PROBLEM FOR SOLUTION

P. 162. Let G be a finite abelian group, written additively, and S a subset of G. S is said to be a sum-free set in G if $(S + S) \cap S = \phi$. Let $\lambda(G)$ denote the largest possible order of a sum-free set in G.

For which abelian groups G does there exist a sum-free set S such that (i) $|S| = \lambda(G)$

and (ii) $|S + S| = \frac{\lambda(G)[\lambda(G) + 1]}{2}$?

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SOLUTIONS

P. 154. Let n identical weighted coins, each falling heads with probability x, be tossed, and let $p_i(x)$ be the probability that exactly i of them fall heads. Evaluate

$$f_n = \min_{0 \leq x \leq 1} \max_{i = 0, 1, \ldots, n} p_i(x)$$

W. Moser, McGill University

Solution by D. Ž. Djoković, University of Waterloo

Let $f_n(x) = \max_{i = 0, 1, \ldots, n} p_i(x)$.

Since

$$p_i(x) = \binom{n}{i} x^i (1 - x)^{n-i}$$

and

$$\frac{p_i(x)}{p_{i+1}(x)} = \frac{i+1}{n-1} \cdot \frac{1-x}{x} \qquad (i = 0, 1, \ldots, n-1)$$

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we infer that

$$f_n(x) = \binom{n}{i} x^i (1 - x)^{n-i}, \quad \frac{i}{n+1} \leq x \leq \frac{i+1}{n+1}$$

for each $i = 0, 1, \ldots, n$. We see that $f_n(x)$ is decreasing in $\left(0, \frac{1}{n+1}\right)$ and increasing in $\left(\frac{n}{n+1}, 1\right)$. In the interval $\left(\frac{i}{n+1}, \frac{i+1}{n+1}\right)$ $(i = 1, \ldots, n-1)$ it is increasing for $\frac{i}{n+1} < x < \frac{i}{n}$ and decreasing for $\frac{i}{n} < x < \frac{i+1}{n+1}$. Therefore

$$f_n = \min_{i=1,\ldots,n} a_i$$

where

$$a_i = f_n\left(\frac{i}{n+1}\right) = \binom{n}{i} \frac{i^i (n+1-i)^{n-i}}{(n+1)^n}$$

We have

$$\frac{a_i}{a_{i+1}} = \frac{\left(1 + \frac{1}{n-i}\right)^{n-i}}{\left(1 + \frac{1}{i}\right)^i} > 1, \qquad \text{if} \quad i < \frac{n}{2}.$$

Taking into account the symmetry $(a_i = a_{n+1-i})$, we get

$$f_n = \binom{n}{k} \frac{k^k (n+1-k)^{n-k}}{(n+1)^n}$$

where k is the integral part of $\frac{n}{2}$.

Also solved by G. Letac, I.B. MacNeill, K.G. Miller, R.A. Schaufele, and the proposer.

P. 155. If $a_1 < a_2 < \ldots < a_k \leq n$ is a sequence of positive integers such that $[a_i, a_j] > n$ for all $i \neq j$, show that $\sum_{i=1}^{k} \frac{1}{a_i} < 2$,

($[a_i, a_j]$ means "the least common multiple of a_i and a_j").

Anonymous

P. Erdős and A. Makowski have pointed out that this problem was posed for solution in the Amer. Math. Monthly, 56 (1949) p. 637, by Erdős. A solution with a better constant than 2, by R. Lehman appeared ibid., 58 (1951), p 345. Finally, A. Schinzel and G. Szekeres, Acta Sci. Math. (= Acta Szeged) 20 (1959) pp. 221-229, prove that the sum considered is $\leq 31/30$ and that this constant is best possible.

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