual basis. We argue analogously in the Hermitian case.

A refined statement is true for possibly reducible subvarieties $X \subset \mathbb{P}^n$ that are not necessarily hypersurfaces (see [KS20, Theorem 5.7]). These correspond to so-called determinantal representations of Livšic-type introduced in [SV18] which are closely related to determinantal representations of the Chow form of X (see also [ESW03]. The main applications of this article, however, concern irreducible varieties only.

Example 8.3 The positive symmetric f-Ulrich sheaf on the reducible cubic hypersurface $X = \mathcal{V}((x_0 - x_1)\cdot(x_0^2 - (x_1^2 + x_2^2 + x_3^2)) \subset \mathbb{P}^3$ from Example 7.8 gives the following symmetric definite determinantal representation:

$$\begin{pmatrix} x_0 + x_1 & x_3 & -x_2 \\ x_3 & x_0 - x_1 & 0 \\ -x_2 & 0 & x_0 - x_1 \end{pmatrix}.$$

Now we consider again the situation of Example 6.1, when h is of the form $y^2 - p(x)$, where $p \in \mathbb{R}[x_1,\dots,x_n]_{2e}$ is a globally nonnegative polynomial.

Lemma 8.4 *Let $p \in \mathbb{R}[x_1,\dots,x_n]_{2e}$ be a globally nonnegative polynomial which is not a square. Let $h = y^2 - p(x)$, where y has degree e. If $h^r = \det(y\cdot I - A)$ for some $r \geq 1$ and a symmetric or Hermitian matrix A of size $2r$ whose entries are homogeneous of degree e, then p is a sum of $2r$ squares in the symmetric case and a sum of $4r - 1$ squares in the Hermitian case.*

Proof Since p is not a square, we find that h is irreducible. Thus, $h^r = \det(y\cdot I - A)$ implies that h is the minimal polynomial of A, i.e., $A^2 = p\cdot I$. Letting a_i be the ith column of A, we then have $p = a_i^t a_i$ in the symmetric case and $p = a_i^t\overline{a_i}$ in the Hermitian case. Thus, p is a sum of $2r$ squares in the symmetric case and a sum of $4r - 1$ squares in the Hermitian case. ∎

A converse of this last conclusion is given by the following lemma.

Lemma 8.5 *Let $P = G_1^2 + \cdots + G_n^2$, where the $G_i \in A$ are elements of some commutative ring A. There is a symmetric square matrix Q of some size $m \in \mathbb{N}$ whose entries are $\mathbb{Z}$-linear combinations of the G_i such that $P\cdot I_m = Q^2$.*

Proof This follows from the basic properties of Clifford algebras. Recall that the Clifford algebra $Cl_{0,n}(\mathbb{R})$ is generated by $e_1, \dots, e_n$ satisfying $e_i \cdot e_j = -e_j \cdot e_i$ if $i \neq j$ and $e_i^2 = -1$. In particular, we have

$$x_1^2 + \cdots + x_n^2 = -(x_1 e_1 + \cdots + x_n e_n) \cdot (x_1 e_1 + \cdots + x_n e_n)$$

for all $x_k \in \mathbb{R}$. For $1 \leq i \leq n$, let A_i be the representing matrix of the map

$$Cl_{0,n}(\mathbb{R}) \to Cl_{0,n}(\mathbb{R}),\ a \mapsto e_i \cdot a$$

with respect to the basis $e_{i_1} \dots e_{i_r}$ of $Cl_{0,n}(\mathbb{R})$ with $1 \leq i_1 < \cdots < i_r \leq n$ and $r \geq 0$. Then one immediately verifies that A_i is a matrix having only entries in $\{0, \pm 1\}$ satisfying $A_i^T = -A_i$. It follows that

$$(x_1^2 + \cdots + x_n^2) \cdot I_N = S \cdot S^t$$

for all $x_k \in \mathbb{R}$, where $S = x_1 A_1 + \cdots + x_n A_n$ and N is the dimension of $Cl_{0,n}(\mathbb{R})$. Now we can choose

$$Q = \begin{pmatrix} 0 & S \\ S^t & 0 \end{pmatrix}$$

and we get $(x_1^2 + \cdots + x_n^2) \cdot I_N = Q^2$ for all $x_k \in \mathbb{R}$. Since the entries of the A_i are integers, the identity holds over every commutative ring. ∎

Putting all this together, we get the following connection of sums of squares to Ulrich sheaves.

Theorem 8.6 *Let $p \in \mathbb{R}[x_1, \dots, x_n]_{2e}$ be a homogeneous polynomial of degree $2e$ which is not a square. Inside the weighted projective space $\mathbb{P}(e, 1, \dots, 1)$, we consider the hypersurface X defined by $y^2 = p(x_0, \dots, x_n)$ and the natural projection $\pi\colon X \to \mathbb{P}^n$ onto the x-coordinates. Then p is a sum of squares of polynomials if and only if there is a positive symmetric (or Hermitian) π-Ulrich sheaf $\mathcal{F}$ on X. In that case, if $\operatorname{rank}(\mathcal{F}) = r$ then p is a sum of $2r$ squares in the symmetric case and a sum of $4r - 1$ squares in the Hermitian case.*

Proof First assume that p is a sum of squares. By Lemma 8.5, there is a symmetric square matrix A of some size $m \in \mathbb{N}$ whose entries are homogeneous of degree e in the variables $x_1, \dots, x_n$ such that $p \cdot I_m = A^2$. Because p is not a square, the polynomial $h = y^2 - p$ is irreducible. Thus, $A^2 - p \cdot I_m = 0$ shows that h is the minimal polynomial of A. This implies that $h^r = \det(y \cdot I - A)$ for some $r > 0$. Thus, there is a positive symmetric π-Ulrich sheaf on X by Remark 8.2.

Now assume that there is a positive symmetric (or Hermitian) π-Ulrich sheaf $\mathcal{F}$ on X. Then, by Proposition 8.1, there is a symmetric (resp. Hermitian) matrix A of size $2 \cdot r$ whose entries are homogeneous polynomials in the x_i of degree e such that $h^r = \det(y \cdot I - A)$. Now the claim follows from Lemma 8.4. ∎

As a special case, we get the following result by Netzer and Thom [NT12].

Corollary 8.7 *Let $h \in \mathbb{R}[x_0, \dots, x_n]_2$ be a quadratic hyperbolic polynomial. Then h^r has a definite symmetric determinantal representation for some $r > 0$.*

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Proof If $X = \mathcal{V}(h)$ is hyperbolic with respect to $e \in \mathbb{P}^n$, then the linear projection $\pi_e: X \to \mathbb{P}^{n-1}$ is a real-fibered double cover ramified along the zero set of a nonnegative quadratic polynomial p. Since p is a sum of squares, there is a positive symmetric π_e-Ulrich sheaf on X by Theorem 8.6. Proposition 8.1 implies the claim. ∎

Remark 8.8 Positive Ulrich sheaves on *reciprocal linear spaces*, i.e., the closure of the image of a linear space under coordinatewise inversion, were used in [KV19] to prove that a certain polynomial associated with a hyperplane arrangement, called the *entropic discriminant*, is a sum of squares. The relation of Ulrich sheaves and sums of squares used in [KV19] is a generalization of one direction of Theorem 8.6. Namely, let $f: X \to \mathbb{P}^n$ be a finite surjective real-fibered morphism such that $f_*\mathcal{O}_X$ is a sum of line bundles. Then, if there is a positive f-Ulrich sheaf, then the polynomial defining the branch locus of f is a sum of squares [KV19, Theorem 6.1].

9 Positive Ulrich sheaves of rank one on irreducible varieties

In this section, let $f: X \to Y$ always denote a finite surjective morphism of geometrically irreducible varieties which are proper over $\mathbb{R}$. We assume that Y is smooth and has a real point. We further assume that the singular locus of X has codimension at least two. This allows us to speak about Weil divisors on X. For a given Weil divisor D on X, we denote by $\mathcal{L}(D)$ the subsheaf of $\mathcal{K}_X$ consisting of all rational functions with pole and zero orders prescribed by D. We further denote $\ell(D) = \dim(\Gamma(X, \mathcal{L}(D)))$. Let $Z \subset Y$ be a closed subset of codimension at least two such that $f^{-1}(Z)$ contains X_{sing}. Letting $V = Y \setminus Z$ and $U = f^{-1}(V)$, the restriction $f|_U: U \to V$ is a finite surjective morphism of smooth irreducible varieties. Thus, $\Delta(U/V)$ is invertible by Proposition 5.7 and thus corresponds to a Weil divisor R on U. Since $X \setminus U$ has codimension at least two, we can also consider R as a Weil divisor on X which we call the ramification divisor of f. Lemma 5.8 shows that the associated subsheaf $\mathcal{L}(R)$ of $\mathcal{K}_X$ is exactly $\Delta(X/Y)$. All this holds true for the complexification $X_{\mathbb{C}}$ as well. Complex conjugation gives an involution $\sigma: X_{\mathbb{C}} \to X_{\mathbb{C}}$ and for a Weil divisor D on $X_{\mathbb{C}}$ we denote $\sigma(D)$ by $\overline{D}$. We have $\mathcal{L}(\overline{D}) = \overline{\mathcal{L}(D)}$ as subsheaves of $\mathcal{K}_{X_{\mathbb{C}}}$.

Remark 9.1 If D_1 and D_2 are Weil divisors on X, then we have

$$\mathcal{L}(D_1) \cdot \mathcal{L}(D_2) \subset \mathcal{L}(D_1 + D_2)$$

considered as subsheaves of $\mathcal{K}_X$. But since $\mathcal{L}(D_i)$ is not necessarily an invertible sheaf, we do not have equality in general.

The relative notion of positive semidefiniteness introduced in Section 7 relates to the notion of being real fibered in the following way.

Proposition 9.2 [KS20, Theorem 5.11] *The following are equivalent:*

(i) *There is a coherent sheaf $\mathcal{F}$ on X with* $\operatorname{Supp}(\mathcal{F}) = X$ *and a symmetric nonzero $f^!\mathcal{O}_Y$-valued bilinear form on $\mathcal{F}$ such that the induced bilinear form on $f_*\mathcal{F}$ is positive semidefinite.*

(ii) *f is real fibered, i.e., $f^{-1}(Y(\mathbb{R})) = X(\mathbb{R})$.*

In this situation, we get the following convenient criterion for a Weil divisor to give rise to a positive Ulrich sheaf.

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Theorem 9.3 *Let $f: X \to Y$ be a real-fibered finite surjective morphism of geometrically irreducible varieties which are proper over $\mathbb{R}$. Let Y be smooth and have a real point. Further, let the singular locus of X have codimension at least two. Let R be the ramification divisor of f, and let s be a rational function on X which is nonnegative on $X(\mathbb{R})$.*

(1) *If D is a Weil divisor on X such that $2D + (s) = R$ and $\ell(D) \geq \deg(f)$, then the sheaf $\mathcal{L}(D)$ on X is a positive symmetric f-Ulrich sheaf.*
(2) *If D is a Weil divisor on $X_{\mathbb{C}}$ that satisfies $D + \overline{D} + (s) = R$ and $\ell(D) \geq \deg(f)$, then the sheaf $\mathcal{L}(D)$ on $X_{\mathbb{C}}$ is a positive Hermitian f-Ulrich sheaf.*

Proof Let $\mathcal{L} = \mathcal{L}(D)$ be the subsheaf of $\mathcal{K}_X$ corresponding to D. By assumption, we can define the symmetric (resp. Hermitian) $\Delta(X/Y)$-valued bilinear form on $\mathcal{L}$ that maps a pair of sections (g, h) to the product $s \cdot g \cdot h$. This is clearly nondegenerate at the generic point of X and it is positive semidefinite by Proposition 6.4. Then the claim follows from Theorems 7.2 and 7.4, respectively. ∎

Remark 9.4 Let D be a Weil divisor on X or $X_{\mathbb{C}}$ such that $2D$ or $D + \overline{D}$, respectively, is linearly equivalent to R, i.e., they differ only by a principal divisor (g). The signs that g takes on real points of X (up to global scaling) do only depend on the divisor class of D. Indeed, if $D' = D + (f)$ for some rational function f, then $2D'$ resp. $D' + \overline{D'}$ differs from R by $g \cdot f^2$ or $g \cdot f\overline{f}$, respectively.

Example 9.5 Let $L \subset \mathbb{P}^n$ be a linear subspace of dimension $d < n$ that is not contained in any coordinate hyperplane. We denote by L^{-1} its *reciprocal*, i.e., the (Zariski closure of the) image of L under the rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined by coordinatewise inversion. It was shown by Varchenko [Var95] that L^{-1} is hyperbolic with respect to the orthogonal complement $L^{\perp}$ of L. Further, it was shown in [KV19] that there is a symmetric positive f-Ulrich sheaf of rank one on L^{-1}, where $f: X \to \mathbb{P}^d$ is the linear projection from $L^{\perp}$. We want to outline how this also follows from Theorem 9.3, at least for generic L. To this end, let L be the row span of a matrix $A = (a_{ij})$ of size $(d+1) \times (n+1)$ and assume that every maximal minor of A is nonzero. Letting $l_j = \sum_{i=1}^{d+1} a_{ij} x_i$, for $j = 1, \ldots, n+1$, we can describe L^{-1} as the image of the rational map

$$\psi: \mathbb{P}^d \dashrightarrow \mathbb{P}^n,\ x \mapsto \left(\frac{l_1 \cdots l_{n+1}}{l_1} : \cdots : \frac{l_1 \cdots l_{n+1}}{l_{n+1}} \right).$$

Note that ψ is defined in all points where at most one of the l_j vanishes. It follows from the proof of [SSV13, Corollary 5] and [SSV13, Remark 31] that the ramification divisor R of f on L^{-1} is the proper transform under ψ of the zero set $Z \subset \mathbb{P}^d$ of

$$P = \sum_I \det(A_I)^2 \prod_{j \in I} l_j^2,$$

where the sum is taken over all $I \subset \{1, \ldots, n+1\}$ of size $n - d$ and A_I denotes the submatrix of A obtained from erasing all columns indexed by I. Let $H \subset \mathbb{P}^d$ be a generic hyperplane defined by a linear form G, and let D be the divisor on L^{-1} defined as the proper transform of H under ψ. On $\mathbb{P}^d$, we have $2(n-d)H = Z + \left(\frac{P}{G^{2(n-d)}}\right)$ and our genericity assumption on A implies that P does not vanish entirely on any of the $\mathcal{V}(l_i, l_j) \subset \mathbb{P}^d$ which comprise the locus where ψ is not regular. Therefore, we have

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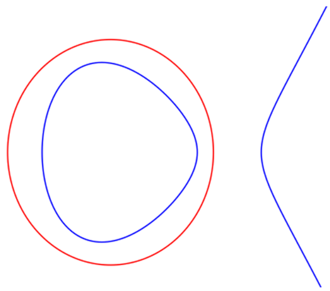

Figure 2: A cubic hyperbolic plane curve (blue) interlaced by a plane hyperbolic conic (red).

$2(n-d)D = R + \left(\frac{P}{G^{2(n-d)}}\right)$ as divisors on L^{-1}. Clearly, the rational function $\frac{P}{G^{2(n-d)}}$ is nonnegative and $\ell((n-d)\cdot D)$ equals $\binom{n}{d}$, the number of monomials of degree $n-d$ in $d+1$ variables. Since this is also the degree of L^{-1} [PS06], Theorem 9.3(1) implies that $\mathcal{L}((n-d)\cdot D)$ is a positive symmetric f-Ulrich sheaf on L^{-1}.

The previous example leads to the following question. Let $X \subset \mathbb{P}^n$ be an irreducible variety not contained in any coordinate hyperplane which is hyperbolic with respect to every linear subspace of codimension $\dim(X)+1$ all of whose Plücker coordinates are positive. Then the image X^{-1} of X under coordinatewise inversion is hyperbolic with respect to all these subspaces as well [KV19, Proposition 1.4].

Problem 9.6 *Given a symmetric positive Ulrich bundle on a variety $X \subset \mathbb{P}^n$ as above. Does there exist one on X^{-1} of the same rank as well?*

Remark 9.7 Using [Kum13, Proposition 3.3.8] one can show that the answer to Problem 9.6 is *yes* for hypersurfaces. It is also true for X a linear subspace by [KV19].

In the case of hypersurfaces, Theorem 9.3 has a geometric interpretation in terms of so-called interlacers.

Definition 9.1 Let $g, h \in \mathbb{R}[x_0, \dots, x_n]$ with $d = \deg(g) + 1 = \deg(h)$ be hyperbolic with respect to e. If, for all $v \in \mathbb{R}^{n+1}$, we have that

$$a_1 \le b_1 \le a_2 \le \cdots \le b_{d-1} \le a_d,$$

where the a_i and b_i are the zeros of $h(te-v)$ and $g(te-v)$, respectively, we say that g *interlaces* h, or that g is an *interlacer* of h. This definition carries over to hyperbolic hypersurfaces in the obvious way.

Example 9.8 If $h \in \mathbb{R}[x_0, \dots, x_n]$ is hyperbolic with respect to e, then the directional derivative $\mathrm{D}_e h$ of h in direction e is an interlacer of h. This follows from Rolle's theorem.

Corollary 9.9 *Let $h \in \mathbb{R}[x_0, \dots, x_n]_d$ be hyperbolic with respect to e, and let $X = \mathcal{V}(h) \subset \mathbb{P}^n$ be the corresponding hypersurface. Assume that the singular locus of X has dimension at most $n-3$. Let g be an interlacer of h and denote by G the Weil divisor it defines on X.*

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(1) *Assume that $G = 2D$ for some Weil divisor D on X. If the vector space of all $p \in \mathbb{R}[x_0, \dots, x_n]_{d-1}$ that vanish on D has dimension at least d, then h has a definite symmetric determinantal representation.*
(2) *Assume that $G = D + \overline{D}$ for some Weil divisor D on $X_{\mathbb{C}}$. If the vector space of all $p \in \mathbb{C}[x_0, \dots, x_n]_{d-1}$ that vanish on D has dimension at least d, then h has a definite Hermitian determinantal representation.*

Proof The ramification divisor of the linear projection with center e is the zero set of the directional derivative $\mathrm{D}_e h$ on X. Since g is an interlacer, we have that the rational function $\frac{\mathrm{D}_e h}{g}$ is nonnegative on $X(\mathbb{R})$ by [KPV15, Lemma 2.4]. Then the claim follows from Theorem 9.3 and Proposition 8.1 by our dimensional assumption. ∎

Remark 9.10 When $n = 2$, the case of plane curves, the dimensional condition in Corollary 9.9 is automatically satisfied by Riemann–Roch. If $g = \mathrm{D}_e h$, then $G = D + \overline{D}$ by Theorem 6.3. In that case, this recovers the result from [PV13, Section 4]. Note that the proof in [PV13] does not seem to generalize to higher dimensions as it uses Max Noether's AF+BG theorem. It would be interesting to know when we actually need the dimensional condition in the case of higher-dimensional hypersurfaces.

Problem 9.11 *Find hyperbolic hypersurfaces $X \subset \mathbb{P}^n$ for which there is an interlacer whose zero divisor on X is of the form $D + \overline{D}$ with $\ell(D) < \deg(X)$.*

Example 9.12 Let $h \in \mathbb{R}[x_0, \dots, x_n]_n$ be the elementary symmetric polynomial of degree n in $x_0, \dots, x_n$. It is hyperbolic with respect to every point in the positive orthant and $g = \frac{\partial}{\partial x_0} h$ is an interlacer. The zero divisor of g on the hypersurface $X = \mathcal{V}(h) \subset \mathbb{P}^n$ is of the form $2D$, where $D = \sum_{1 \le i < j \le n} L_{ij}$. Here, L_{ij} is the linear subspace $\mathcal{V}(x_i, x_j) \subset X$. Each monomial $\frac{x_1 \cdots x_n}{x_i}$ for $1 \le i \le n$ vanishes on D. Thus, by Corollary 9.9, the polynomial h has a definite symmetric determinantal representation. This was known before and follows, e.g., from the more general result that the bases generating polynomial of a regular matroid (in our case $U_{n,n+1}$) has a definite symmetric determinantal representation (see [COSW04, Section 8.2]).

10 Smooth curves

In this section, let $f \colon X \to Y$ be a real-fibered morphism between smooth irreducible curves that are projective over $\mathbb{R}$ with $Y(\mathbb{R})$ Zariski dense in Y. Let R be the ramification divisor of f. We want to apply Theorem 9.3.

Lemma 10.1 *There is a divisor M on X and a nonnegative s in the function field of X such that $R + (s) = 2M$.*

Proof The proof is a projective version of the proof of [Han17, Corollary 4.2]. By Theorem 6.3, we have that f is unramified at real points. Thus R, considered as a Weil divisor, is a sum of nonreal points. Therefore, the Weil divisor R' that we obtain on the complexification $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ of X is of the form $R' = Q + \overline{Q}$, where Q is some effective divisor and $\overline{Q}$ its complex conjugate. Since the group $\mathrm{Pic}^0(X_{\mathbb{C}})$ is divisible,

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there is a g in the function field of $X_{\mathbb{C}}$ and a divisor N on $X_{\mathbb{C}}$ such that $Q - nP = 2N + (g)$, where $n = \deg(Q)$ and P is any point on $X_{\mathbb{C}}$ with $P = \overline{P}$. Thus $(Q - nP) + (\overline{Q - nP}) = 2(N + \overline{N}) + (g \cdot \overline{g})$ which implies

$$R' = Q + \overline{Q} = 2(N + \overline{N} + nP) + (g \cdot \overline{g}).$$

Since $N + \overline{N} + nP$ is fixed by conjugation, it descends to a divisor M on X. The function $s = g \cdot \overline{g}$ is a sum of two squares and thus nonnegative. ∎

From this, we get the following theorem.

Theorem 10.2 *Let X be a smooth irreducible curve that is projective over $\mathbb{R}$. For every real fibered $f\colon X \to \mathbb{P}^1$, there is a positive symmetric f-Ulrich line bundle.*

Proof Let M be the divisor from Lemma 10.1. By Theorem 9.3, we have to show that $\ell(M) \geq \deg(f)$. By Hurwitz's Theorem, we have that $2g - 2 = \deg(R) - 2\deg(f)$, where g is the genus of X. Thus $\deg(M) = \deg(f) + g - 1$ and by Riemann–Roch

$$\ell(M) \geq \deg(M) - g + 1 = \deg(f).$$ ∎

Corollary 10.3 [SV18, Theorem 7.2] *The Chow form of every hyperbolic curve $X \subset \mathbb{P}^n$ has a symmetric and definite determinantal representation.*

Proof By [KS20, Theorem 5.7] and [KS20, Remark 4.4], it suffices to show that there is a positive Ulrich bundle of rank one on X. But this follows from the preceding theorem applied to the linear projection from an $n-2$-space of hyperbolicity if X is smooth. Otherwise we can pass to the normalization of X. ∎

Corollary 10.4 (Helton–Vinnikov Theorem [HV07]) *Every hyperbolic polynomial in three variables has a definite determinantal representation.*

Example 10.5 If the target is not $\mathbb{P}^1$ as in Theorem 10.2, then there is, in general, no (positive symmetric) f-Ulrich sheaf on X. This fails already in the next easiest case, namely, when X and Y both have genus one. Indeed, let $f\colon X \to Y$ be an unramified and real-fibered double cover of an elliptic curve Y. Such maps exist, see, for example, [KS22, Lemma 6.5], and by Riemann–Hurwitz X is an elliptic curve as well. We claim that in this case there is actually no f-Ulrich sheaf at all. Indeed, there is a line bundle $\mathcal{L}$ on Y which is nontrivial and 2-torsion such that $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L}$. Then we have $f^*\mathcal{L} = \mathcal{O}_X$ and the projection formula implies that $f_*\mathcal{F} = \mathcal{L} \otimes f_*\mathcal{F}$ for all coherent sheaves $\mathcal{F}$ on X. This excludes $f_*\mathcal{F} = \mathcal{O}_Y^r$.

11 del Pezzo surfaces

Recall that a *del Pezzo surface* is a smooth projective and geometrically irreducible surface whose anticanonical class is ample. We are interested in morphisms $f\colon X \to \mathbb{P}^2$, where X is a del Pezzo surface and the pullback $f^*\mathcal{O}_{\mathbb{P}^2}(1)$ is the anticanonical line bundle. It was shown in [Bea18, Proposition 4.1] that for such f there exist f-Ulrich line bundles. We will show that if f is real fibered, then there are even positive Hermitian f-Ulrich line bundles. An introduction to the classical theory of del Pezzo surfaces can be found, for example, in [Dol12, Chapter 8] or [KSC04, Section 3.5]. This section further relies on the classification of real del Pezzo surfaces due to Comessatti [Com12, Com28] (see also [Rus02] for a survey in English).

Definition 11.1 The *degree* of a del Pezzo surface X is the self-intersection number $K_X.K_X$ of its canonical class K_X. A *line* on X is an irreducible curve $L \subset X$ such that $L.L = L.K_X = -1$.

Remark 11.1 Note that if the anticanonical class $-K_X$ of a del Pezzo surface X is very ample, then a line L on X is mapped by the associated embedding to an actual line, i.e., a linear subspace of dimension one because $L.K_X = -1$.

Example 11.2 These are examples of del Pezzo surfaces [KSC04, Theorem 3.36(7)]:

(1) A smooth hypersurface of degree four in the weighted projective space $\mathbb{P}(2,1,1,1)$ is a del Pezzo surface of degree two.
(2) A smooth cubic hypersurface in $\mathbb{P}^3$ is a del Pezzo surface of degree three.
(3) A smooth complete intersection of two quadrics in $\mathbb{P}^4$ is a del Pezzo surface of degree four.

Furthermore, in case (1), the (complete) anticanonical linear system corresponds to the projection $\mathbb{P}(2,1,1,1) \dashrightarrow \mathbb{P}^2$ restricted to our surface. Moreover, in the cases (2) and (3), the embeddings of the surfaces to $\mathbb{P}^3$ and $\mathbb{P}^4$, respectively, correspond to the (complete) anticanonical linear system. These statements are, for example, shown in the course of the proof of [KSC04, Theorem 3.36].

Remark 11.3 If X is a del Pezzo surface over an algebraically closed field, then X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blowup of $\mathbb{P}^2$ in $n \leq 8$ points (see [KSC04, Exercise 3.56] or [Dol12, Corollary 8.1.17 and Proposition 8.1.25]. A straightforward calculation shows that the degree of X is eight in the former and $9 - n$ in the latter case.

Lemma 11.4 *Let $X \subset \mathbb{P}^4$ be a smooth complete intersection of two quadrics in $\mathbb{P}^4$ such that $X(\mathbb{R})$ is homeomorphic to the disjoint union of two spheres. Then:*

(1) *X is contained in exactly five real singular quadrics.*
(2) *One of these five quadrics has signature $(2,2)$, and the other four have signature $(3,1)$.*
(3) *For exactly two of these singular quadrics, the linear projection from its vertex realizes X as a real-fibered double cover of a hyperbolic quadratic hypersurface $Q \subset \mathbb{P}^3$.*

Proof The complex pencil $\lambda q_0 + \mu q_1$ contains five singular quadrics as the q_i can be represented by symmetric 5×5 matrices. We will show that all of them are real. To this end, recall that by Remark 11.3, the complexification $X_\mathbb{C}$ of X is isomorphic to the blowup of $\mathbb{P}^2$ at five points $p_0, \dots, p_4$. The 16 lines on $X_\mathbb{C}$ correspond to the exceptional divisors $E_0, \dots, E_4$, the lines l_{ij} through p_i and p_j for $0 \leq i < j \leq 4$ and the conic C through all five points $p_0, \dots, p_4$ (see, for example, [Man86, Theorem 26.2]). After relabeling, if necessary, the complex conjugation on $X_\mathbb{C}$ interchanges E_0 with C, E_i with l_{0i} for $1 \leq i \leq 4$ and l_{ij} with l_{kl} for $\{i,j,k,l\} = \{1,2,3,4\}$ and $i < j$, $k < l$ (see [Rus02, Example 2, case $n = 3$]). We write

$$A_1 = E_0 + C, A_2 = l_{12} + l_{34}, A_3 = l_{13} + l_{24}, A_4 = l_{14} + l_{23}$$

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(divisors on X) and note that all A_i belong to the same linear system. Similarly, the divisors $B_i = E_i + l_{0i}$ for $1 \le i \le 4$ on X are also linearly equivalent to each other. Note that we did write $A_i = L_i + \overline{L_i}$ and $B_i = L_i' + \overline{L_i'}$ for suitable lines L_i and L_i' on $X_{\mathbb{C}}$. Each of the two linear systems realize X as a conic bundle, i.e., define a morphism $X \to \mathbb{P}^1$ all of whose fibers are isomorphic to a plane conic curve [Rus02, Example 2]. The four singular fibers of each bundle are exactly the A_i and B_i, respectively. Therefore, for each connected component S of $X(\mathbb{R})$, there are exactly two values for i and j such that $L_i \cap \overline{L_i}$ resp. $L_j' \cap \overline{L_j'}$ is a point on S. Our two conic bundle structures on X induce a map $X \to \mathbb{P}^1 \times \mathbb{P}^1$ which is a double cover since $A_i \cdot B_j = 2$. Since $A_i + B_j$ is an anticanonical divisor, this double cover is a linear projection of X to $\mathbb{P}^3$ whose image is a hypersurface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., defined by a quadric with signature $(2,2)$. The cone over this quadric in $\mathbb{P}^4$ is one of our singular quadrics. The four divisors

$$D_j = L_1 + \overline{L_j'} = E_0 + l_{0j}$$

on $X_{\mathbb{C}}$ for $1 \le j \le 4$ also realize $X_{\mathbb{C}}$ as a conic bundle $X_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ and, for each j, we consider the map $X_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ associated with D_j in the first coordinate and $\overline{D_j}$ on the second. This corresponds to a morphism $f_j\colon X \to Q$, where $Q \subset \mathbb{P}^3$ is a hypersurface defined by a quadric of signature $(3,1)$. Since $D_j \cdot \overline{D_j} = 2$, this is a double cover, and since $D_j + \overline{D_j}$ is anticanonical, the maps f_j correspond to linear projections of X. This shows (2) and (3). We have

$$D_j \cdot \overline{D_j} = L_1 \cdot \overline{L_1} + L_j' \cdot \overline{L_j'}.$$

Thus, in order to determine whether $f_i\colon X \to Q$ is real-fibered or not, we have to check whether the two intersection points $L_1 \cap \overline{L_1}$ and $L_i' \cap \overline{L_i'}$ lie on the same (not real fibered) or different connected components (real fibered) of $X(\mathbb{R})$. As noted above, both cases occur for exactly two values of j. ∎

With this preparation, we are able to determine for which del Pezzo surfaces X there is a real-fibered morphism $X \to \mathbb{P}^2$ whose corresponding linear system is anticanonical.

Proposition 11.5 *Let X be a real del Pezzo surface, and let K be a canonical divisor on X. There is a real-fibered morphism $f\colon X \to \mathbb{P}^2$ such that the pullback of a line is linearly equivalent to $-K$ on X if and only if we have one of the following:*

(1) *X is a quartic surface in $\mathbb{P}^4$ such that $X(\mathbb{R})$ is homeomorphic to a disjoint union of two spheres.*
(2) *X is a cubic hypersurface in $\mathbb{P}^3$ such that $X(\mathbb{R})$ is homeomorphic to a disjoint union of a sphere and a real projective plane.*
(3) *X is a double cover of $\mathbb{P}^2$ branched along a smooth plane quartic curve C with $C(\mathbb{R}) = \varnothing$ so that $X(\mathbb{R})$ is homeomorphic to a disjoint union of two real projective planes.*

In particular, in each case, $X(\mathbb{R})$ has two connected components.

Proof Let $d = K.K$. For $d = 2$, the anticanonical map is a double cover of $\mathbb{P}^2$ branched along a plane quartic curve C (see Example 11.2). This is real fibered if and only if $C(\mathbb{R}) = \varnothing$ and $X(\mathbb{R})$ is homeomorphic to a disjoint union of two real projective planes. In general, if there exists such a morphism f, then $X(\mathbb{R})$ must be homeomorphic to the disjoint union of s spheres and r real projective planes such that $d = 2s + r$ by [KS20, Corollary 2.20]. Going through the classification of real del Pezzo surfaces in [Rus02], we see that for $d \neq 2$ this is only possible for a complete intersection of two quadrics in $\mathbb{P}^4$ such that $X(\mathbb{R})$ is homeomorphic to a disjoint union of two spheres ($d = 4$) or a cubic hypersurface in $\mathbb{P}^3$ such that $X(\mathbb{R})$ is homeomorphic to a disjoint union of a sphere and a real projective plane ($d = 3$). This shows the "only if part." It thus remains to show that the embedded surfaces in (1) and (2) are hyperbolic as these embeddings correspond to the anticanonical linear system (see Example 11.2). The case $d = 3$ is covered by [Vin12, Proposition 2.2]. For the case $d = 4$, we can compose a real-fibered linear projection $X \to Q$ from Lemma 11.4 to a hyperbolic hypersurface $Q \subset \mathbb{P}^3$ with the linear projection $Q \to \mathbb{P}^2$ from a point with respect to which Q is hyperbolic. ■

Lemma 11.6 *Let X be a del Pezzo surface, and let K be a canonical divisor on X. Let $f: X \to \mathbb{P}^2$ be a real-fibered morphism of degree d such that the pullback of a line is linearly equivalent to $-K$ as constructed in the previous proposition.*

(a) *The ramification divisor R is linearly equivalent to $-2K$.*
(b) *Let g be a rational function on X whose principal divisor is $R + 2K$. Then g has constant sign on each of the two connected components of $X(\mathbb{R})$ and these signs are not the same.*

Proof The ramification divisor is linearly equivalent to the canonical divisor on X minus the pullback of the canonical divisor on $\mathbb{P}^2$. As the latter is $\mathcal{O}_{\mathbb{P}^2}(-3)$ its pullback is $3K$. This shows (a).

For $d = 2$, part (b) follows from the fact X is a double cover of $\mathbb{P}^2$ of the form $y^2 = p(x)$, where $p \in \mathbb{R}[x_0, x_1, x_2]_4$ is a globally positive quartic curve. The ramification divisor is given as the zero locus of y, whereas $-K$ is the zero set of a linear form $l \in \mathbb{R}[x_0, x_1, x_2]$. Clearly, $\frac{y}{l^2}$ has the desired properties.

In the case $d = 3$, our surface X is the zero set of a hyperbolic polynomial h and our morphism f is the linear projection from a point $e \in \mathbb{P}^3$ of hyperbolicity. Its ramification divisor is thus cut out by the interlacer $\mathrm{D}_e h$. Again $-K$ is the zero set of a linear form $l \in \mathbb{R}[x_0, x_1, x_2]$ and $\frac{\mathrm{D}_e h}{l^2}$ has different sign on the two connected components of $X(\mathbb{R})$.

In the case $d = 4$, we first note that by Lemma 11.4, we can assume (after a linear change of coordinates) that X is cut out by $q_1 = x_0^2 - (x_1^2 + x_2^2 + x_3^2)$ and $q_2 = x_4^2 - p$ for some quadratic form $p \in \mathbb{R}[x_0, x_1, x_2, x_3]_2$ that is nonnegative on $Q = \mathcal{V}(q_1) \subset \mathbb{P}^3$. Our real-fibered morphism $f: X \to \mathbb{P}^2$ is then the composition of the two linear projections $X \to Q$ with center $[0:0:0:0:1]$ and $Q \to \mathbb{P}^2$ with center $[1:0:0:0]$. Thus the ramification locus of f is cut out by $x_0 \cdot x_4$. Again $-K$ is the zero set of a linear form $l \in \mathbb{R}[x_0, x_1, x_2]$ and $\frac{x_0 \cdot x_4}{l^2}$ has different sign on the two connected components of $X(\mathbb{R})$. ■

Let X_4 be a complete intersection of two quadrics in $\mathbb{P}^4$ such that $X(\mathbb{R})$ is homeomorphic to a disjoint union of two spheres. We fix a sequence of morphisms

$$X_2 \to X_3 \to X_4, \tag{11.1}$$

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where each map $f_i: X_i \to X_{i+1}$ is the blowup of X_i at a real point on a connected component of $X_i(\mathbb{R})$ that is homeomorphic to a sphere. Further, let E_i be the exceptional divisor of f_i. By the classification of real del Pezzo surfaces, we have that X_3 is a cubic hypersurface in $\mathbb{P}^3$ such that $X(\mathbb{R})$ is homeomorphic to a disjoint union of a sphere and a real projective plane and X_2 is a double cover of $\mathbb{P}^2$ branched along a smooth plane quartic curve C with $C(\mathbb{R}) = \varnothing$ so that $X(\mathbb{R})$ is homeomorphic to a disjoint union of two real projective planes. Conversely, every such real del Pezzo surface fits in such a sequence of blowups.

Lemma 11.7 *Consider the cubic hypersurface $X_3 \subset \mathbb{P}^3$.*

(a) *In addition to E_3, there are two more real lines L and L' on X_3. These three lines lie on a common plane.*
(b) *There are two different hyperplanes that contain L and are tangential to a real point on the connected component of $X_3(\mathbb{R})$ that is homeomorphic to the sphere.*
(c) *The divisors H_1 and H_2 on X_3 that are defined as the intersections with the hyperplanes from part (b) are of the form $H_i = L + L_i + \overline{L_i}$ for some nonreal lines L_i on $(X_3)_{\mathbb{C}}$.*
(d) *The lines L_i and $\overline{L_i}$ are disjoint from E_3. Furthermore, L_1 and L_2 are disjoint.*
(e) *Let f be a rational function on X_3 whose principal divisor is $L_2 + \overline{L_2} - L_1 - \overline{L_1}$. Then f has constant sign on each of the two connected components of $X(\mathbb{R})$ and these signs are not the same.*

Proof The number of real lines on X_3 can be found, for example, in [Rus02, p. 302]. As they all must lie in the component of $X(\mathbb{R})$ that is homeomorphic to $\mathbb{RP}^2$, each two of them intersect in a point. Thus they all lie in a common plane H_0 which proves (a).

In the affine chart $\mathbb{R}^3 = (\mathbb{P}^3 \setminus H_0)(\mathbb{R})$, the connected component of $X_3(\mathbb{R})$ that is homeomorphic to a sphere is the boundary of a compact convex set $K \subset \mathbb{R}^3$, namely, K is an affine slice of the hyperbolicity cone of the cubic that defines X_3. The hyperplanes containing L correspond to a family of parallel affine hyperplanes in $\mathbb{R}^3$. Thus exactly two of them are tangent to K. This shows (b).

The zero divisors of these hyperplanes H_i contain besides L a plane conic which has an isolated real point, namely, the point of tangency. Thus the conic is a complex conjugate pair of lines L_i and $\overline{L_i}$ which shows part (c).

In order to show (d), assume for the sake of a contradiction that L_i intersects E_3. Since $L_i \subset H_i$ and $E_3 \subset H_0$, this intersection point must lie on $E_3 \cap H_0 \cap H_i = E_3 \cap L$ which implies that it is real. But the only real point of L_i lies on the spherical component of $X_3(\mathbb{R})$. An analogous argument shows that L_1 and L_2 are disjoint.

Finally, let l_i be the linear form that cuts out H_i. Then, by construction, $p = l_1 l_2$ is an interlacer of the polynomial defining X_3. Thus the rational function $f = \frac{p}{l_1^2}$ has constant sign on each of the two connected components of $X(\mathbb{R})$ and these signs are not the same. Clearly the principal divisor corresponding to f is $L_2 + \overline{L_2} - L_1 - \overline{L_1}$. Therefore, we have shown part (d). ■

Theorem 11.8 *Let X be a del Pezzo surface, and let $f: X \to \mathbb{P}^2$ be a real-fibered morphism of degree d such that the pullback of a line is the anticanonical divisor class $-K$. Then there is a positive Hermitian f-Ulrich line bundle.*

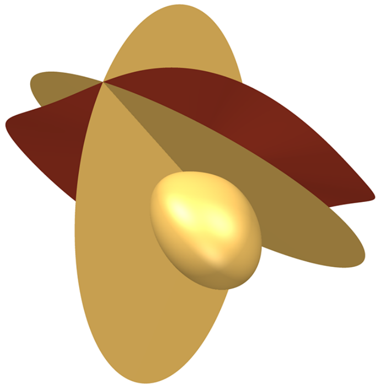

Figure 3: A cubic hyperbolic hypersurface with two planes that contain a line on the pseudo-plane (red) and are tangent to the spherical component (yellow).

Proof We put X into a sequence of blowups as in (11.1). Since the lines L_i and $\overline{L_i}$ on $(X_3)_\mathbb{C}$ from Lemma 11.7 are disjoint from E_3, they can be identified with some lines on $(X_4)_\mathbb{C}$ which we, by abuse of notation, also denote by L_i and $\overline{L_i}$. The same we do for the proper transforms of L_i and $\overline{L_i}$ in $(X_2)_\mathbb{C}$. We want to apply part (b) of Theorem 9.3 to the divisor $M = L_2 - L_1 - K$, where K is a canonical divisor on X. Since X is birational to X_3, the rational function f from part (e) of Lemma 11.7 is also a rational function on X and we have $M + \overline{M} = (f) - 2K$. By Lemma 11.6, there is a rational function g on X such that $(g) = R + 2K$, where R is the ramification divisor. Furthermore, we can choose g to have the same sign on each connected component of $X(\mathbb{R})$ as f. Thus $M + \overline{M} = (\frac{f}{g}) + R$ and $\frac{f}{g}$ is nonnegative. It thus remains to show that the dimension $\ell(M)$ of the space of global sections of $\mathcal{L}(M)$ is at least d. To this end, we invoke the theorem of Riemann–Roch for surfaces [Har77, Theorem 1.6]:

$$\ell(M) + \ell(K - M) = \frac{1}{2}M.(M - K) + 1 + p_a + s(M) \geq \frac{1}{2}M.(M - K) + 1 = d.$$

Now, since the intersection product of $(K - M)$ with the ample divisor $-K$ equals $-2d < 0$, it cannot be effective. Thus $\ell(K - M) = 0$ and the claim follows. ■

We now apply Theorem 11.8 to the three cases from Proposition 11.5. The following consequence is originally due to Buckley and Koşir [BK07].

Corollary 11.9 *Every hyperbolic polynomial $h \in \mathbb{R}[x_0, x_1, x_2, x_3]$ of degree three has a definite Hermitian determinantal representation.*

Proof First assume that the zero set of h is smooth. Then we are in case (2) of Proposition 11.5 and the claim follows from Theorem 11.8 and Proposition 8.1. For the singular case, note that by [Nui68], the set of all hyperbolic polynomials is the closure of the smooth ones. Furthermore, by [PV13, Lemma 3.4], the set of hyperbolic polynomials with a definite Hermitian determinantal representation is closed. ■

The following consequence is originally due to Hilbert [Hil88].

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Corollary 11.10 *Every nonnegative ternary quartic is a sum of three squares.*

Proof First consider a nonnegative ternary quartic p with smooth zero set. The hypersurface defined by $y^2 - p$ in $\mathbb{P}(2,1,1,1)$ is an instance of Proposition 11.5(3). Thus, the claim follows from Theorems 8.6 and 11.8. The general case now follows from a limit argument as the set of sums of squares is closed in $\mathbb{R}[x_0, x_1, x_2]_4$. ■

Remark 11.11 A recent generalization of Hilbert's result on nonnegative ternary quartics was established in [BSV16]. Among others, they show that every quadratic polynomial $p \in \mathbb{R}[x_0, \dots, x_n]$ which is nonnegative on the real part of an irreducible variety $X \subset \mathbb{P}^n$ with $\deg(X) = \operatorname{codim}(X) + 1$ and Zariski dense real part is a sum of squares modulo the ideal I of X. Analogously to Theorem 8.6, one can show that this is equivalent to the existence of a certain positive Ulrich sheaf. Namely, let Y be the zero set in $\mathbb{P}^{n+1}$ of the ideal generated by I and the polynomial $y^2 - p$. Let $f\colon Y \to X$ the natural projection. Analogously to Theorem 8.6, the polynomial p is a sum of squares modulo I if and only if there is a positive f-Ulrich sheaf. It would be interesting to construct these sheaves directly.

Corollary 11.12 *The Chow form of a smooth hyperbolic surface in $\mathbb{P}^4$ of degree four, which is a complete intersection of two quadrics, has a definite Hermitian determinantal representation.*

Proof Here, we are in case (1) of Proposition 11.5. The claim follows from Theorem 11.8 together with a straightforward adaption of the proof of [KS20, Theorem 5.7] to the Hermitian case and [KS20, Remark 4.4]. ■

Remark 11.13 We have seen that every nonnegative polynomial $p \in \mathbb{R}[x_0, x_1, x_2]_4$ is a sum of squares and every hyperbolic polynomial $h \in \mathbb{R}[x_0, x_1, x_2, x_3]_3$ has a definite Hermitian determinantal representation, i.e., the associated real-fibered morphisms admit a positive Ulrich sheaf. This is no longer true if we increase the degrees: not every nonnegative polynomial $p \in \mathbb{R}[x_0, x_1, x_2]_6$ is a sum of squares [Hil88] and there are hyperbolic polynomials $h \in \mathbb{R}[x_0, x_1, x_2, x_3]_4$ such that no power h^r has a definite determinantal representation, take, for example, the polynomial considered in [Kum16b]. Double covers of $\mathbb{P}^2$ ramified along plane sextic curves and quartic hypersurfaces in $\mathbb{P}^3$ both belong to the class of K3 surfaces. So it would be very interesting to understand which real-fibered morphisms $X \to \mathbb{P}^2$ from a K3 surface X admit a positive Ulrich bundle. Note that (not necessarily positive) Ulrich sheaves of rank two on K3 surfaces have been constructed in [AFO17]. Similarly, we cannot increase the dimensions: not every nonnegative polynomial $p \in \mathbb{R}[x_0, x_1, x_2, x_3]_4$ is a sum of squares [Hil88] and it is not known whether there are hyperbolic polynomials $h \in \mathbb{R}[x_0, x_1, x_2, x_3, x_4]_3$ such that no power h^r has a definite determinantal representation (see [Sau19, Section 5] for cubic hyperbolic hypersurfaces). Double covers of $\mathbb{P}^3$ ramified along quartic surfaces and cubic hypersurfaces in $\mathbb{P}^4$ both belong to the class of Fano threefolds of index two. In [Bea18, Section 6], Ulrich sheaves of rank two on such threefolds have been constructed.

Problem 11.14 *Understand which finite surjective and real-fibered morphisms $X \to \mathbb{P}^n$ admit a positive Ulrich sheaf for X a K3 surface or a Fano threefold of index two. If such* *exist, what are their ranks? Are there hyperbolic cubic hypersurfaces in $\mathbb{P}^4$ that do not carry a positive Ulrich sheaf?*

We conclude this section with some examples.

Example 11.15 Let $h = x_0^3 - x_0(2x_1^2 + 2x_2^2 + x_3^2) + x_1^3 + x_1x_2^2$. The hypersurface $X = \mathcal{V}(h) \subset \mathbb{P}^3$ is hyperbolic with respect to $e = [1:0:0:0]$ and contains the real line $L = \mathcal{V}(x_0, x_1)$. The hyperplanes $H_1 = \mathcal{V}(x_0)$ and $H_2 = \mathcal{V}(x_0 - x_1)$ contain L and are tangent to the hyperbolicity cone of h. The quadratic polynomial $p = x_0(x_0 - x_1)$ is an interlacer of h and its zero divisor on X is

$$D = 2L + L_1 + \overline{L_1} + L_2 + \overline{L_2},$$

where $L_1 = \mathcal{V}(x_0, x_1 + \mathrm{i}x_2)$ and $L_2 = \mathcal{V}(x_0 - x_1, x_2 + \mathrm{i}x_3)$. Thus, we have $D = M + \overline{M}$ with $M = L + L_1 + L_2$. The space of all quadrics vanishing on L, L_1, and L_2 is spanned by

$$x_0(x_0 - x_1), x_0(x_2 + \mathrm{i}x_3), (x_0 - x_1)(x_1 + \mathrm{i}x_2).$$

The minimal free resolution over $S = \mathbb{R}[x_0, x_1, x_2, x_3]$ of the ideal in $S/(h)$ generated by these quadrics has length one and is given by the matrix

$$\begin{pmatrix} x_0 + x_1 & -x_2 - \mathrm{i}x_3 & -x_1 - \mathrm{i}x_2 \\ -x_2 + \mathrm{i}x_3 & x_0 - x_1 & 0 \\ -x_1 + \mathrm{i}x_2 & 0 & x_0 \end{pmatrix}.$$

This matrix is Hermitian and positive definite at e. Its determinant is indeed h.

Example 11.16 Consider the following nonnegative ternary quartic

$$p = x_0^4 + 2x_0^2x_1^2 + 2x_0x_1^3 + x_1^4 + x_0^2x_2^2 - 2x_0x_1x_2^2 - x_1^2x_2^2 + x_2^4,$$

and let $X \subset \mathbb{P}(2,1,1,1)$ be the corresponding double cover defined by $y^2 = p$. On X, we have the two lines

$$L_1 = \mathcal{V}(y + (1 - \mathrm{i})x_1^2 - x_2^2, x_0 + \mathrm{i}x_1) \text{ and } L_2 = \mathcal{V}(y + \mathrm{i}x_1x_2 - x_1^2 + x_2^2, x_0 + \mathrm{i}x_2).$$

The principal divisor associated with the rational function

$$f = \frac{x_0^2 + x_2^2}{y - x_0x_1 - x_1^2 + x_2^2}$$

is $L_2 - L_1 + \overline{L_2} - \overline{L_1}$. As L_1 and L_2 are both nonreal lines, this implies that f has constant sign on each of the two connected components of $X(\mathbb{R})$. Evaluating f at points from the two different components, for example, at

$$[y : x_0 : x_1 : x_2] = [\pm 1 : 1 : 0 : 0],$$

shows that f changes sign. Letting K be a canonical divisor on X, we have for $M = L_2 - L_1 - K$ that $M + \overline{M} = (f) - 2K$ as in the proof of Theorem 11.8. We can realize the divisor class of M by the ideal in $\mathbb{R}[y, x_0, x_1, x_2]/(y^2 - p)$ that is generated by

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$y - x_0x_1 - x_1^2 + x_2^2$ and $(x_0 + \mathrm{i}x_1)(x_0 + \mathrm{i}x_2)$. The minimal free resolution over the ring $\mathbb{R}[y, x_0, x_1, x_2]$ has length one and is given by the matrix

$$\begin{pmatrix} y - x_0x_1 - x_1^2 + x_2^2 & x_0^2 - x_1x_2 + \mathrm{i}(x_0x_1 + x_0x_2) \\ x_0^2 - x_1x_2 - \mathrm{i}(x_0x_1 + x_0x_2) & y + x_0x_1 + x_1^2 - x_2^2 \end{pmatrix}.$$

Indeed, we have that $A^2 = p \cdot I$. Therefore,

$$p = (x_0x_1 + x_1^2 - x_2^2)^2 + (x_0^2 - x_1x_2)^2 + (x_0x_1 + x_0x_2)^2.$$

The key ingredient for this construction was the rational function f. One way to find it is to blow down a real line of X and then proceed as in Lemma 11.7.

Example 11.17 The del Pezzo surface

$$X = \mathcal{V}(x_0^2 + x_1^2 + x_2^2 - x_3^2, x_0^2 + 4x_1^2 + 9x_2^2 - x_4^2) \subset \mathbb{P}^4$$

of degree four is hyperbolic with respect to the line E spanned by $[0{:}0{:}0{:}1{:}0]$ and $[0{:}0{:}0{:}0{:}1]$. The rational function $\frac{(x_3-x_0)(x_4-x_0)}{x_0^2}$ has different signs on the two connected components of $X(\mathbb{R})$. Its corresponding divisor is of the form $M + \overline{M} + 2K$ for a suitable divisor M on $X_{\mathbb{C}}$ with $\ell(M) = 4$. Thus, $\mathcal{L}(M)$ is a positive Hermitian Ulrich line bundle by Theorem 9.3. As in [ESW03, Theorem 0.3], we obtain from that the following determinantal representation of the Chow form of X, written in the Plücker coordinates:

$$\begin{pmatrix} 2x_{03} - 2x_{04} + 2x_{34} & 4x_{01} - 6\mathrm{i}x_{02} + 4x_{13} - 6\mathrm{i}x_{23} & -2x_{01} + 2\mathrm{i}x_{02} - 2x_{14} + 2\mathrm{i}x_{24} & -2\mathrm{i}x_{12} \\ 4x_{01} + 6\mathrm{i}x_{02} + 4x_{13} + 6\mathrm{i}x_{23} & -2x_{03} - 2x_{04} + 2x_{34} & 10\mathrm{i}x_{12} & 2x_{01} - 2\mathrm{i}x_{02} - 2x_{14} + 2\mathrm{i}x_{24} \\ -2x_{01} - 2\mathrm{i}x_{02} - 2x_{14} - 2\mathrm{i}x_{24} & -10\mathrm{i}x_{12} & 2x_{03} + 2x_{04} + 2x_{34} & -4x_{01} + 6\mathrm{i}x_{02} + 4x_{13} - 6\mathrm{i}x_{23} \\ 2\mathrm{i}x_{12} & 2x_{01} + 2\mathrm{i}x_{02} - 2x_{14} - 2\mathrm{i}x_{24} & -4x_{01} - 6\mathrm{i}x_{02} + 4x_{13} + 6\mathrm{i}x_{23} & -2x_{03} + 2x_{04} + 2x_{34} \end{pmatrix}.$$

We observe that it is Hermitian and positive definite at E.

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