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α-REPRESENTABLE COPRODUCTS OF DISTRIBUTIVE LATTICES

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There are a number of classes of distributive lattices whose members can be characterised as the coproduct $A*L$ of suitable distributive lattices A and L. For example, Post algebras [**1**], pseudo-Post algebras [**4**], Post L-algebras ([**6**], [**8**], [**9**]) and the lattices $[D]_n$ of [**4**]. Moreover, the α-completeness and α-representability of some (though not all) of these algebras have been investigated in [**7**], [**2**], [**6**], and [**10**].

In this note we investigate the α-representability of the coproduct $A*L$ of two distributive lattices. In Section 2 we show (Theorem 2.3) that if L is finite, then $A*L$ is α-complete if and only if A is α-complete, and (Theorem 2.6) if L is arbitrary and B is a Boolean algebra, then $B*L$ is α-complete if and only if both B and L are α-complete and at least one of them is finite. The α-representability of $A*L$, where L is finite, is discussed in Section 3 where we show (Theorem 3.2) that $A*L$ is an α-homomorphic image of an α-ring of sets if and only if A has the same property, and (Theorem 3.5) $A*L$ is isomorphic to an α-ring of sets modulo an α-ideal if and only if A has the same property. The specialisation of these results to Post algebras and their generalisations yields the known, as well as some new, results concerning the α-representability of these algebras. (see Corollary 3.4 and Remark 3.6.)

1. Notation

All lattices considered below will be distributive lattices with 0 and 1 and all lattice homomorphisms will preserve 0 and 1. The least upper bound and greatest lower bound of two elements x and y will be denoted by $x+y$ and xy respectively. If L' is a sublattice of L and $S\subseteq L'$, then the least upper bounds of S in L' and L will be denoted (whenever they exist) by $\sum_{x\in S}^{L'} x$ and $\sum_{x\in S}^{L} x$ respectively. Similar notations will be used for the greatest lower bounds of S in L' and L.

Let L_1, L_2 and L be distributive lattices and let $i_1\colon L_1\to L$ and $i_2\colon L_2\to L$ be lattice monomorphisms. The pair $(L,\{i_1,i_2\})$ will be called the *coproduct* (=*free product*) of L_1 and L_2 if for every distributive lattice D and every pair of lattice homomorphisms $h_1\colon L_1\to D$ and $h_2\colon L_2\to D$, there is a unique lattice homomorphism $h\colon L\to D$ such that $hi_1=h_1$ and $hi_2=h_2$. The coproduct of L_1 and L_2 will be denoted by L_1*L_2. We shall often identify L_1 and L_2 with their isomorphic images in L_1*L_2 and thus consider them as sublattices of L_1*L_2. With this convention, L_1*L_2 can be characterised as follows (cf. [**1**], Theorem VII.1.):

Lemma 1.1. *Let L be a distributive lattice generated by the union $L_1 \cup L_2$ of two sublattices L_1 and L_2. Then L is the coproduct of L_1 and L_2 if and only if for every $a_1, b_1 \in L_1$ and $a_2, b_2 \in L_2$, $a_1 a_2 \leqq b_1 + b_2$ implies $a_1 \leqq b_1$ or $a_2 \leqq b_2$.*

Let A and L be distributive lattices. We shall make use of the following specific representation of $A * L$ by a ring of sets (cf. [5]): Let the mapping $a \to X_a$ be an isomorphism of A onto a lattice of subsets of a set X and $l \to Y_l$ be an isomorphism of L onto a lattice of subsets of a set Y. For every $S \subseteq X$ and $T \subseteq Y$, let $S^* = S \times Y$ and $T^* = X \times T$. Then the lattice (of subsets of $X \times Y$) generated by $\{X_a^* : a \in A\} \cup [Y_l^* : l \in L\}$ is the coproduct $A * L$. We note that every element E of $A * L$ can be expressed as

$$E = \bigcup_{i=1}^{n} (X_{a_i} \times Y_{l_i}) = \bigcup_{i=1}^{n} (X_{a_i}^* \cap Y_{l_i}^*), \tag{1}$$

where n depends on E and each $a_i \in A$ and $l_i \in L$. Using the distributive law, the expression (1) can also be written as

$$E = \bigcap_{i=1}^{k} (X_{b_i}^* \cup Y_{m_i}^*), \tag{2}$$

where each $b_i \in A$ and $m_i \in L$.

2. α-complete lattices

In this section we shall investigate when the coproduct of two distributive lattices is α-complete. Henceforth, A and L will always denote distributive lattices and α an infinite cardinal. We begin with the following:

Lemma 2.1. *If $P = A * L$ is α-complete, then both A and L are α-complete.*

Proof. Let $S \subseteq A$ such that $|S| \leqq \alpha$ and let $\sum_{x \in S}^{P} x = a_1 l_1 + a_2 l_2 + \ldots + a_n l_n$, where $a_i \in A$, $l_i \in L$, $l_i \neq 0$, $1 \leqq i \leqq n$. Then $a = \sum_{i=1}^{n} a_i$ is an upper bound of S. If $u \in A$ is another upper bound of S, then $u \geqq \sum_{i=1}^{n} a_i l_i$. Hence for every i, $1 \leqq i \leqq n$, $a_i l_i \leqq u + 0$ and by Lemma 1.1, $a_i \leqq u$. Hence a is the least upper bound of S. Similarly we show that the greatest lower bound of S is in A and hence A is an α-complete lattice. Similarly, L is α-complete.

Lemma 2.2. *Let $P = A * L$, $\{a_i : i \in I\} \subseteq A$, and $l \in L$. Then*

(i) *if $\sum_{i \in I}^{A} a_i$ exists, then $\sum_{i \in I}^{P} a_i l$ exists and $l \sum_{i \in I}^{A} a_i = \sum_{i \in I}^{P} a_i l$,*

(ii) *if $\prod_{i \in I}^{A} a_i$ exists, then $\prod_{i \in I}^{P} (a_i + l)$ exists and $l + \prod_{i \in I}^{A} a_i = \prod_{i \in I}^{P} (a_i + l)$.*

Proof. (i) Let $a = \sum_{i \in I}^{A} a_i$. Then la is an upper bound of $\mathscr{S} = \{a_i l : i \in I\}$ in P. Moreover, if $u = \prod_{j=1}^{k} (b_j + m_j)$ is another upper bound of $\mathscr{S}$ in P then for all $i \in I$ and all $j \in K = \{1, 2, \ldots, k\}$, $a_i l \leqq b_j + m_j$ and hence $a_i \leqq b_j$ or $l \leqq m_j$. Let $J = \{j \in K : l \leqq m_j\}$ and $J' = \{j \in K : b_j \geqq \sum_{i \in I}^{A} a_i = a\}$. Then

$$la \leqq m_j \leqq b_j + m_j, \text{ when } j \in J,$$

and

$$la \leqq b_j \leqq b_j + m_j, \text{ when } j \in J'.$$

Therefore $la \leqq b_j + m_j$, for all $j \in J \cup J' = K$, so that $la \leqq u$. Thus la is the least upper bound of $\mathscr{S}$ in P.

(ii) Dualise the proof of (i).

Theorem 2.3. *Let $P = A * L$ be the coproduct of a distributive lattice A and a finite distributive lattice L. Then P is α-complete if and only if A is α-complete.*

Proof. Let $L = \{l_1, l_2, \ldots, l_n\}$. Suppose first that A is α-complete and let $\{x_i : i \in I\} \subseteq P$, where $|I| \leqq \alpha$. For every $i \in I$, let $x_i = \sum_{j=1}^{n} a_{ij} l_j$, where each $a_{ij} \in A$ and $l_j \in L$. We shall show that

$$\sum_{i \in I}^{P} x_i = \sum_{j=1}^{n} \left(\left(\sum_{i \in I}^{A} a_{ij} \right) l_j \right). \tag{3}$$

For every j, $1 \leqq j \leqq n$, let $u_j = (\sum_{i \in I}^{A} a_{ij}) l_j$; then by Lemma 2.2(i), $u_j = \sum_{i \in I}^{P} a_{ij} l_j$. But $\sum_{j=1}^{n} u_j = \sum_{i \in I}^{P} x_i$. Hence (3) holds. To show that $\prod_{i \in I}^{P} x_i$ exists, we express each x_i as $x_i = \prod_{j=1}^{n} (b_{ij} + m_j)$, where each $b_{ij} \in A$ and $m_j \in L$. Then by using Lemma 2.2(ii) and dualising the above argument, we conclude that

$$\prod_{i \in I}^{P} x_i = \prod_{j=1}^{n} \left(\left(\prod_{i \in I}^{A} b_{ij} \right) + m_j \right). \tag{4}$$

The converse follows from Lemma 2.1.

We recall that a lattice L is said to satisfy the *ascending chain condition* if every increasing chain of L terminates; that is, if for every chain $x_1 \leqq x_2 \leqq x_3 \leqq \ldots \leqq x_n \leqq \ldots$ of elements of L, there is an m such that $x_i = x_m$ for all $i \geqq m$. A lattice with the *descending chain condition* is defined similarly.

The converse of the last theorem is false; that is, if $A * L$ is α-complete, then neither A nor L need be finite. (For example, the coproduct of two infinite, σ-complete, increasing chains is σ-complete). However, the next lemma will show that the α-completeness of $A * L$ implies that A or L must satisfy one of the chain conditions.

Lemma 2.4. *If $P = A * L$ is α-complete for any infinite cardinal α, then A satisfies the descending chain condition or L satisfies the ascending chain condition.*

Proof. We identify P with a lattice of subsets of $X \times Y$ (cf. Section 1). Then it suffices to show that if A has a non-terminating decreasing chain $a_1 \geqq a_2 \geqq \ldots \geqq a_n \geqq \ldots$ and L has a non-terminating increasing chain $l_1 \leqq l_2 \leqq \ldots \leqq l_n \leqq \ldots$, then the set $\mathscr{S} = \{x_{a_n} \times Y_{l_n} : n = 1, 2, 3, \ldots\}$ has no least upper bound in P. Suppose on the contrary that $\mathscr{S}$ has a least upper bound $U \in P$. Then $U = \bigcup_{i=1}^{k} (X_{b_i} \times Y_{m_i})$, where $b_i \in A$, $m_i \in L$, $1 \leqq i \leqq k$. Since $X_{a_1}^{*} = X_{a_1} \times Y$ is an upper bound of $\mathscr{S}$, $U \subseteq X_{a_1}^{*}$ and it follows that each $X_{b_i} \subseteq X_{a_1} = \bigcup_{n=1}^{\infty} (X_{a_n} - X_{a_{n+1}})$. Hence for every i, $1 \leqq i \leqq k$, there is an $n(i)$ such that $X_{b_i} \cap (X_{a_{n(i)}} - X_{a_{n(i)+1}}) \neq \emptyset$. We shall show that $Y_{m_i} \subseteq Y_{l_{n(i)}}$, $1 \leqq i \leqq k$. Suppose this is not the case for some i, and let $(x,y)\in(X_{b_i}\cap(X_{a_{n(i)}}-X_{a_{n(i)+1}})\times(Y_{m_i}-Y_{l_{n(i)}})$. Let $V=(U\cap X^*_{a_{n(i)+1}})\cup(\bigcup_{j=1}^{n(i)}(X_{a_j}\times Y_{l_j})$. Then V is an upper bound of $\mathscr{S}$, and V is properly contained in U since $(x,y)\in U-V$. This contradiction shows that $Y_{m_i}\subseteq Y_{l_{n(i)}}$, $1\leqq i\leqq k$. Now, let $m=\max\{n_i: 1\leqq i\leqq k\}$. Then $Y_{m_i}\subseteq Y_{l_m}$, for all $i\in\{1,2,\ldots,k\}$, so that $U\subseteq Y^*_{l_m}$. Hence U cannot be an upper bound of $\mathscr{S}$; otherwise the ascending chain $l_1\leqq l_2\leqq\ldots\leqq l_n\leqq\ldots$ would terminate at l_m.

Lemma 2.5. *If A has an infinite disjoint subset and L is infinite, then $A*L$ is not α-complete for any α.*

Proof. Since A has an infinite disjoint subset, A does not satisfy the ascending chain condition. Hence by Lemma 2.4, if L does not satisfy the descending chain condition, then $A*L$ is not α-complete for any α. On the other hand, if L satisfies the descending chain condition, then (cf. Theorem III.2.2 of [1]) L has an infinite ascending chain $l_1\leqq l_2\leqq\ldots\leqq l_n\leqq\ldots$. Let $\{a_n: n=1,2,3,\ldots\}$ be an infinite disjoint subset of A and let $S=\{a_nl_n: n=1,2,3,\ldots\}$. Then it follows by an argument similar to the one used in the proof of Lemma 2.4 that S does not have a least upper bound in $A*L$. Hence $A*L$ is not α-complete for any α in this case also.

We can now show that the converse of Theorem 2.3 holds when A is a Boolean algebra.

Theorem 2.6. *Let B be a Boolean algebra, L a distributive lattice, and α an infinite cardinal. Then $B*L$ is α-complete if and only if both B and L are α-complete and at least one of them is finite.*

Proof. The sufficiency follows from Theorem 2.3. For the necessity, suppose $B*L$ is α-complete. Then by Lemma 2.1, B and L are both α-complete. If B is infinite, then (cf. [3]) B has an infinite disjoint subset; hence by Lemma 2.5, L is finite.

Since a Post L-algebra (B,L) is isomorphic to the coproduct of a Boolean algebra B and a distributive lattice L, the last theorem yields the following result which is a generalisation of Theorem 6.1 of [6]:

Corollary 2.7. *A Post L-algebra (B,L) is α-complete if and only if both B and L are α-complete and at least one of them is finite.*

3. α-representable lattices

Following [1], we define a distributive lattice L to be *α-representable* if there exists an α-ring of sets R and an α-homomorphism of R onto L. This is a weaker condition than requiring L to be isomorphic to an α-ring of set modulo an α-ideal (i.e. $L\cong R/I$, where R is an α-ring of sets and I an α-ideal of R). We shall investigate in this section when $A*L$, where L is finite, is α-representable and when it is isomorphic to an α-ring of sets module an α-ideal.

Lemma 3.1. *Let R be an α-ring of sets and let $L=\{l_1,l_2,\ldots,l_n\}$ be a finite distributive lattice. Then $P=R*L$ is isomorphic to an α-ring of sets.*

Proof. Let R be an α-ring of subsets of a set X and represent P by a ring of subsets of $X\times Y$ (cf. Section 1). Let $\{E_i:i\in I\}\subseteq P$, $|I|\leqq\alpha$, and for every $i\in I$, let $E_i=\bigcup_{j=1}^{n}(X_{A_{ij}}\times Y_{l_j})$, where each $A_{ij}\in R$. Then it follows from (3) and the fact that R is an α-ring of sets that

$$\sum_{i\in I}^{P}E_i=\bigcup_{j=1}^{n}\left(\left(\sum_{i\in I}^{R}X_{A_{ij}}\right)\times Y_{l_j}\right)=\bigcup_{j=1}^{n}\left(\left(\bigcup_{i\in I}X_{A_{ij}}\right)\times Y_{l_j}\right)=\bigcup_{i\in I}E_i.$$

Similarly we show that $\prod_{i\in I}^{P}E_i=\bigcap_{i\in I}E_i$.

Theorem 3.2. *Let A and L be distributive lattices where $L=\{l_1,l_2,\ldots,l_n\}$ is finite. Then $P=A*L$ is α-representable if and only if A is α-representable.*

Proof. Suppose first that A is α-representable and let h_1' be an α-homomorphism of an α-ring of sets R onto A. Let $P'=R*L$. Then by Lemma 3.1, P' is isomorphic to an α-ring of sets, and we shall exhibit an α-homomorphism of P' onto P. By the definition of the coproduct there are imbedding monomorphisms $i_1:A\to P$, $i_2:L\to P$, $i_1':R\to P'$, and $i_2':L\to P'$. Let $\bar{A}=i_1(A)$, $\bar{L}=i_2(L)$, $R'=i_1'(R)$, and $L'=i_2'(L)$. Then the α-homomorphism h_1' of R onto A induces an α-homomorphism h_1 of R' onto $\bar{A}$, and the identity automorphism of L induces an isomorphism h_2 of L' onto $\bar{L}$. Moreover, h_1 and h_2 can be extended to a homomorphism h of P' onto P. We shall show that h is an α-homomorphism. Let $\{x_i:i\in I\}\subseteq P'$, $|I|\leqq\alpha$, and for every $i\in I$, let $x_i=\sum_{j=1}^{n}a_{ij}l_j$, where each $a_{ij}\in R'$. Then

$$h\left(\sum_{i\in I}^{P'}x_i\right)=h\left(\sum_{j=1}^{n}\left(\left(\sum_{i\in I}^{R'}a_{ij}\right)l_j\right)\right)=\sum_{j=1}^{n}\left(h\left(\sum_{i\in I}^{R'}a_{ij}\right)h(l_j)\right)$$

$$=\sum_{j=1}^{n}\left(h_1\left(\sum_{i\in I}^{R'}a_{ij}\right)h_2(l_j)\right)=\sum_{j=1}^{n}\left(\left(\sum_{i\in I}^{\bar{A}}h_1(a_{ij})\right)h_2(l_j)\right)$$

(since h_1 is an α-homomorphism)

$$=\sum_{j=1}^{n}\left(\left(\sum_{i\in I}^{\bar{A}}h(a_{ij})\right)h(l_j)\right)=\sum_{i\in I}^{P}\left(\sum_{j=1}^{n}h(a_{ij})h(l_j)\right)$$

(by (3))

$$=\sum_{i\in I}^{P}\left(\sum_{j=1}^{n}h(a_{ij}l_j)\right)=\sum_{i\in I}^{P}h(x_i).$$

Thus h preserves α-sums. To show that h preserves α-products, we express each x_i by $x_i=\prod_{j=1}^{n}(a_{ij}+l_j)$ and dualise the above argument using (4) instead of (3).

Conversely, suppose that $P=A*L$ is α-representable. Then there is an α-ring of sets T and an α-homomorphism g of T onto P. Let $T'=\{E\in T: g(E)\in \bar{A}\}$. Since α-sums and α-products in $\bar{A}$ agree with those in P, T' is an α-subring of T. Moreover, the restriction of g to T' is an α-homomorphism. Therefore $\bar{A}$, and hence A, is α-representable. This completes the proof of the theorem.

A Post algebra $P=(B,C)$ of order n is called *α-representable* if there is an α-Post ring of sets $R=(F,C)$ of order n and an α-Post homomorphism of R onto P (cf. [**2**]). α-representable pseudo-Post algebras and α-representable Post L-algebras are defined similarly (cf. [**10**]).

It is clear from the proof of Lemma 3.1 that if F is an α-field of sets and L is a finite distributive lattice, then $A*L$ is isomorphic to an α-Post field of sets. Moreover, the proof of Theorem 3.2 yields the following:

Corollary 3.3. *Let B be a Boolean algebra and L a finite distributive lattice. Then $B*L$ is α-representable* (*i.e. the α-homophorphic image of an α-Post field of sets*) *if and only if B is α-representable.*

The following follows from the proof of Theorem 3.2.

Corollary 3.4. (i) ([7], [2]) *A Post algebra (B,C) is α-representable if and only if B is α-representable.*

(ii) [**10**] *A Post L-algebra (B,L) with finite lattice of constants L is α-representable if and only if B is α-representable.*

(iii) *A pseudo-Post algebra (D,L) is α-representable if and only if D is α-representable.*

We shall now examine when $A*L$, where L is finite, is isomorphic to an α-ring of sets modulo an α-ideal. If I is an ideal of a distributive lattice L, then I determines a congruence relation of L; namely, the relation $\theta(I)=\{(x,y)\in L^2: x+u=y+u$ for some $u\in I\}$. We shall denote the quotient lattice $L/\theta(I)$ by L/I and the elements of L/I by $[x]_I$, where $x\in L$.

Theorem 3.5. *Let A and L be distributive lattices where L is finite and let $P=A*L$. Then P is isomorphic to an α-ring of sets modulo an α-ideal if and only if A is isomorphic to an α-ring of sets modulo an α-ideal.*

Proof. Suppose first that $A\cong R/I$, where R is an α-ring of sets and I is an α-ideal of R. We consider R as a sublattice of $Q=R*L$ and let $I^*=\{x\in Q: x\leqq u$ for some $u\in I\}$. Since Q is α-complete (Theorem 2.3), I^* is an α-ideal of Q. We shall show that $Q/I^*\cong(R/I)*L$. Let $R'=\{[x]_{I^*}: x\in R\}$ and let $i_1: R/I\to Q/I^*$ be defined by $i_1([x]_I)=[x]_{I^*}$. Then i_1 is a homomorphism of R/I onto R'. Moreover, if $i_1([x]_I)=i_1([y]_I)$, then $x+u^*=y+u^*$ for some $u^*\in I^*$. But $u^*\leqq u$ for some $u\in I$. Hence $x+u=y+u$; thus $[x]_I=[y]_I$ and it follows that i_1 is an isomorphism of R/I onto R'. Next let $i_2: L\to Q/I^*$ be defined by $i_2(l)=[l]_{I^*}$. Then i_2 is a homomorphism of L onto $L'=\{[x]_{I^*}: x\in L\}$. Moreover, if $i_2(l)=i_2(m)$, then $l+u=m+u$ for some $u\in I$. Hence $l\cdot 1\leqq m+u$ and it follows from Lemma 1.1. and the fact that I is a proper ideal that $l\leqq m$. Similarly, $m\leqq l$ so $l=m$ and i_2 is an isomorphism of L onto L'. Thus to complete the proof that $Q/I^*\cong(R/I)*L$, it suffices to show that the criterion of Lemma 1.1. is satisfied. Let $[x]_{I*}[l]_{I*} \leqq [y]_{I*} + [m]_{I*}$, where $x, y \in R$ and $l, m \in L$. Then $[xl]_{I*} \leqq [y+m]_{I*}$. Hence $xl \leqq y+m+u^*$ for some $u^* \in I^*$, so $xl \leqq (y+u)+m$ for some $u \in I$. Thus applying Lemma 1.1 to $Q = R * L$, we have $x \leqq y+u$ or $l \leqq m$ and this implies $[x]_{I*} \leqq [y]_{I*}$ or $[l]_{I*} \leqq [m]_{I*}$. Hence $Q/I^* \cong (R/I) * L \cong P = A * L$. But by Lemma 3.1, Q is isomorphic to an α-ring of sets, hence P is isomorphic to an α-ring modulo an α-ideal.

Conversely, suppose that P is isomorphic to T/J, where T is an α-ring of sets and J is an α-ideal of T. Let $g: T \to T/J$ be the α-homomorphism defined by $g(x) = [x]_J$ and let $h = ig$, where i is an isomorphism of T/J onto $P = A * L$. Let $T' = \{x \in T : h(x) \in A\}$. Then as was shown in the proof of Theorem 3.3, T' is an α-subring of T, and it is not difficult to show that $A \cong T'/J'$, where J' is the α-ideal of T' defined by $J' = J \cap T'$. This completes the proof of the theorem.

Remark 3.6. It is clear from the proof of the last theorem that Corollary 3.3 remains valid if "α-representable" is replaced by "isomorphic to an α-field of sets modulo an α-ideal". Moreover, a Post algebra (B, C) is isomorphic to an α-Post ring of sets modulo an α-Post ideal if and only if B is isomorphic to an α-field of sets modulo an α-ideal. The remaining two results in Corollary 3.4 also remain valid after similar changes are made.

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