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THE RADIUS OF CONVEXITY OF A LINEAR COMBINATION OF FUNCTIONS IN $\mathfrak{K}$, $CV_k(\beta)$, $\mathfrak{S}$ OR $\mathfrak{U}_\alpha$

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Labelle and Rahman [**4**] showed that if f, $g \in \mathfrak{K}$, the normalized convex functions in the unit disc D, then $\frac{1}{2}(f(z) + g(z))$ has a radius of convexity at least as large as the smallest root of $1 - 3r + 2r^2 - 2r^3 = 0$. Their method requires neither the properties of the arithmetic mean nor the strong geometric properties of $\mathfrak{K}$; indeed, the procedure works for a linear combination of functions from any linear invariant family of finite order.

We examine three general classes of linear invariant families with varying degrees of control on $|\arg f'(z)|$. All families considered are subsets of $\mathfrak{L} . \mathfrak{S}.$, the normalized locally univalent analytic functions in D. A survey of relevant properties of linear invariant families can be found in [**1**] or [**5**].

THEOREM. *Let $\mathfrak{M}$ be a linear invariant family of finite order α. Let*

$$H = \{h(z) : h(z) = tf(z) + (1 - t)g(z), f, g \in \mathfrak{M}, t \in \mathbf{R}\}.$$

Let $\arg (g'(z)/f'(z)) = \gamma(r, \theta)$, $z = re^{i\theta} \in D$, *where* $\gamma(0, 0) = 0$. *Then for any* $h(z) \in H$ *and any* $z \in D$ *such that* $-\pi < \gamma(r, \theta) < \pi$, *we have*

$$\operatorname{Re}\{1 + zh''/h'(z)\} \geqq \frac{(1 + r^2)\cos(\gamma/2) - 2\alpha r}{(1 - r^2)\cos(\gamma/2)}.$$

Proof. We have $h(z) = tf(z) + (1 - t)g(z)$. Since either t or $1 - t$ is not zero, we assume that $t \neq 0$. Then

$$z\frac{h''(z)}{h'(z)} = z \cdot \frac{tf''(z) + (1 - t)g''(z)}{tf'(z) + (1 - t)g'(z)}$$

$$= z\frac{f''(z)}{f'(z)} \cdot \frac{1}{1 + Ae^{i\gamma}} + z\frac{g''(z)}{g'(z)} \cdot \frac{1}{1 + A^{-1}e^{-i\gamma}},$$

where $Ae^{i\gamma} = ((1 - t)/t)|g'(z)/f'(z)|\arg\{g'(z)/f'(z)\}$. It is clear, as in [**4**], that if $|w - a| \leqq d$, $a \geqq 0$, and w_0 is an arbitrary complex number, then $|ww_0 - a|w_0|e^{i\arg w_0}| \leqq d|w_0|$; that is,

$$\operatorname{Re}\{ww_0\} \geqq |w_0|\{a \cos(\arg w_0) - d\}.$$

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Therefore, for $|z| = r < 1$ satisfying $-\pi < \gamma(r, \theta) < \pi$, we have

$$\mathrm{Re}\left\{z\frac{f''(z)}{f'(z)}\cdot\frac{1}{1+Ae^{i\gamma}}\right\} \geqq \frac{1}{(1+A^2+2A\cos\gamma)^{\frac{1}{2}}} \times\left\{\frac{2r^2}{1-r^2}\cdot\frac{1+A\cos\gamma}{(1+A^2+2A\cos\gamma)^{\frac{1}{2}}}-\frac{2\alpha r}{1-r^2}\right\},$$

$$\mathrm{Re}\left\{z\frac{g''(z)}{g'(z)}\cdot\frac{1}{1+A^{-1}e^{-i\gamma}}\right\} \geqq \frac{A}{(1+A^2+2A\cos\gamma)^{\frac{1}{2}}} \times\left\{\frac{2r^2}{1-r^2}\cdot\frac{A+\cos\gamma}{(1+A^2+2A\cos\gamma)^{\frac{1}{2}}}-\frac{2\alpha r}{1-r^2}\right\},$$

since for any linear invariant family of order α we have [**5**, p. 115]

$$\left|z\frac{f''(z)}{f'(z)}-\frac{2r^2}{1-r^2}\right| \leqq \frac{2\alpha r}{1-r^2}.$$

Consequently,

$$\begin{aligned}\mathrm{Re}\{1+zh''(z)/h'(z)\} &\geqq 1+\frac{2r^2}{1-r^2}-\frac{2\alpha r}{1-r^2}\left(1+\frac{2A}{(1+A)^2}(\cos\gamma-1)\right)^{-\frac{1}{2}}\\ &\geqq \frac{1+r^2}{1-r^2}-\frac{2\alpha r}{1-r^2}[(1+\cos\gamma)/2]^{-\frac{1}{2}}\\ &\geqq \frac{(1+r^2)\cos(\gamma/2)-2\alpha r}{(1-r^2)\cos(\gamma/2)}.\end{aligned}$$

The second line in the chain of inequalities follows since the minimum occurs when A is 1. This concludes the proof of the theorem.

COROLLARY 1. *Let* $F(z) = \frac{1}{2}(f(z) + g(z))$ *and* R_c *denote the radius of convexity of* $F(z)$.

(a) If $f, g \in \mathfrak{S}$, *the normalized univalent analytic functions, then* R_c *is no less than the smallest positive root of* $1 - 4r - 7r^2 + 8r^6 = 0$, *that is* $R_c \geqq .185$.

(b) *If* $f, g \in CV_k(\beta)$, *the normalized* β*-close-to-*V_k *functions, then* R_c *is no less than the smallest positive root of* $(1 + r^2)\cos((k + 2\beta)\sin^{-1}r) - (k + 2\beta)r = 0$

(c) *If* $f, g \in U_\alpha$, *the normalized universal linear invariant family of order* α, *then* R_c *is no less than the smallest positive root of*

$$(1+r^2)\cos\left[2\int_0^r\frac{(\alpha^2-x^2)^{\frac{1}{2}}}{1-x^2}\,dx\right]-2\alpha r=0.$$

Proof. These conclusions are immediate from Theorem 1 and the definition of R_c upon noting that:

(a) If $f \in \mathfrak{S}$, $|\arg f'(z)| \leqq 4\sin^{-1}|z|$, for $|z| < 1/\sqrt{2}$ [**3**, p. 115].

(b) If $f \in CV_k(\beta)$, $|\arg f'(z)| \leqq (k + 2\beta)\sin^{-1}|z|$, for $z \in D$ [**2**, Corollary 4.6].

(c) If $f \in U_\alpha$,

$$|\arg f'(z)| \leqq 2\int_0^r\frac{(\alpha^2-x^2)^{\frac{1}{2}}}{1-x^2}\,dx,$$

for $|z| = r$, $0 < r < 1$ [**5**, Theorem 2.1].

COROLLARY 2. *With the notation of Corollary* 1:

(a) *If* $f, g \in \mathfrak{K}$, *the normalized convex univalent fnuctions, then* R_c *is no less than the smallest positive root of* $1 - 3r + 2r^2 - 2r^3 = 0$, *that is* $R_c \geqq .395$.

(b) *If* f, g *are close-to-convex, then* R_c *is no less than the smallest positive root of* $1 - 4r - 7r^2 + 8r^6 = 0$, *that is* $R_c \geqq .185$.

(c) *If* $f, g \in V_k$, *the functions whose bounded boundary rotation is* $\leqq k\pi$, *then* R_c *is no less than the smallest positive root of* $(1 + r^2) \cos (k \sin^{-1} r) - kr = 0$.

Proof. This is immediate from Corollary 1(b), since $CV_2(0) = \mathfrak{K}$, $CV_2(1) =$ close-to-convex functions, $CV_k(0) = V_k$ as proved in [**2**]. Note that the lower bound for R_c for the class $\mathfrak{S}$ and the class of close-to-convex functions is given by the same expression which is obtained by two different types of arguments. Finally, we see that Labelle's and Rahman's result is a special case of Corollary 2(c), $k = 2$.

The results of Corollaries 1 and 2 are 'nearly' best possible. If one lets $F(z) = (2\alpha)^{-1}[((1 + z)/(1 - z))^\alpha - 1]$, $\alpha \geqq 1$, then, as is well-known, $F(z) \in V_{2\alpha}$. Furthermore, $G(z) = -F(-z) \in V_{2\alpha}$. Letting $H(z) = \frac{1}{2}(F(z) + G(z))$ yields for $z = re^{i\theta}$ that

$$\mathrm{Re}\{1 + zH''(z)/H'(z)\} = 1 + \frac{2r(\alpha^2 + r^2 + 2\alpha r \cos \theta)^{\frac{1}{2}} \cos \theta_1}{(1 + r^4 - 2r^2 \cos 2\theta)^{\frac{1}{2}} \cos \theta_2}$$

where $\theta_1 = \theta + \arg (\alpha + re^{i\theta}) + (\alpha - 2) \arg (1 + re^{i\theta}) + (\alpha + 2) \arg (1 - re^{-i\theta})$, $\theta_2 = (\alpha - 1) \arg (1 + re^{i\theta}) + (\alpha + 1) \arg (1 - re^{-i\theta})$. Using an

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$\alpha(k = 2\alpha)$	$\leqq R_c \leqq$	r	θ	$\mathrm{Re}\,\{1 + zH''(z)/H'(z)\}$, $z = re^{i\theta}$
1.0	$.395 \leqq R_c \leqq .405$	.405	1.2252	$-.0003$
1.1	$.355 \leqq R_c \leqq .375$	.375	1.3823	$-.00026$
1.2	$.320 \leqq R_c \leqq .350$	.350	1.3823	$-.005$
1.3	$.295 \leqq R_c \leqq .325$	.325	1.4451	$-.0001$
1.4	$.270 \leqq R_c \leqq .305$	.305	1.4451	$-.006$
1.5	$.250 \leqq R_c \leqq .285$	.285	1.4765	$-.001$
1.6	$.235 \leqq R_c \leqq .265$	.265	1.6022	$-.001$
1.7	$.220 \leqq R_c \leqq .250$	.250	1.6022	$-.004$
1.8	$.205 \leqq R_c \leqq .235$	.235	1.6650	$-.004$
1.9	$.195 \leqq R_c \leqq .225$	.225	1.5707	$-.006$
2.0	$.185 \leqq R_c \leqq .210$	.210	1.7278	$-.001$
2.1	$.175 \leqq R_c \leqq .200$	.200	1.7278	$-.004$
2.2	$.170 \leqq R_c \leqq .190$	.190	1.7592	$-.0008$
2.3	$.160 \leqq R_c \leqq .185$	.185	1.6022	$-.001$
2.4	$.155 \leqq R_c \leqq .175$	.175	1.6964	$-.003$
2.5	$.145 \leqq R_c \leqq .170$	.170	1.6336	$-.01$
2.6	$.140 \leqq R_c \leqq .160$	.160	1.7907	$-.005$
2.7	$.135 \leqq R_c \leqq .155$	.155	1.7278	$-.007$
2.8	$.130 \leqq R_c \leqq .150$	.150	1.6964	$-.008$
2.9	$.125 \leqq R_c \leqq .145$	.145	1.6964	$-.01$

elementary Fortran program Mr. Russell Anderson was able to find where $Re\{1 + zH''(z)/H'(z)\}$ was negative. The results are summarized in the table above (θ is in radians).

This table can also be used for functions in $CV_k(\beta)$ $(k + 2\beta \leqq 5.8)$, $\mathfrak{S}$ or the close-to-convex functions ($\alpha = 2$) to show that R_c is determined to within .035. A similar analysis is possible for U_α.

References

1. D. M. Campbell, *Locally univalent functions with locally univalent derivatives*, Trans. Amer. Math. Soc. *162* (1971), 395–409.
2. ——— *β-close-to-bounded boundary rotation functions* I. (to appear).
3. G. M. Goluzin, *Geometric theory of functions of a complex variable*, Amer. Math. Soc. Vol. *26* (Providence, R. I., 1969).
4. G. Labelle and Q. I. Rahman, *Remarque sur la moyenne arithmétique de fonctions univalentes convexes*, Can. J. Math. *21* (1969), 977–981.
5. C. Pommerenke, *Linear-invariante Familien Analytischer Funktionen*. I, Math. Ann. *155* (1964), 108–154.

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