BULL. AUSTRAL. MATH. SOC. 26B25, 49J52
VOL. 51 (1995) [249–262]

INVEXITY CRITERIA FOR A CLASS OF VECTOR-VALUED FUNCTIONS

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This paper gives criteria, necessary or sufficient for a vector-valued function $F = (f_1, f_2, \dots, f_k)$ to be invex. Here each f_i is of the $C^{11}_{x_0}$-class (that is, each f_i is a function whose gradient mapping is locally Lipschitz in a neighbourhood of x_0) and the invexity of F means that $F(x) - F(x_0) \subset \overset{o}{F}'(X) + Q$ for a fixed convex cone Q of R^k and every x near x_0 ($\overset{o}{F}'$ being the Jacobian matrix of F at x_0).

1. INTRODUCTION

Let $X = R^n, Y = R^k$ be Euclidean spaces of dimensions n and k, respectively. Let $F : X \to Y$ be a vector-valued function with Fréchet differentiable components $f_i(\cdot)$ $(i = 1, 2, \dots, k)$. Given a convex cone $Q \subset Y$ and a vector-valued function $\eta : X \to X$, we say that F is Q-invex at x_0 in a neighbourhood of x_0, with respect to η, if there is a positive number γ such that $\|x - x_0\| \leqslant \gamma$ implies

$$F(x) - F(x_0) \in \overset{o}{F}'\eta(x) + Q \tag{1.1}$$

where $\overset{o}{F}'$ denotes the Fréchet derivative of F at x_0 and $\|x\|$ stands for the norm of x. The idea of invexity for a function was first introduced by Hanson [3] who showed that the requirement of invexity is weaker than the requirement of convexity, but it still assures the validity of the converse Kuhn-Tucker condition and the Wolfe duality theory [11] for mathematical programming problems. The above Q-invexity concept was given in [2] where a condition, necessary or sufficient , for invexity of F is expressed in terms of its second derivative. This result was obtained assuming that $f_i(\cdot)$ are C^2-functions . But the twice continuous differentiability hypothesis used in [2] is too strong a requirement and, as was shown in [5], it is not satisfied for many optimisation problems such as problems with C^{11}-data (that is, problems with functions whose gradient mappings are locally Lipschitz). The aim of this paper is to give some necessary or sufficient criteria of invexity for C^{11}-maps, one of which includes

Received 11th May, 1994
This paper was supported in part by the National Basic Research Program in Natural Sciences, Vietnam.

as a special case the above result of Craven [**2**]. The organisation of the paper is as follows. Section 2 recalls the concept of C^{11}-functions and their properties [**5**]. Section 3 presents two invexity criteria, the first of which deals with the case where η is a C^{11}-map while in the second one this property of η is not assumed to be satisfied. Section 4 gives some examples. Section 5 discusses other sufficient invexity conditions and their relationship with those of Jeyakumar [**6**] and Hanson-Rueda [**4**]. The reader who is interested in generalisations of invexity for nonsmooth maps and multifunctions, and their applications to optimisation problems and duality theories is referred to [**1, 7, 9, 10, 12**].

2. Preliminaries

In this paper, elements of finite-dimensional spaces are identified with column vectors. The symbols $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are used to denote the inner product and the norm in these spaces. For $x_0 \in X = R^n$ and $\gamma > 0$, $B(x_0, \gamma)$ is the ball of radius γ centered at x_0. The closure and the interior of a set $A \subset X$ are denoted by $\overline{A}$ and $\operatorname{int} A$, respectively. The cone generated by A is the set

$$\operatorname{cone} A = \{\alpha x : \alpha > 0,\ x \in A\}.$$

Instead of $\overline{\operatorname{cone} A}$ we write $\overline{\operatorname{cone}}\, A$. The positive polar cone of the cone $Q \subset Y = R^k$ is the set

$$Q^+ = \{y \in Y : \langle y, \widetilde{y}\rangle \geqslant 0 \qquad (\forall \widetilde{y} \in Q)\}.$$

The gradient vector and the Hessian matrix of a twice differentiable real-valued function f at x are denoted by $f'(x)$ and $f''(x)$. The Hessian matrix is an element of the space E of $n \times n$ matrices which is topologised by taking a matricial norm $|||\cdot|||$ on it. To every element A of E we associate a bilinear form $A[.,.]$.

Let $A = A_1 \times A_2 \times \ldots \times A_k$ be an element of the direct product E^k of k spaces E. For $v \in X$, we denote the vector in Y with components $A_i[v,v]$ by the symbol $A[v,v]$. For $y = (y_1, y_2, \ldots, y_k)^T \in Y$, we set

$$yA = \sum_{i=1}^{k} y_i A_i, \tag{2.1}$$

that is, yA is a linear combination of the matrices A_i with coefficients y_i. Hence the inner product of y and $A[v,v]$ is equal to $yA[v,v]$. If G is a $k \times n$-matrix and $A \in E^n$, then the symbol GA stands for the element of E^k with components $g_i^T A$ where ^{T} denotes the transpose and g_i are the row vectors of G:

$$GA = (g_1^T A) \times (g_2^T A) \times \ldots \times (g_k^T A) \in E^k.$$

Using (2.1) we can write

$$y(GA) = \sum_{i=1}^{k} y_i(g_i^T A) = (G^T y)A \tag{2.2}$$

for every $y \in Y$. Fix $x_0 \in X$. Let $C_{x_0}^{11}$ be the class of all real-valued functions which are differentiable in a neighbourhood of x_0 and whose gradient mapping $f'(\cdot)$ is locally Lipschitz on this neighbourhood. The generalised Hessian matrix of f at x_0, denoted by $\partial^2 f(x_0)$, is defined in [5] as the convex hull of the matrices each of which can be expressed as the limit of a sequence $f''(x_i)$ where f is twice differentiable at x_i and $x_i \to x_0$.

The following properties of the generalised Hessian matrix of f were established in [5]:

1. $\partial^2 f(x_0)$ is a nonempty compact convex set of E.
2. The set-valued map $x \mapsto \partial^2 f(x)$ is locally bounded at x_0, that is

$$\sup\{|||A||| : A \in \partial^2 f(x),\ x \in V\} \leqslant \beta$$

where V is a neighbourhood of x_0 and $\beta > 0$ is some constant.

3. The set-valued map $x \mapsto \partial^2 f(x)$ is upper semicontinuous at x_0 in the sense that

$$[x_i \to x_0,\ A_i \to A_0,\ A_i \in \partial^2 f(x_i)] \Rightarrow A_0 \in \partial^2 f(x_0).$$

4. (Second order Taylor expansion Theorem) If f is a $C_{x_0}^{11}$-function then there is a neighbourhood V of x_0 such that for any pair $x_1 \in V$, $x_2 \in V$ there are $\alpha \in (0,1)$ and $A \in \partial^2 f(x_1 + \alpha(x_2 - x_1))$ such that

$$f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) + 2^{-1}A[x_2 - x_1, x_2 - x_1].$$

If $F = (f_1, f_2, \ldots, f_k)^T$ is a $C_{x_0}^{11}$-map (that is, a map with f_i being $C_{x_0}^{11}$-functions in a neighbourhood of x_0), then we set $\partial^2 F(\cdot) = \partial^2 f_1(\cdot) \times \partial^2 f_2(\cdot) \times \ldots \times \partial^2 f_k(\cdot)$. Hence, for any x in a neighbourhood of x_0, each element of $\partial^2 F(x)$ is a point of E^k with components belonging to $\partial^2 f_i(x)$.

We conclude this section with the following technical lemma.

Lemma 2.1. *Let $P : X \to E^k$ be a multifunction such that P is upper semicontinuous and locally bounded at $x_0 \in X$. Let $Q \subset Y$ be a convex cone with nonempty interior. If*

$$(\forall v \in X_0)(\forall A \in P(x_0)) \qquad A[v,v] \in \operatorname{int} Q \tag{2.3}$$

then there is a positive number γ such that

$$(2.4) \qquad (\forall v \in X_0)(\forall x \in B(x_0,\gamma))(\forall A \in P(x)) \qquad A[v,v] \in \operatorname{int} Q.$$

Here and in the sequel $X_0 = X \setminus \{0\}$.

PROOF: Assume to the contrary that the lemma fails to hold. Then there are sequences $\gamma_i \downarrow 0$, $v_i \in X_0$, $A_i \in P(x_i)$, $x_i \in B(x_0,\gamma_i)$ such that $A_i[v_i,v_i] \in Y \setminus \operatorname{int} Q$ or, equivalently,

$$(2.5) \qquad A_i[u_i,u_i] \in Y \setminus \operatorname{int} Q$$

where $u_i = v_i \left\|v_i\right\|^{-1}$. Using subsequences if necessary, we may assume that u_i and A_i converge to elements $u \in X$ $(\|u\| = 1)$ and $A \in P(x_0)$, respectively. Letting $i \to \infty$ in (2.5) and noting that $Y \setminus \operatorname{int} Q$ is a closed set, we get $A[u,u] \in Y \setminus \operatorname{int} Q$, a contradiction to (2.3). ∎

3. Invexity criteria

Let $x_0 \in X = R^n$ and let $Q \subset Y = R^k$ be a convex cone. Throughout this paper, unless otherwise specified, we assume that $F = (f_1, f_2, \dots, f_k)^T$ is a $C^{11}_{x_0}$-map. For simplicity we shall write $\overset{o}{F}{}', \partial^2 \overset{o}{F}$ instead of $F'(x_0)$ and $\partial^2 F(x_0)$. Similarly, for the $C^{11}_{x_0}$-map $\eta : X \to X$, we shall use $\overset{o}{\eta}, \overset{o}{\eta}{}'$ and $\partial^2 \overset{o}{\eta}$ in place of $\eta(x_0), \eta'(x_0)$ and $\partial^2 \eta(x_0)$, respectively. We set $Q_0 = \overline{\operatorname{cone}}\left(Q + \overset{o}{F}{}'\overset{o}{\eta}\right)$ and we denote by I the identity map. We write $\overset{o}{P}$ in place of $\partial^2 \overset{o}{F} - \overset{o}{F}{}'\partial^2 \overset{o}{\eta}$. Observe that $D \in \overset{o}{P}$ means that $D = M - \overset{o}{F}{}'N$ with suitable $M \in \partial^2 \overset{o}{F}$ and $N \in \partial^2 \overset{o}{\eta}$.

THEOREM 3.1.

1. Assume that F is Q-invex in a neighbourhood of x_0, with respect to a $C^{11}_{x_0}$-map η. Then

$$(3.1) \qquad (\forall v \in X) \qquad \overset{o}{F}{}'\left(I - \overset{o}{\eta}{}'\right)v \in Q_0$$

$$(3.2) \qquad (\forall v \in X_0)\left(\exists D \in \overset{o}{P}\right) \quad D[v,v] \in Q_0.$$

2. Conversely, assume that there is a $C^{11}_{x_0}$-map η such that the condition (3.1) and, instead of (3.2), the condition

$$(3.3.) \qquad (\forall v \in X_0)\left(\forall D \in \overset{o}{P}\right) \quad D[v,v] \in \operatorname{int} Q_0$$

are satisfied. Then there is a function $\lambda : X \to R$ such that F is Q-invex at x_0 in a neighbourhood of x_0, with respect to the map $\eta_1(\cdot) = \eta(\cdot) + \lambda(\cdot)\overset{o}{\eta}$.

PROOF:

1. *Sufficiency.* Consider the map $P(\cdot) = \partial^2 F(\cdot) - \overset{o}{F}{}'\partial^2\eta(\cdot)$. Since $\overset{o}{P} = P(x_0)$ and since condition (3.3) holds, by Lemma 2.1 there is $\gamma > 0$ such that

$$(\forall v \in X_0)(\forall x \in B(x_0,\gamma)) \quad (\forall A \in P(x)) \quad A[v,v] \in \operatorname{int} Q_0. \tag{3.4}$$

Let us fix $v \in B(0,\gamma)$. By the second order Taylor expansion theorem we have

$$f_i(x_0+v) - f_i(x_0) = f'(x_0)v + 2^{-1}M_i[v,v] \tag{3.5}$$

where M_i is a suitable element of $\partial^2 f_i(x_0+\theta_i v)$ $(0 < \theta_i < 1)$. Denoting by M the element of E^k with components M_i we can write

$$F(x_0+v) - F(x_0) = \overset{o}{F}{}'v + 2^{-1}M[v,v]. \tag{3.6}$$

This equality together with (3.1) yields

$$F(x_0+v) - F(x_0) = \overset{o}{F}{}'\overset{o}{\eta}{}'v + 2^{-1}M[v,v] + q \tag{3.7}$$

where q is some point of Q_0.

On the other hand, applying the second order Taylor expansion theorem to $\eta(\cdot)$ gives

$$\overset{o}{\eta}{}'v = \eta(x_0+v) - \eta(x_0) - 2^{-1}N[v,v] \tag{3.8}$$

where N is an element of E^n with suitable components $N_i \in \partial^2\eta_i(x_0+\beta_i v)$ $(0 < \beta_i < 1)$. Therefore, (3.7) can be rewritten as

$$\begin{aligned} F(x_0+v) - F(x_0) - \overset{o}{F}{}'(\eta(x_0+v) - \eta(x_0)) &= q + 2^{-1}\left(M - \overset{o}{F}{}'N\right)[v,v] \\ &\in q + \operatorname{int} Q_0 \subset \dot{Q}_0 + \operatorname{int} Q_0 \subset \operatorname{int} Q_0, \end{aligned} \tag{3.9}$$

since (3.4) implies that $\left(M - \overset{o}{F}{}'N\right)[v,v] \in \operatorname{int} Q_0$. Using (3.9) we can easily verify that

$$F(x_0+v) - F(x_0) - \overset{o}{F}{}'[\eta(x_0+v) - \eta(x_0)] \in \operatorname{cone}\left(Q + \overset{o}{F}{}'\overset{o}{\eta}\right),$$

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that is, for some $\widetilde{\lambda} = \widetilde{\lambda}(v) > 0$, we have

$$F(x_0 + v) - F(x_0) - \overset{o}{F}'[\eta(x_0 + v) - \eta(x_0)] - \widetilde{\lambda}\overset{o}{F}'\overset{o}{\eta} \in Q,$$

that is

$$F(x_0 + v) - F(x_0) \in \overset{o}{F}'\eta_1(x_0 + v) + Q$$

with $\eta_1(x_0 + v) = \eta(x_0 + v) + \lambda\overset{o}{\eta}$ where $(\lambda = \widetilde{\lambda} - 1)$. This shows that F is Q-invex at x_0 in $B(x_0, \gamma)$, with respect to η_1.

2. *Necessity.* By assumption we have, for v sufficiently small,

$$F(x_0 + v) - F(x_0) - \overset{o}{F}'\eta(x_0 + v) \in Q. \tag{3.10}$$

Using (3.6) and (3.8) we derive from (3.10)

$$\overset{o}{F}'\left(I - \overset{o}{\eta}'\right)v + 2^{-1}\left(M - \overset{o}{F}'N\right)[v, v] \in Q + \overset{o}{F}'\overset{o}{\eta} \subset Q_0. \tag{3.11}$$

For fixed $\overline{v} \in X_0$, taking a sequence $\alpha_j \downarrow 0$, applying (3.11) to $v = \alpha_j\overline{v}$ and dividing by α_j, we get

$$\overset{o}{F}'\left(I - \overset{o}{\eta}'\right)\overline{v} + 2^{-1}\alpha_j\left(\widetilde{M}_j - \overset{o}{F}'\widetilde{N}_j\right)[\overline{v}, \overline{v}] \in \alpha_j^{-1}Q_0 \subset Q_0 \tag{3.12}$$

where $\widetilde{M}_j$ is an element of E^k whose i^{th} component is a $n \times n$-matrix $\widetilde{M}_{ij} \in \partial^2 f_i(x_0 + \theta_{ji}\alpha_j v)$ $(0 < \theta_{ji} < 1)$, and similarly for $\widetilde{N}_j$. Letting $j \to \infty$ in (3.12) and noting the local boundedness property of generalised Hessian matrices (see Section 2), we obtain (3.1) for $v = \overline{v}$. The first property of Theorem 3.1 is thus proved.

Now we rewrite (3.12) as

$$\left(\widetilde{M}_j - \overset{o}{F}'\widetilde{N}_j\right)[\overline{v}, \overline{v}] \in 2\alpha_j^{-1}\left(Q_0 + \overset{o}{F}'\left(I - \overset{o}{\eta}'\right)(-\overline{v})\right) \in 2\alpha_j^{-1}(Q_0 + Q_0) \subset Q_0, \tag{3.13}$$

using (3.1) with $v = -\overline{v}$. Based on the second and third properties of generalised Hessian matrices (see Section 2), we may assume (by taking subsequences if necessary) that the sequences $\widetilde{M}_j$ and $\widetilde{N}_j$ converge to some elements M and N belonging to $\partial^2\overset{o}{F}$ and $\partial^2\overset{o}{\eta}$, respectively. To complete the proof of (3.2) with $v = \overline{v}$ and $D = M - \overset{o}{F}'N$, it remains to let $j \to \infty$ in (3.13) and to observe the closedness of Q_0. $\square$

REMARK 3.1. If $\overset{o}{\eta} = 0$ then the conclusion of the sufficiency part of Theorem 3.1 means that F is Q-invex at x_0 in a neighbourhood of x_0 with respect to the same η for which conditions (3.1) and (3.3) are satisfied. Example 4.3 of Section 4 shows that, for the case $\overset{o}{\eta} \neq 0$, this conclusion fails to hold (that is, F can be Q-invex with respect to $\eta_1(\cdot) = \eta(\cdot) + \lambda(\cdot)\overset{o}{\eta}$, but cannot be Q-invex with respect to η).

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REMARK 3.2. Theorem 3.1 includes as a special case Theorem 2 in [**2**] where Craven assumes that Q is a closed convex cone with $Q \cap (-Q) = \{0\}$, F and η are twice continuously differentiable, and $\eta(x_0) = 0$, $\eta'(x_0) = I$.

REMARK 3.3. The requirement of the existence of a $C^{11}_{x_0}$-map η satisfying the assumptions of Theorem 3.1 is a difficult matter. On the other hand, in the definition of Q-invexity of F, the map η is not required to be differentiable . In fact, we know from [**3**] that even the continuity property of η is not used for proving the converse Kuhn-Tucker condition and the Wolfe duality in mathematical programming . It is then natural to ask whether we can find invexity conditions without assuming continuity or differentiability properties of η. Before giving such a condition, we set

$$\begin{aligned} Q_1 &= \{y \in Q^+ : \|y\| = 1,\ \overset{o}{F}{}'^T y = 0\}. \\ \Gamma &= \overset{o}{F}{}'(X) + Q := \{y : y = \overset{o}{F}{}' x + q,\ x \in X,\ q \in Q\}. \end{aligned} \tag{3.14}$$

THEOREM 3.2.

1. If F is Q-invex at x_0 in a neighbourhood of x_0 (with respect to some η) then

$$(\forall v \in X_0)\left(\exists M \in \partial^2 \overset{o}{F}\right) \qquad \inf_{y \in Q_1} yM[v,v] \geqslant 0. \tag{3.15}$$

2. Conversely, if

$$(\forall v \in X_0)\left(\forall M \in \partial^2 \overset{o}{F}\right) \qquad \inf_{y \in Q_1} yM[v,v] > 0 \tag{3.16}$$

then F is Q-invex at x_0 in a neighbourhood of x_0 (with respect to some η).

PROOF:

1. *Sufficiency.* It follows from (3.16) that there exists $\gamma > 0$ such that

$$(\forall v \in X_0)(\forall x \in B(x_0,\gamma))\big(\forall M \in \partial^2 F(x)\big) \qquad M[v,v] \in \Gamma. \tag{3.17}$$

Indeed, otherwise there are sequences $\gamma_i \downarrow 0$, $v_i \in X_0$, $x_i \to x_0$, $M_i \in \partial^2 F(x_i)$ such that $M_i[v_i,v_i] \notin \Gamma$. Since Γ is a cone we derive from this condition that $M_i[u_i,u_i] \notin \Gamma$ where $u_i = v_i \|v_i\|^{-1}$. In view of the separation theorem we can find $y_i \in Y$, $\|y_i\| = 1$, such that

$$y_i M_i[u_i,u_i] \leqslant 0 \leqslant \langle y_i, y\rangle \qquad \forall y \in \Gamma. \tag{3.18}$$

The second inequality in (3.18) implies that $y_i \in Q_1$. Taking subsequences if necessary, we may assume that sequences u_i, M_i and y_i converge to some elements $u \in X_0$, $M \in$

$\partial^2 \overset{o}{F}$ and $y \in Q_1$, respectively. Hence, by letting $i \to \infty$ in the first inequality in (3.18) we obtain $yM[v,v] \leqslant 0$, a contradiction to (3.16).

We now claim that F is Q-invex at x_0 in $B(x_0,\gamma)$. Indeed, for every $v \in B(0,\gamma)$, consider (3.6). Using (3.17) we derive from (3.6) that

$$F(x_0+v)-F(x_0) \in \overset{o}{F}{}'v + \overset{o}{F}{}'(X) + Q = \overset{o}{F}{}'(X) + Q,$$

showing the Q-invexity of F at x_0.

2. *Necessity.* By invexity, for v sufficiently small, $F(x_0+v)-F(x_0) \in \Gamma$. This condition together with (3.6) yields

$$\begin{aligned} M[v,v] &\in 2\left(\Gamma - \overset{o}{F}{}'v\right) = 2[\overset{o}{F}{}'(X) + Q + \overset{o}{F}{}'(-v)] \\ &\in 2[\overset{o}{F}{}'(X-v) + Q] = \overset{o}{F}{}'(X) + Q = \Gamma. \end{aligned} \tag{3.19}$$

For fixed $\overline{v} \in X$, take a sequence $\alpha_i \downarrow 0$. Applying (3.19) to $v = \alpha_i \overline{v}$ and using an argument similar to that given in the proof of Theorem 3.2, we can show that there is $\widetilde{M} \in \partial^2 \overset{o}{F}$ such that $\widetilde{M}[\overline{v},\overline{v}] \in \overline{\Gamma}$. Since $y \in Q_1$ implies that $y \in \Gamma^+$ we conclude that $y\widetilde{M}[v,v] \geqslant 0$ for all $y \in Q_1$. This completes the proof of the necesssity part of the theorem. ∎

Remark 3.4. Condition (3.16) may be satisfied while condition (3.3) of Theorem 3.1 fails to hold for any $C^{11}_{x_0}$-map η (see Example 4.4 of Section 4).

Corollary 3.1. *Assume that there are $N \in E^n$ and $\xi \in X$ such that*

$$(\forall v \in X_0)\left(\forall M \in \partial^2 \overset{o}{F}\right) \quad \left(M - \overset{o}{F}{}'N\right)[v,v] \in \operatorname{int} \overline{\operatorname{cone}}\left(Q + \overset{o}{F}{}'\xi\right). \tag{3.20}$$

Then F is Q-invex at x_0 in a neighbourhood of x_0, with respect to some η.

Proof: Since $\overline{\operatorname{cone}}\left(Q + \overset{o}{F}{}'\xi\right) \subset \overline{\Gamma}$, (3.20) implies that $\left(M - \overset{o}{F}{}'N\right)[v,v] \in \operatorname{int} \overline{\Gamma}$. Hence, for any $y \in \left(\overline{\Gamma}\right)^+$ with $\|y\| = 1$, we have

$$yM[v,v] - y\left(\overset{o}{F}{}'N\right)[v,v] = y\left(M - \overset{o}{F}{}'N\right)[v,v] > 0. \tag{3.21}$$

On the other hand, we verify easily that

$$\{y \in \left(\overline{\Gamma}\right)^+ : \|y\| = 1\} = Q_1.$$

So, using (3.21) and (2.2) we obtain (3.16), and the corollary follows now from Theorem 3.2. ∎

REMARK 3.5. When comparing sufficiency conditions given by Theorem 3.1 and Corollary 3.1 we see that, if we are not interested in the form of the map with respect to which F is Q-invex, then the sufficiency part of Theorem 3.1 can be weakened, assuming that

$$\left(\exists N \in \partial^2 \overset{o}{\eta}\right)(\forall v \in X_0)\left(\forall M \in \partial^2 \overset{o}{F}\right) \qquad \left(M - \overset{o}{F}{}' N\right)[v, v] \in \operatorname{int} Q_0.$$

4. EXAMPLES

Some examples will be given to illustrate the results of the previous section. Observe from the first example that sufficient condition (3.3) of Theorem 3.1 (respectively, sufficient condition (3.16) of Theorem 3.2) cannot be replaced by

$$(4.1) \qquad (\forall v \in X_0)\left(\exists D \in \overset{o}{P}\right) \qquad D[v, v] \in \operatorname{int} Q_0,$$

$$(4.2) \qquad \left(\text{respectively } (\forall v \in X_0)\left(\exists M \in \partial^2 \overset{o}{F}\right) \qquad \inf_{y \in Q_1} yM[v, v] > 0\right).$$

The second example shows that necessary condition (3.2) of Theorem 3.1 (respectively, necessary condition (3.15) of Theorem 3.2) cannot be replaced by

$$(4.3) \qquad (\forall v \in X_0)\left(\forall D \in \overset{o}{P}\right) \qquad D[v, v] \in Q_0,$$

$$(4.4) \qquad \left(\text{respectively } (\forall v \in X_0)\left(\forall M \in \partial^2 \overset{o}{F}\right) \qquad \inf_{y \in Q_1} yM[v, v] \geqslant 0\right).$$

In examples 4.1-4.3 we shall assume that $X = Y = R$ (the real line), $Q = R_+$ (the nonnegative half-line), $x_0 = 0$ and $F = f$ is a real-valued function.

EXAMPLE 4.1. Let

$$f(x) = \begin{cases} 2^{-1}\alpha x^2 & \text{if} \quad x \geqslant 0, \\ 2^{-1}\beta x^2 & \text{if} \quad x < 0 \end{cases}$$

where α, β are real parameters such that $\beta \leqslant \alpha$. Obviously, f is a $C^{11}_{x_0}$-function and $\partial^2 \overset{o}{f} = \{\gamma : \beta \leqslant \gamma \leqslant \alpha\}$. We shall set $\eta(x) = x$. The R_+-invexity property of f depends on the choice of β and α. Indeed,

- If $\beta > 0$, then f is R_+-invex, using the sufficient condition of Theorem 3.1 or Theorem 3.2.
- If $\alpha < 0$, then f is not R_+-invex since the necessary condition of Theorem 3.1 or Theorem 3.2 is violated.
- If $\beta < 0$ and $\alpha > 0$, then (4.1) and (4.2) are satisfied with $M = \gamma \in (0, \alpha)$, but f is not R_+-invex.

EXAMPLE 4.2. [5] Let

$$f(x) = \int_0^{|x|} \varphi(t)dt$$

where

$$\varphi(t) = \begin{cases} 2t^2 + t^2 \sin t^{-1} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t = 0. \end{cases}$$

It was shown in [5] that $x_0 = 0$ is a local minimum of f and $\partial^2 \overset{o}{f} = [-1, +1]$. Hence f is R_+-invex in a neighbourhood of x_0 with respect to $\eta(x) \equiv 0$, but (4.3) and (4.4) cannot be satisfied.

EXAMPLE 4.3. Let $f(x) = 2^{-1}x^2 + \sin x$ and $\eta(x) = 4^{-1}x^2 + \sin x + \cos x$. A simple calculation yields $\overset{o}{f} = 0$, $\overset{o}{f}{}' = 1$, $\partial^2 \overset{o}{f} = f''(0) = \{1\}$; $\overset{o}{\eta} = 1$, $\overset{o}{\eta}{}' = 1$, $\partial^2 \overset{o}{\eta} = \eta''(0) = \{-2^{-1}\}$. Hence, $\overset{o}{\eta} = 1 \neq 0$, and conditions (3.1) and (3.3) of Theorem 3.1 are satisfied. But F cannot be Q-invex at x_0 in a neighbourhood of x_0, with respect to η. Indeed, otherwise we must have $f(x) \geqslant \eta(x)$, that is $\cos x \leqslant 4^{-1}x^2$ for all x near $x_0 = 0$. Of course, this condition is impossible. Observe that by using $\eta_1(x) = \eta(x) - \cos x$ instead of η we can verify that f is R_+-invex at x_0 in a neighbourhood of x_0 with respect to η_1 (that is, the conclusion of the sufficiency part of Theorem 3.1 holds for $\lambda(x) = -\cos x$).

EXAMPLE 4.4. Let $X = R$, $Y = R^2$, $Q = \{y = (y_1, y_2) : y_2 = 0, y_1 \geqslant 0\}$, $x_0 = 0$ and $F = (f_1, f_2)^T$ where $f_1(x) = x$, $f_2(x) = x^2$. Since int $Q_0 =$ int Q is empty, assumption (3.3) of Theorem 3.1 cannot be satisfied for any $C^{11}_{x_0}$-map η. But sufficient condition (3.16) of Theorem 3.2 holds. Indeed, in our case $\partial^2 \overset{o}{F} = F''(x_0)$, Q_1 consists of the unique element $y = (0,1)^T \in R^2$ and a simple calculation gives $yF''(x_0)[v,v] = 2v^2$ which is positive for all $v \neq 0$.

5. OTHER SUFFICIENT INVEXITY CRITERIA

This section gives some invexity criteria which are closely related to those of Jeyakumar [6] and Hanson-Rueda [4]. We say that F is Q-invex at x_0 in a set $V \subset X$ with respect to η, if (1.1) holds for any $x \in V$. Obviously, Q-invexity in a neighbourhood of x_0 corresponds to the case $V = B(x_0, \gamma)$ for some $\gamma > 0$.

For $\theta = (\theta_1, \theta_2, \dots, \theta_k) \in R^k$ we set

$$\partial^2 F(x_0 + \theta v) = \partial^2 f_1(x_0 + \theta_1 v) \times \partial^2 f_2(x_0 + \theta_2 v) \times \dots \times \partial^2 f_k(x_0 + \theta_k v)$$

and we write $0 \leqslant \theta \leqslant 1$ if $0 \leqslant \theta_i \leqslant 1$ for all i.

Theorem 5.1. *Let V be a subset of X. Assume that either $Q_1 = \emptyset$ or the following two conditions are fulfilled:*

1. *$\Gamma := \overset{o}{F}'(X) + Q$ is a closed cone.*
2. *There is a multifunction $G : V \to Y = R^k$ such that*

$$(5.1) \qquad (\forall v \in V)(\forall \widetilde{y} \in G(v)) \qquad \inf_{y \in Q_1} < y, \widetilde{y} > \geqslant 0.$$

$$(5.2) \qquad (\forall v \in V)(\forall 0 \leqslant \theta \leqslant 1)\big(\forall M \in \partial^2 F(x_0 + \theta v)\big) \quad M[v,v] \subset G(v) + Q.$$

Then F is Q-invex at x_0 in the set $x_0 + V$ (with respect to some η).

Proof: If $Q_1 = \emptyset$ then $\Gamma = Y$ and the invexity property is obvious. Consider now the second case. Using a separation theorem and noting the closedness of Γ we can derive from (5.1) that $G(v) \subset \overset{o}{F}'(X) + Q$. Combining this inclusion with (3.6) and (5.2) yields

$$\begin{aligned} F(x_0 + v) - F(x_0) &\in \overset{o}{F}'v + 2^{-1}[G(v) + Q] \\ &\in \overset{o}{F}'v + \overset{o}{F}'X + Q = \overset{o}{F}'(X) + Q, \end{aligned}$$

showing the Q-invexity of F, as desired. ∎

Theorem 5.1 can be extended to differentiable map F which may not be of the $C^{11}_{x_0}$-class:

Theorem 5.2. *Let V be a subset of X and $F : X \to Y$ be an arbitrary differentiable map. Assume that either $Q_1 = \emptyset$ or the following two conditions are fulfilled:*

1. *Γ is a closed cone.*
2. *There is a multifunction $G : V \to Y = R^k$ such that the condition (5.1) and, instead of (5.2), the condition*

$$(5.3) \qquad (\forall v \in V) \qquad F(x_0 + v) - F(x_0) \in \overset{o}{F}'(X) + G(v) + Q$$

are satisfied.

Then F is Q-invex at x_0 in the set $x_0 + V$ (with respect to some η).

The proof is similar to that of Theorem 5.1.

Remark 5.1. Let us write $Y = R^k = R \times R^{k-1}$, let "*pr*" denote the projection of Y on R^{k-1} and let

$$Q_2 = \{y = (y_1, y_2, \dots, y_k)^T \in Q^+ : |y_1| = 1,\ \overset{o}{F}'^T y = 0\}.$$

If

$$pr\Gamma = R^{k-1} \tag{5.4}$$

then it can be verified that

$$[y = (y_1, y_2, \dots, y_k)^T \neq 0,\ y \in Q^+, \quad \overset{o}{F}{}'^T y = 0] \Rightarrow y_1 \neq 0.$$

From this we can deduce that

$$Q_1 = \emptyset \Longleftrightarrow Q_2 = \emptyset,$$
$$\inf_{y \in Q_1} \langle y, \widetilde{y} \rangle \geqslant 0 \Longleftrightarrow \inf_{y \in Q_2} \langle y, \widetilde{y} \rangle \geqslant 0.$$

Hence, under assumption (5.4) the cone Q_1 appearing in the formulation of Theorems 5.1 and 5.2 can be replaced by Q_2. (Similarly, under this assumption the cone Q_1 in (3.15) and (3.16) can be replaced by Q_2).

Observe that (5.4) holds if

$$f_2'(x_0), f_3'(x_0), \dots, f_k'(x_0) \quad \text{are linearly independent.} \tag{5.5}$$

Moreover, if $Q = R^k_+$ and $Q_2 \neq \emptyset$ then (5.5) implies that Q_2 is a singleton.

REMARK 5.2. In [6] a generalised version of invex functions, called ρ-invexity, was introduced. In our notation, the map F is called ρ-invex at x_0 in a set $x_0 + V$ ($V \subset X$) if

$$(\forall v \in V)(\exists \alpha(v) \in X) \qquad F(x_0 + v) - F(x_0) \in \overset{o}{F}{}'(X) + \rho \left\|\alpha(v)\right\|^2 + Q \tag{5.6}$$

where $Q = R^k_+$ is the nonnegative orthant of R^k and $\rho = (\rho_1, \rho_2, \dots, \rho_k)^T$ is a given vector of $Y = R^k$. We shall show that the ρ-invexity of F at x_0 in the set $x_0 + V$ implies the R^k_+-invexity of F provided that assumption (5.5) and the inequality

$$\langle y, \rho \rangle \geqslant 0 \qquad (y \in Q_2) \tag{5.7}$$

hold. Indeed, setting $G(v) = \rho \left\|\alpha(v)\right\|^2$ and taking account of Remark 5.1, we find that the second assumption of Theorem 5.2 is satisfied. The first one follows from [**8**, Theorem 20.3]. The desired conclusion is now a direct consequence of Theorem 5.2. Observe that condition (5.7) must be assumed to be valid when Jeyakumar [**6**] establishes the Wolfe duality theorem with ρ-invex data.

REMARK 5.3. In [4] Hanson and Rueda give a sufficient condition for the R^k_+-invexity of a C^2-map F, assuming that (5.5) holds, (5.6) is satisfied with

$$\alpha(v) = (1,0,\dots,0)^T \in X, \tag{5.8}$$

and

$$\rho_1 - \sum_{i=2}^{k} \rho_i f_1'(x_0)\xi_i \geqslant 0 \tag{5.9}$$

where ρ_i is a lower bound of $2^{-1} f_i''(x_0 + \beta v)[v, v]$ $\quad(\beta \in [0,1], v \in V)$ and $\xi_i \in X$ is a point such that

$$f_i'(x_0)\xi_j = \delta_{ij} \qquad \text{(the Kronecker symbol).} \tag{5.10}$$

As remarked above, if $Q_2 \neq \emptyset$ then Q_2 is a singleton denoted by $y = (y_1, y_2, \dots, y_k)$ with $y_1 = 1$. From (5.10) and condition $y \in Q_2$ it can be seen that $-f_1'(x_0)(\xi_j) = y_j$ for $j = 2, 3, \dots, k$. Hence, condition (5.9) introduced in [4] coincides with (5.7) and the sufficient condition for R^k_+-invexity given in [4] is also a direct consequence of Theorem 5.2.

Observe that, unlike [4], Theorems 5.1 and 5.2 can be used without assuming the linear independence of $f_i'(x_0)$ $\quad(i = 2, 3, \dots, k)$.

EXAMPLE 5.1. Let $X = Y = R^3$, $Q = R^3_+$, $V = B(0,1)$ and $x_0 = 0$. For $x = (x_1, x_2, x_3)^T \in R^3$ we define $F(x) = \left(x_1, x_2^2 + x_3^2, x_1^2 + x_2^2 + x_3^2\right)^T$. In our case, condition (5.5) fails to hold and, therefore, the above result of Hanson-Rueda does not apply. But the R^3_+-invexity of F at $x_0 = 0$ in $B(0,1)$ can be deduced from Theorem 5.1 or 5.2 since all assumptions of these theorems are satisfied, with $G(v) \equiv (0,0,0)^T$.

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