# ON THE ORIGINAL DISTRIBUTION OF THE ASTEROIDS

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**Abstract.** The depletion of an initially uniform distribution of asteroids extending from Mars to Saturn, caused by the gravitational perturbations of Jupiter and Saturn, is calculated by numerical integration of the asteroid orbits. Almost all (about 85%) the asteroids between Jupiter and Saturn are ejected in the first 6000 years. Most of the asteroids between the 2/3 Jupiter resonance (4.0 AU) and Jupiter are ejected in the first 2400 years with the exception of the stable librators (e.g., the Hilda group). Interior to the 2/3 resonance the depletion was small, and interior to the 1/2 resonance (3.3 AU), no asteroids were ejected in the first 2400 years.

## 1. Introduction

The recently reported Palomar-Leiden Survey (PLS) of asteroids (Van Houten *et al.*, 1970) was complete, in a selected area (12° × 18°) of the sky, to apparent magnitude 20. The PLS, apparently free from selection effects, confirmed in some detail the features of the space distribution of the asteroids that were established in previous studies (see, for example, Brown *et al.*, 1967, for a pictorial representation of the qualitative aspects of asteroid statistics).

In Figure 1, we present – in the form of a histogram – the surface number density (i.e., number of asteroids with $a_0 \leqslant a \leqslant a_0 + \delta a_0$ divided by $2\pi a_0 \delta a_0$) for the 981 PLS asteroids with well-determined orbits.

The 'Kirkwood gaps' at positions where the asteroid periods are 1/3, 2/5, and 3/7 of Jupiter's period are apparent in this presentation, and have – in the past – received the most attention from celestial mechanicians. We refer the reader to the classic paper by Brouwer (1963) for an elegant treatment of the gaps.

We were struck by the precipitous fall-off of the surface density at the 1/2 resonance. We find it difficult to imagine any formation mechanism (such as accretion or gravitational instability) that would vary so drastically over so small a scale. Furthermore, the existence of stable librators at the 2/3 Jupiter resonance (the 'Hilda' group) and at the 1/1 Jupiter resonance (the 'Trojans') suggests that whatever mechanisms formed the asteroids operated also at those larger distances. Likewise, we are uncomfortable with the notion that a catastrophic event (e.g., an exploded planet) created the asteroids, because of the low probability of capturing so many fragments in the 2/3 and 1/1 libration regions. By the same token, we reject the idea that a catastrophic event removed asteroids exterior to the 1/2 resonance, leaving the librators intact.

Thus, we are led to suggest that the asteroids had initially a more or less uniform surface density in the region between Mars and Jupiter and that the present asteroids, tightly confined between 2.0 and 3.3 AU, are the remnants that survived ejection by the gravitational perturbations of the planets. However, we hasten to point out that the present surface mass density in the asteroid belt is well below what one would infer from the 'heavy element' content of the planets. Figure 2 presents the surface density

*Y. Kozai (ed.), The Stability of the Solar System and of Small Stellar Systems*, 37–56. 

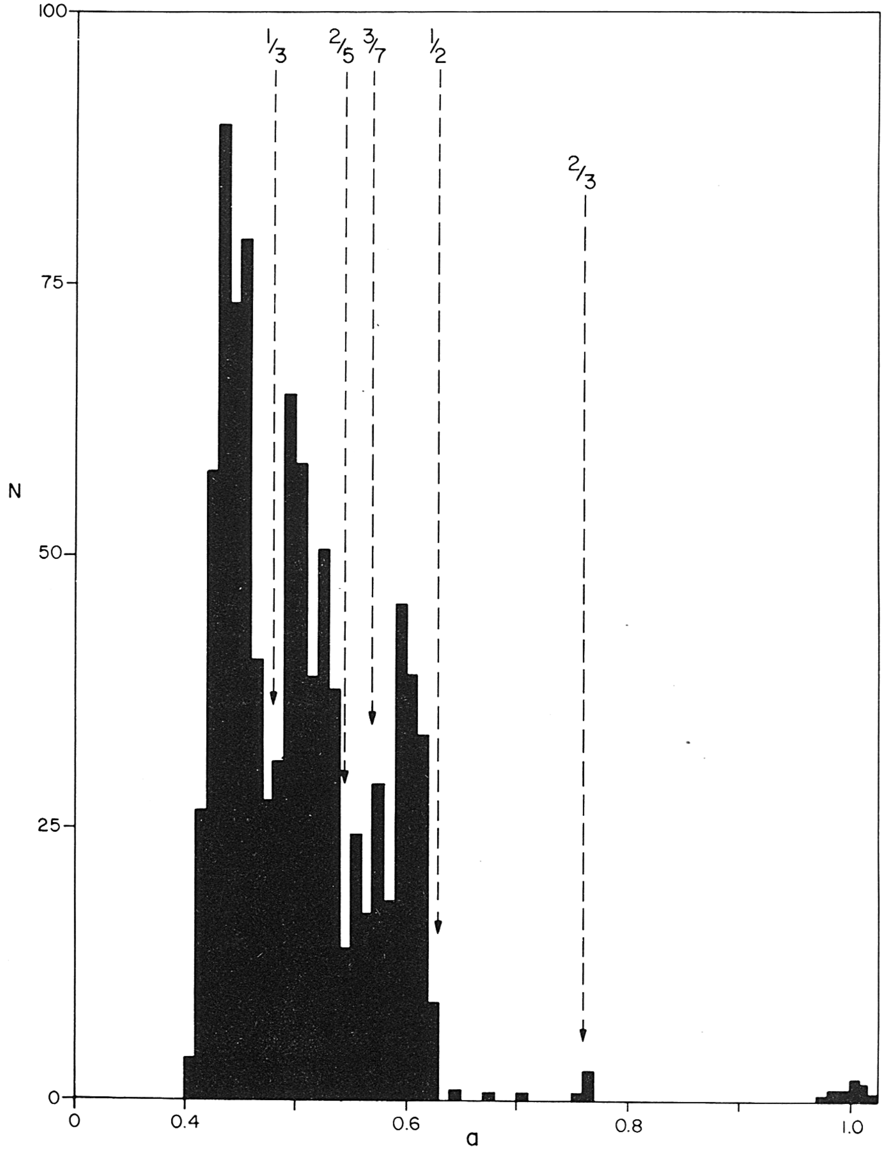


Fig. 1. Histogram of the surface number density vs semimajor axis ($a_J = 1$) of 981 first-class orbits from the Palomar-Leiden Survey of Faint Minor Planets.

heavy elements versus the distance from the Sun on the assumption that the planets collected all the heavy elements inside their present Lagrangian ($L_1$ to $L_2$) points (upper curve) or all the heavy elements between the successive planets (lower curve). The heavy-element content of the giant planets is taken as 1/200 of their present mass. In either case, the mass density in the asteroid belt is anomalously low. Our work sheds no light on this anomaly. We confine ourselves, in this study, to examining the gravita-

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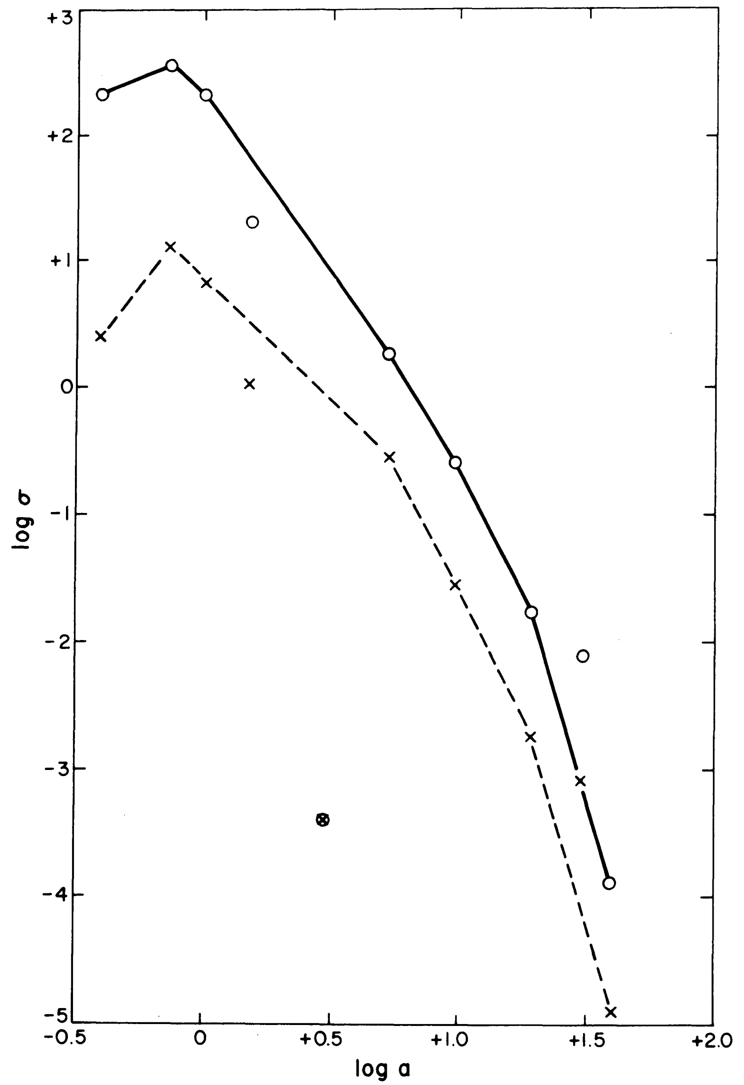


Fig. 2. Surface density in g cm$^{-2}$ vs log of the semimajor axis in AU of the 'heavy element' material, now collected into planets, when spread uniformly between the present values of the $L_1$ and $L_2$ points of each planet (upper curve – ○). Lower curve (×) assumes the material to be spread uniformly to a distance proportional to $L_1$ or $L_2$ but extended to fill the entire area between successive planets. The asteroids are at $\log a \cong +0.5$.

tional perturbations of the planets on an initially uniform (albeit low) distribution of asteroids.

Having supposed that the initial distribution of asteroids extended well past its present boundaries, we are led to ask whether initially there were also asteroids beyond Mars and Jupiter. In this preliminary survey, we look also at the region between Jupiter and Saturn.

We conducted this preliminary survey by numerically integrating the equations of motion (the integration algorithm is described elsewhere, Lecar *et al.*, 1973). Figure 3 gives the integration accuracies (errors in the energy and angular momentum) for a typical example. The errors are growing linearly with time, indicating that round-off errors are still negligible. The integration times are somewhat longer than the shortest

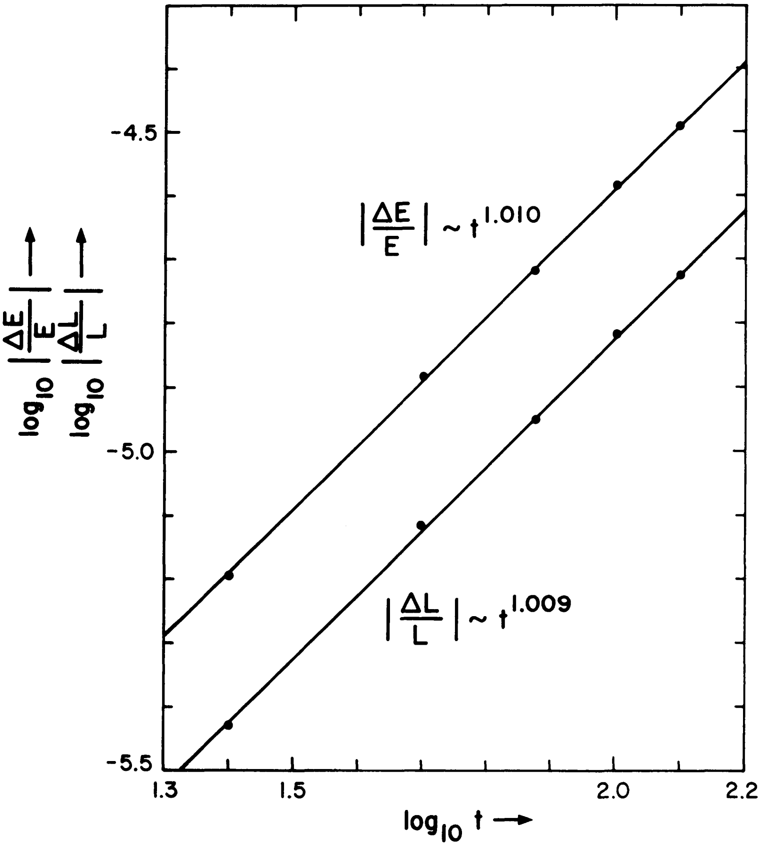


Fig. 3. Errors in energy and angular momentum as a function of time in periods of Jupiter for a typical calculation.

time scale ($\mu^{-1/2}$ Jupiter periods = 375 yr) and less than $\mu^{-1}$ Jupiter periods ≅ 12000 yr. We used numerical integration because, although we trust perturbation theory to give sufficiently accurate trajectories for stable asteroids, we do not trust perturbation theory to single out the unstable orbits of interest in this study. The combined effects of close approaches to Jupiter and overlapping resonances contribute to the invalidity of perturbation theory. We will return to this problem in a subsequent communication.

The results of a typical integration are shown in Figure 4, where the semimajor axes (in units of Jupiter's semimajor axis = 5.203 AU), are shown as a function of time (in units of Jupiter's period = 11.86 yr). The 1/2 resonance at 0.630 and the 2/3 resonance at 0.763 show up clearly in this presentation. Figure 5 is essentially a blowup of

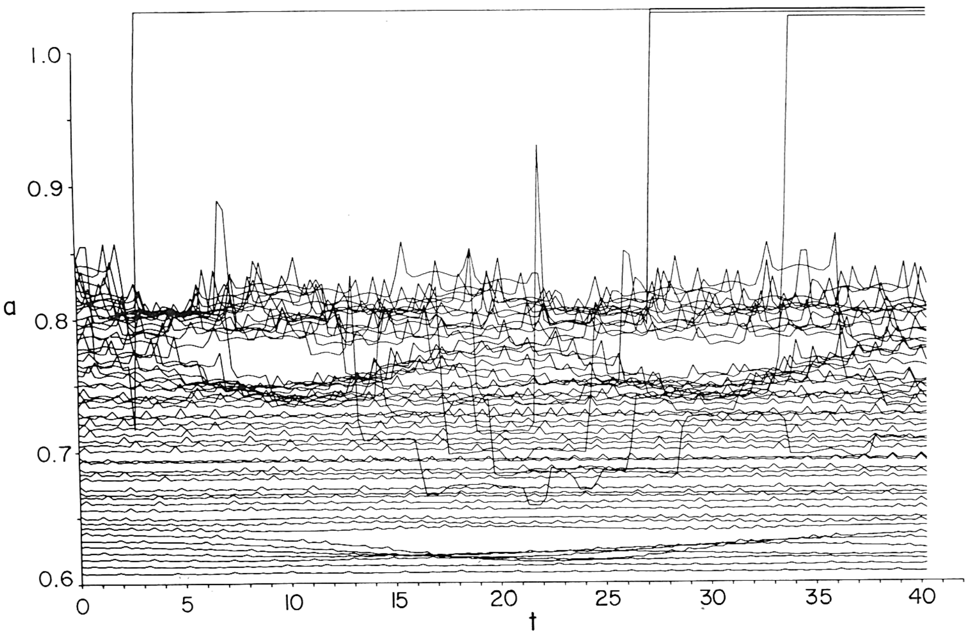


Fig. 4. Semimajor axis vs time in periods of Jupiter, where Jupiter's eccentricity = 0.0, for a set of asteroids whose initial eccentricities are all zero.

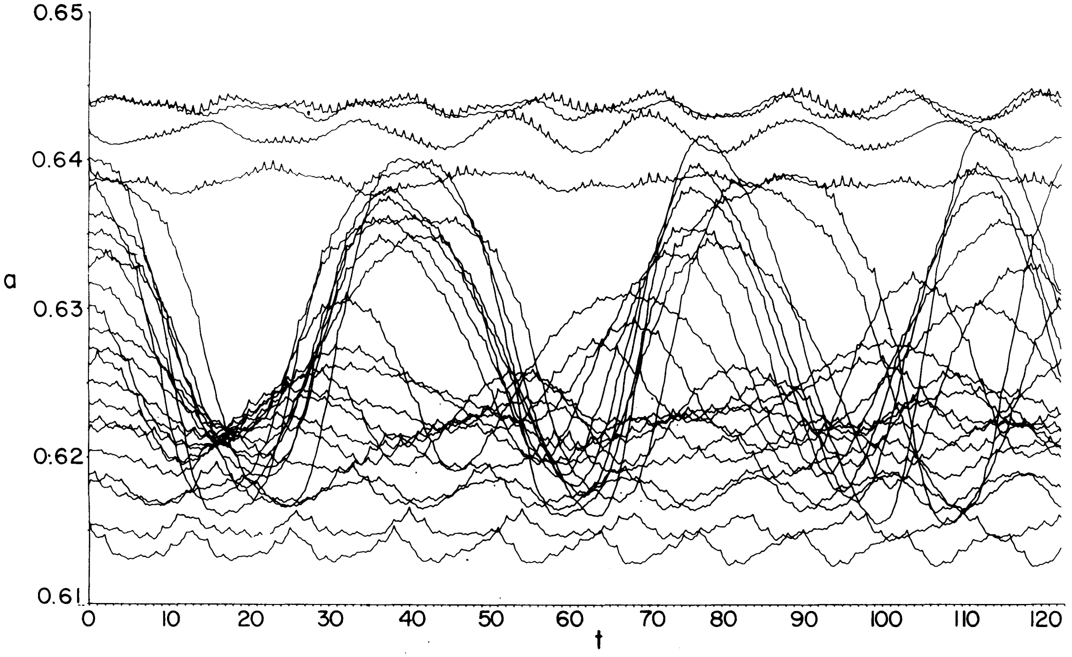


Fig. 5. Semimajor axis vs time in periods of Jupiter, where Jupiter's eccentricity = 0.06, for 25 bodies started near the 1/2 resonance, all with initial eccentricities of zero. Plot is slightly smoothed.

the 1/2 resonance showing the amplitude and width of the 1/2 resonance (0.01 Jupiter units or 0.05 AU) and the libration period, which is of the order of $\mu^{-1/2}$ Jupiter periods. Figure 6 also shows asteroids below the 1/2 resonance and illustrates the weakened effect of Jupiter's perturbations on these objects.

In the surveys to be described below, the ratio of Jupiter's mass to that of the Sun

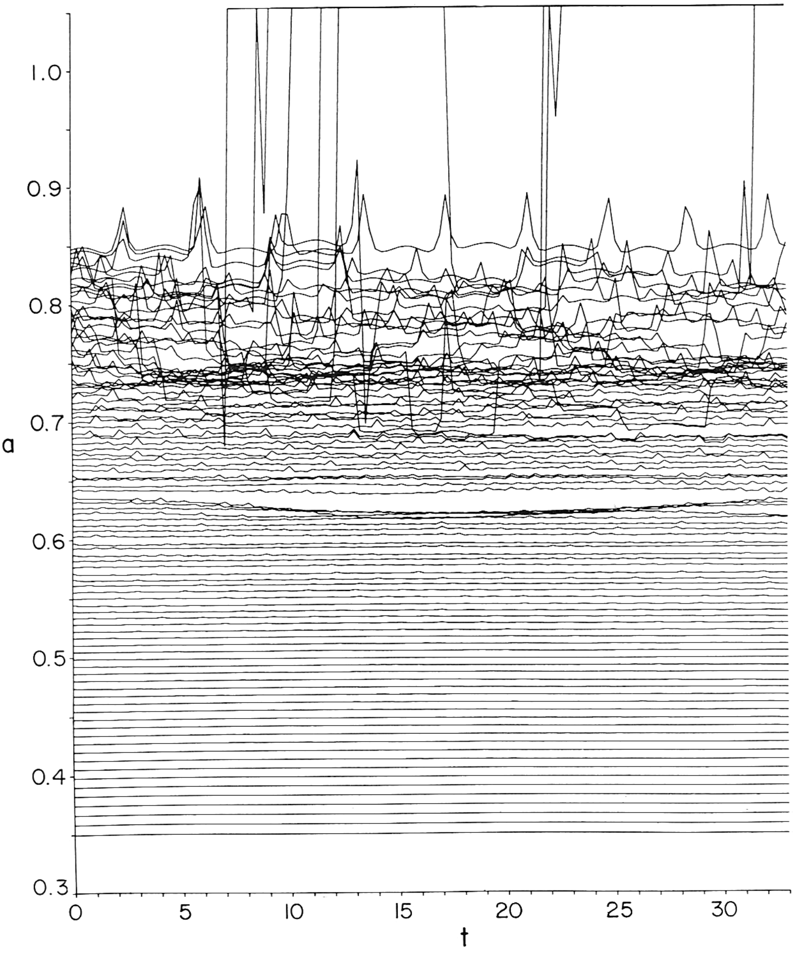


Fig. 6. Identical to Figure 4 except that Jupiter is given its maximum eccentricity of 0.06.

was taken as $10^{-3}$, and Jupiter's eccentricity was given its maximum value of 0.06. Furthermore, all the asteroids are in the Jupiter–Sun orbital plane. We expect that inclined asteroids would behave in a qualitatively similar way, but would evolve on a longer time scale.

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## 2. The Region Between Mars and Jupiter

Two-hundred and sixty asteroids were started with semimajor axes distributed with uniform surface density between 0.55 and 0.85 Jupiter units. All other orbital elements were distributed randomly, with the eccentricities lying between 0.0 and 0.3. From preliminary studies (e.g., Figures 4 and 6), we expected that asteroids interior to 0.55 would not be perturbed much by Jupiter and that asteroids exterior to 0.85 would be ejected immediately. Figure 7 shows the semimajor axis vs time for the first 50 Jupiter periods. The almost vertical lines indicate a very rapid change of semimajor axis (i.e., energy) following a close encounter with Jupiter. Figures 8 and 9 continue the plot of semimajor axis vs time to 200 Jupiter periods. In Figures 8 and 9, bodies with initial semimajor axes below 0.633 have been deleted as they seemed quite stable. 'Ejected' bodies either leave the asteroid domain (i.e., their semimajor axes become greater than 1 or less than 0.55) or approach Jupiter closer than 10 present Jovian radii. In the latter case, the $a(t)$ terminates in a filled circle. Figure 10 gives a histogram of the

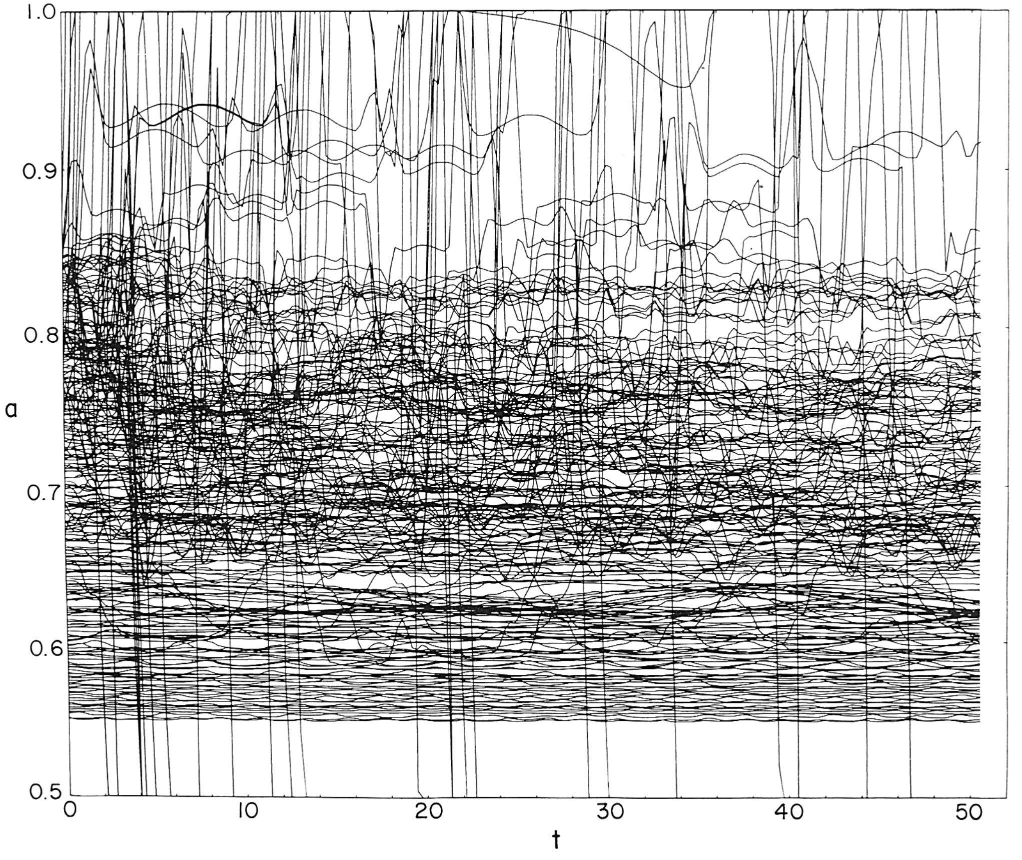


Fig. 7. Semimajor axis vs time in periods of Jupiter (eccentricity = 0.06) for a set of asteroids with initial eccentricities chosen at random between 0.0 and 0.3.

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surface density at various times. The evolution of the surface density is shown in a different presentation in Figure 11. The evolution proceeds very rapidly for the first 50 Jupiter periods and then markedly slows down.

If Jupiter had been in a circular orbit, the theory of the 'restricted three-body problem' would have delineated the escapers from the nonescapers by their value of the

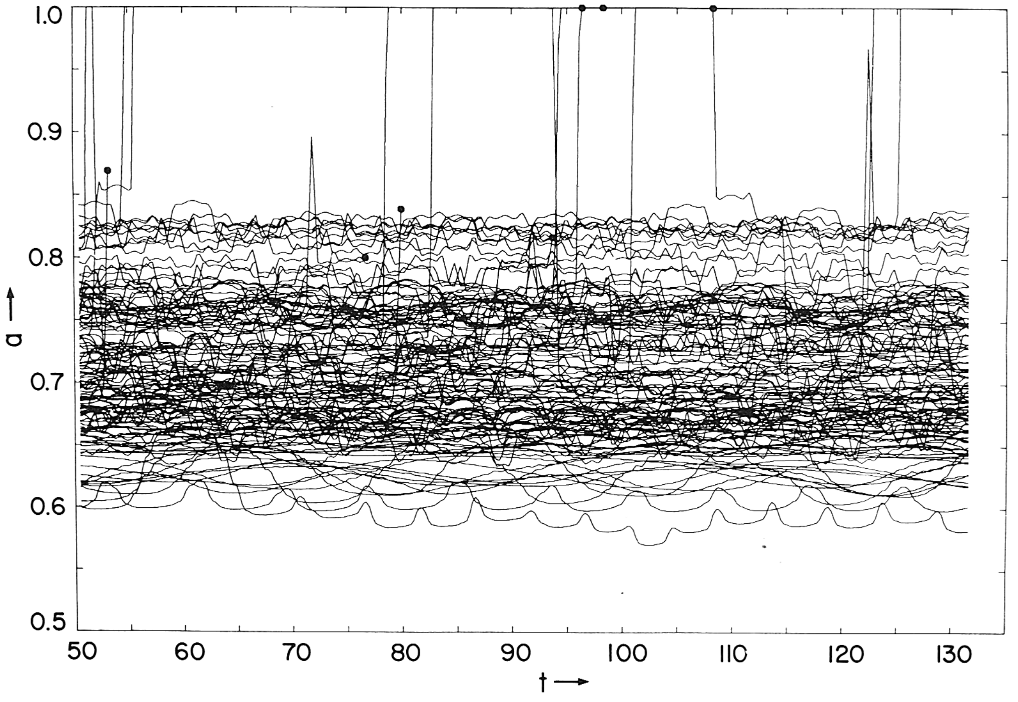


Fig. 8. A continuation, with a slightly different scale, of Figure 7 to 130 revolutions of Jupiter. Solid dots indicate objects that have approached Jupiter to closer than 10 radii and have been dropped from the calculation.

Jacobi integral. Figure 12 is a plot of the initial semimajor axes vs the initial eccentricity of the 260 asteroids except those for which $0.55 < a < 0.60$. The open circles denote those asteroids that subsequently escaped. In Figure 13, only the escapers are shown, and curves of constant Jacobian are given for reference.

The Jacobi constant $J$, in Sun-centered rotating coordinates, is given by Equation (1):

$$J = \tfrac{1}{2}r^2 + \frac{1-\mu}{r} + \frac{\mu}{s} - \mu r \cos\phi + \tfrac{1}{2}\mu^2 - \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2), \qquad (1)$$

where $r$ denotes the Sun-asteroid distance, $s$ denotes the Jupiter-asteroid distance, $\phi$ is the angle between the asteroid and Jupiter as seen from the Sun, and $\mu$ is the ratio of Jupiter's mass to the sum of the masses of Jupiter and the Sun. The sum of the masses, gravitational constant, and Jupiter's semimajor axis have been taken as unity.

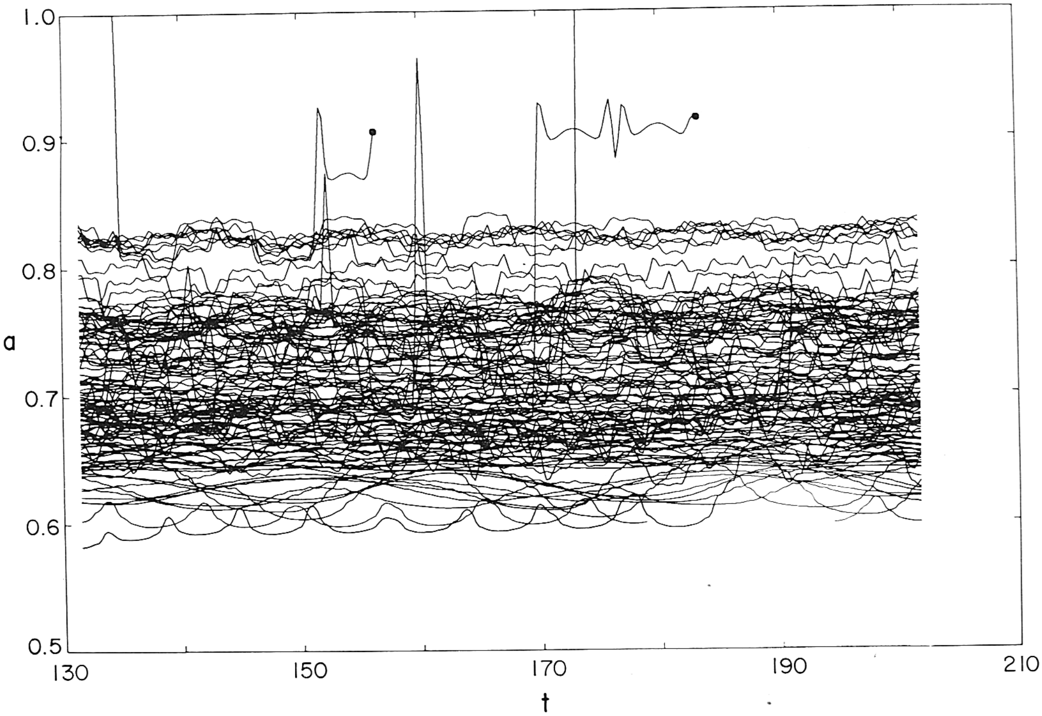


Fig. 9. Figure 8 extended to 200 revolutions of Jupiter. The region between the resonances at 2/3 and 3/4 is now largely clear, although objects can apparently librate stably at these resonances.

Let $F$ denote $J$ with the velocity terms set equal to zero. That is,

$$F=\tfrac{1}{2}r^2+\frac{1-\mu}{r}+\frac{\mu}{s}-\mu r\cos\phi+\tfrac{1}{2}\mu^2. \tag{2}$$

As the velocity terms are positive-definite, motion is possible only where $F\geqslant J$. $F$ assumes its minimum value of 1.51997 at the $L_2$ point, where $r=0.932287$ (and where $\phi=0$, and $s=(1-r)$).

We can approximately include the effect of Jupiter's eccentricity by calculating the instantaneous $L_2$ point when Jupiter is at perihelion. Figure 14 gives the quantities necessary to derive Equation (3), which balances the gravitational forces of the Sun and Jupiter at the distance $r$ from the Sun, viz.:

$$GM_{\odot}/r^2=(r-\mu R)\,\Omega^2+GM_J/(R-r)^2, \tag{3}$$

where $\Omega$ is the Keplerian angular velocity of Jupiter at perihelion. With the substitution $x\equiv r/R$, Equation (2) can be reduced to

$$(x-\mu)(1+e)+\frac{\mu}{(1-x)^2}-\frac{1-\mu}{x^2}=0, \tag{4}$$

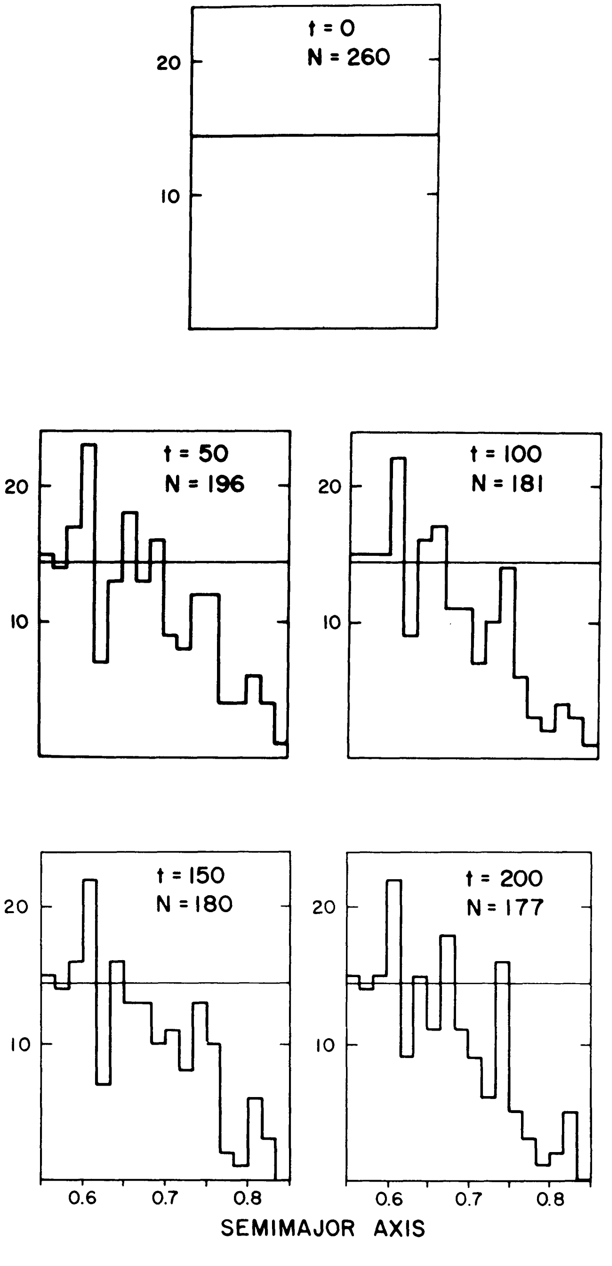


Fig. 10. Histograms of surface number density at different times.

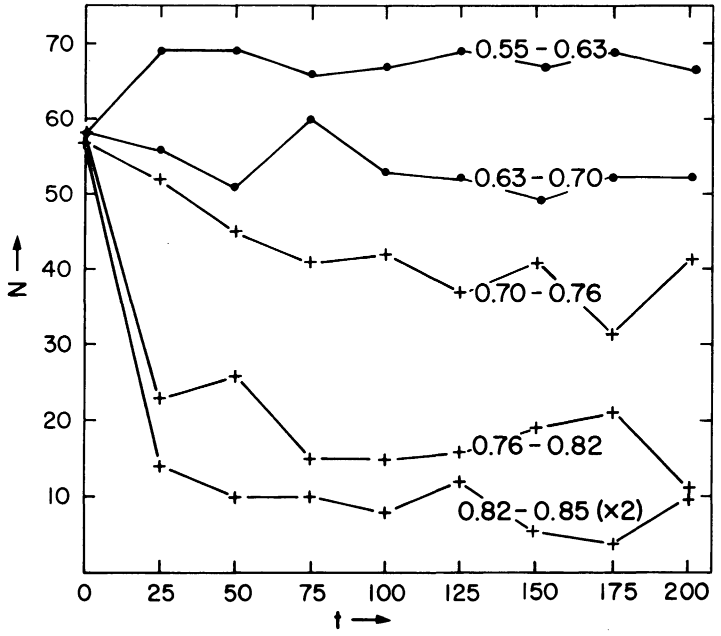


Fig. 11. Surface number density $N$ as a function of time in revolutions of Jupiter for five ranges in semimajor axis.

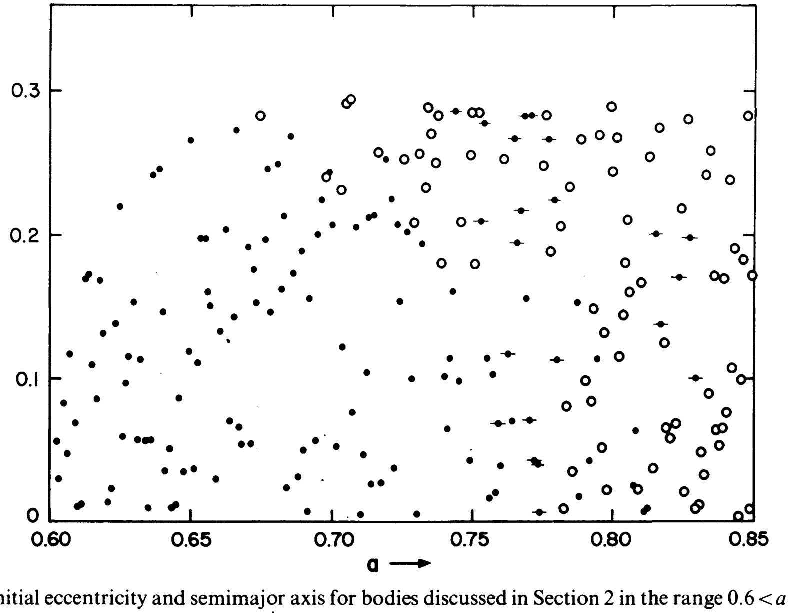


Fig. 12. Initial eccentricity and semimajor axis for bodies discussed in Section 2 in the range $0.6 < a < 0.85$. Open circles represent objects that have escaped during 200 revolutions of Jupiter; solid circles have not. Horizontal lines through a solid dot indicate a librating body.

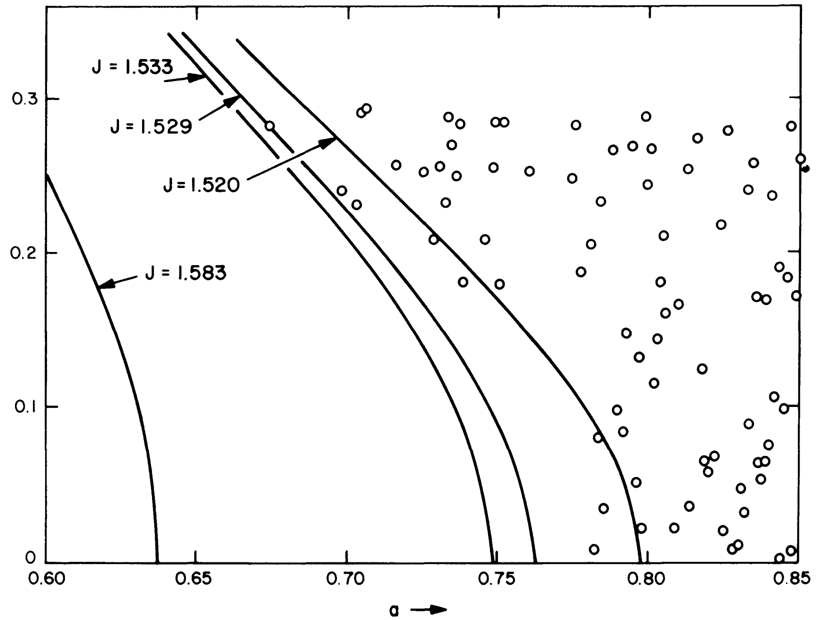


Fig. 13. Circles again give objects that have escaped. Curves for four different Jacobi constants are included.

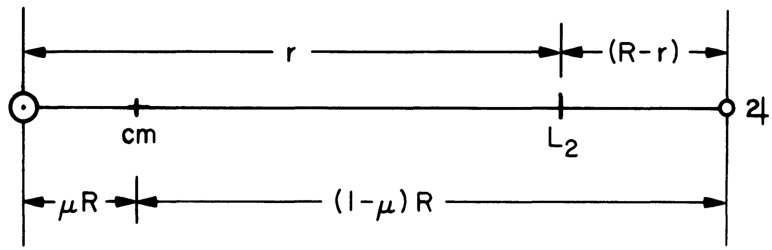


Fig. 14. Geometry useful for the derivation of Equation (2). Jupiter is assumed to move in an eccentric orbit and its 'instantaneous $L_2$ point' to lie at a distance $r$ from the Sun.

where $e$ denotes Jupiter's eccentricity. With the substitutions $\xi = 1 - x$ and $v^3 = (1/3)[\mu/(1-\mu)]$, and assuming $\xi \ll 1$, this can be written as

$$v^3 = \xi^3 [1 + \tfrac{1}{3}e + v^3(1+e)] - \tfrac{1}{3}e\xi^2 + \xi^4 + \tfrac{4}{3}\xi^5 + \cdots. \qquad (5)$$

Letting $\alpha = \frac{1}{3}(e/v)$ be of order unity and neglecting terms of order $e$ or $v$, the root of Equation (5) corresponding to $L_2$ is given by (to an accuracy of 2%)

$$\frac{\xi}{v} = 1 + \tfrac{1}{3}\alpha + \tfrac{1}{9}\alpha^2 + \tfrac{2}{81}\alpha^3 + \cdots \quad \text{for} \quad \alpha \lesssim 1.5$$

$$\frac{\xi}{v} = \alpha + \frac{1}{\alpha^2} - \frac{2}{\alpha^5} + \frac{4}{\alpha^8} + \cdots \quad \text{for} \quad \alpha \gtrsim 1.5.$$

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The solution of Equation (4) (with $e=0.06$) for the instantaneous $L_2$ point gives $r=0.870652$, and the value of $F$, as calculated from Equation (2), is $F=1.53329$. $J=1.529$ corresponds to the largest Jacobi constant to escape after 200 Jupiter periods, and $J=1.583$ corresponds to the Jacobian that bounds the asteroids in the PLS, as is shown in Figure 15. (At large distances from Jupiter, a good approximation to $J$ is given by $J=\frac{1}{2}a+\sqrt{[a(1-e^2)]}$.)

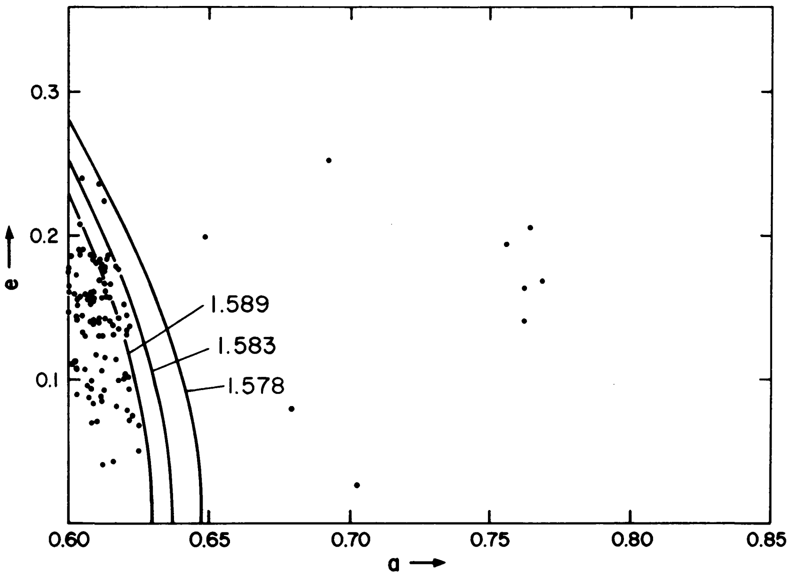


Fig. 15. PLS asteroids with three values of the Jacobi constant for reference.

Figure 16 shows the Jacobi constant as a function of time for the interval 50–130 Jupiter periods. For reference, $J$ is 1.53 at the 2/3 resonance and 1.58 at the 1/2 resonance. If we assume the evolution of the asteroids to be a random walk in 'Jacobian space', then Jupiter has thus far eaten away those asteroids for which $J$ is less than 1.53 and the solar system has retained asteroids with $J$ greater than 1.58. The central problem, which we have not answered in this preliminary survey, is the following: Will Jupiter eventually eat away another 0.05 (3%) units of the Jacobi constant? Typical oscillations of $J$ (Figure 16) are of the order of $10^{-3}$. If this is taken as the order of magnitude of a random impulse, then we would expect to wait $\sim 10^5$ Jupiter periods in order that $J$ be changed by 0.05. However, we suspect that this reasoning is faulty and, instead, that stable orbits are stable just because successive impulses are not independent but correlated. In fact, we expect that interior to the 1/2 resonance, a perturbed Jacobi integral may be valid for all time and that this integral breaks down somewhere exterior to the 1/2 resonance. We shall return to this question in a later study.

We note that the asteroids in the neighborhood of the 2/3 resonance, with the exception of the stable librators, were removed from that neighborhood in a few thousand years. This explains, we think, why there still remains a concentration of asteroids there. There was no possibility for the librators, which have large excursions in eccentricity and semimajor axis, to collide with nonlibrators, because the nonlibrators escaped almost immediately. This is in sharp contrast with the situation at the 1/2 resonance, where a large body of asteroids interior to the 1/2 resonance remained to

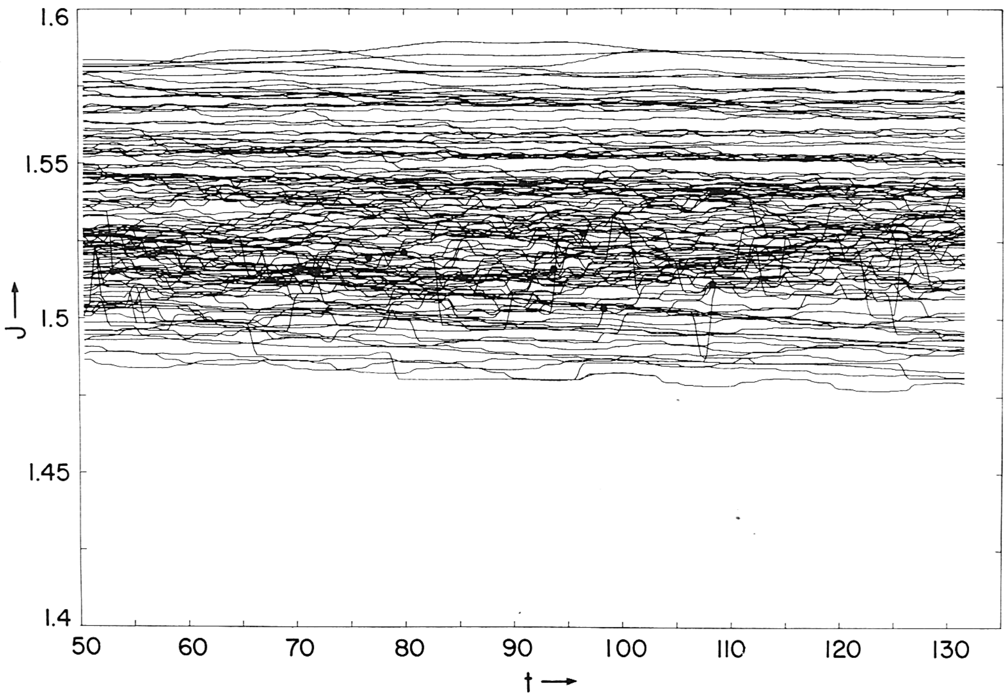


Fig. 16. Variations in the Jacobi 'constant' when Jupiter's eccentricity is 0.06.

collide with librators (see, for example, Figure 5). Thus, at the 1/2 resonance, there is a clean gap rather than a family of bodies. This provides a possible and simple explanation for the curious feature that at some resonances (2/3, 3/4, 1/1) a concentration of bodies exists, while others (1/3, 2/5, 3/7, 1/2) present a gap in the space density.

### 3. The Region Between Jupiter and Saturn

One-hundred asteroids were started with semimajor axes distributed from 1.1–1.75 Jupiter units with uniform surface density and with eccentricities randomly chosen between 0.0 and 0.1. Saturn, introduced only in this calculation, has a semimajor axis of 1.83 Jovian units; its mass was assumed to be 30% of Jupiter's and its eccentricity was taken as 0.08 – near its maximum value. Figures 17, 18, and 19 show the semimajor axes vs time for the intervals 0–160, 160–330, and 330–500 Jupiter periods. Figure 20 presents the surface number density – in the form of a histogram – at

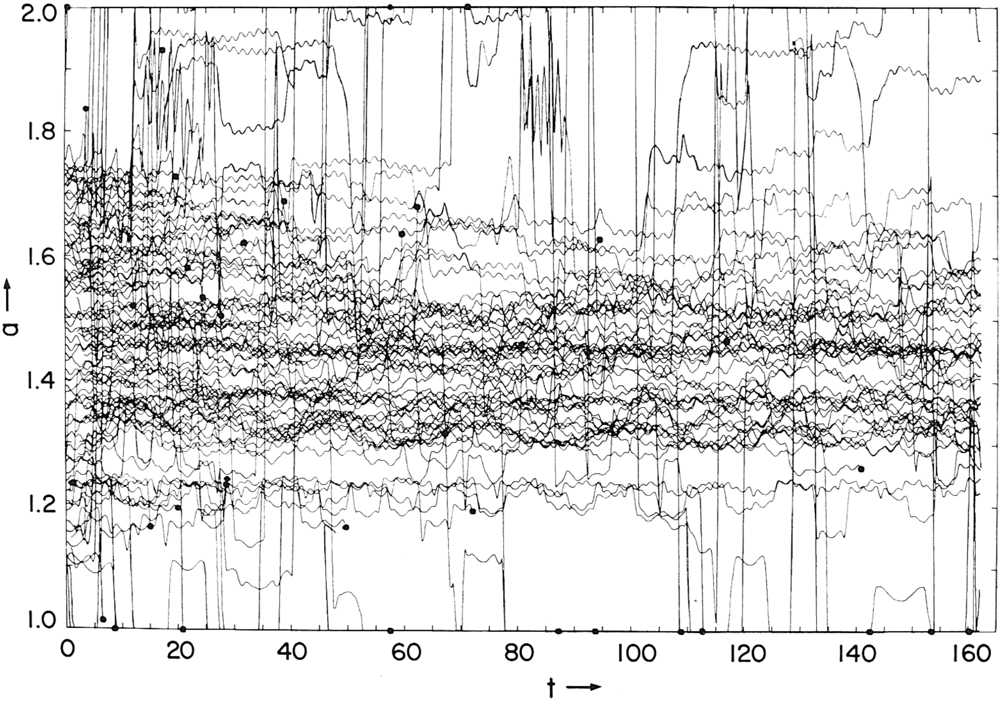


Fig. 17. Semimajor axis vs time in periods of Jupiter for 100 objects started with random eccentricities between 0.0 and 0.1. All other orbital elements, except semimajor axis, are also selected at random.

various times. Inspection of the trajectories leads us to estimate that only the bands at 1.30 and 1.45 Jupiter units (6.8 and 7.5 AU) are possibly stable. Numerous asteroids injected inside Jupiter's orbit in our first study were found to librate, thereby avoiding close approaches to Jupiter. The concentration of bodies at $a \cong 0.825$ in Figure 8, for example, owes its existence to apparently stable librators at the 3/4 resonance with Jupiter. Between Jupiter and Saturn, however, bodies that librate permanently appear to be absent, presumably because of the combined effects of two major planets.

## 4. Conclusions

Figure 21 compares the initial and final surface densities of the computer-simulated asteroids, where the rectangles outline the initial surface density.

We conclude that even if the Jupiter–Saturn region were initially populated with asteroids, objects there would, with the possible exception of bodies in two narrow bands, be ejected in a few thousand years.

In the Mars–Jupiter region, the central problem remains unanswered. In the short interval simulated (200 Jupiter revolutions), the region between the 1/2 and 2/3 resonances remains well populated. Assuming that asteroids were initially present there, we have to look to Jupiter perturbations over much longer times, to collisions be-

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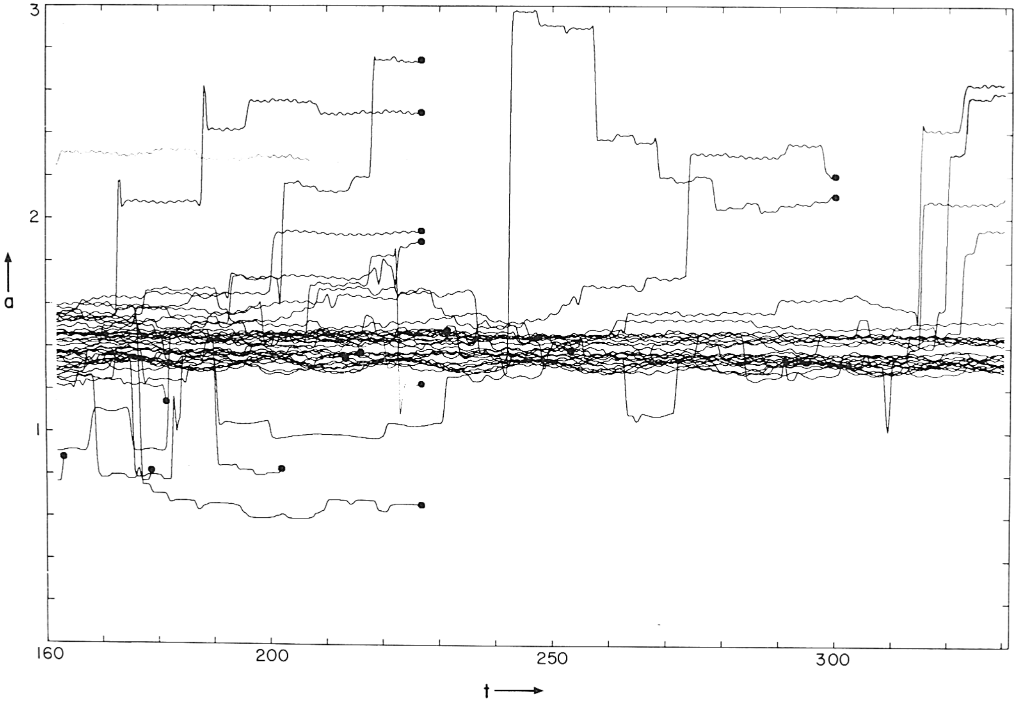


Fig. 18. Continuation of Figure 17 to 330 revolutions of Jupiter. Circles again indicate objects that have approached either planet to 10 radii.

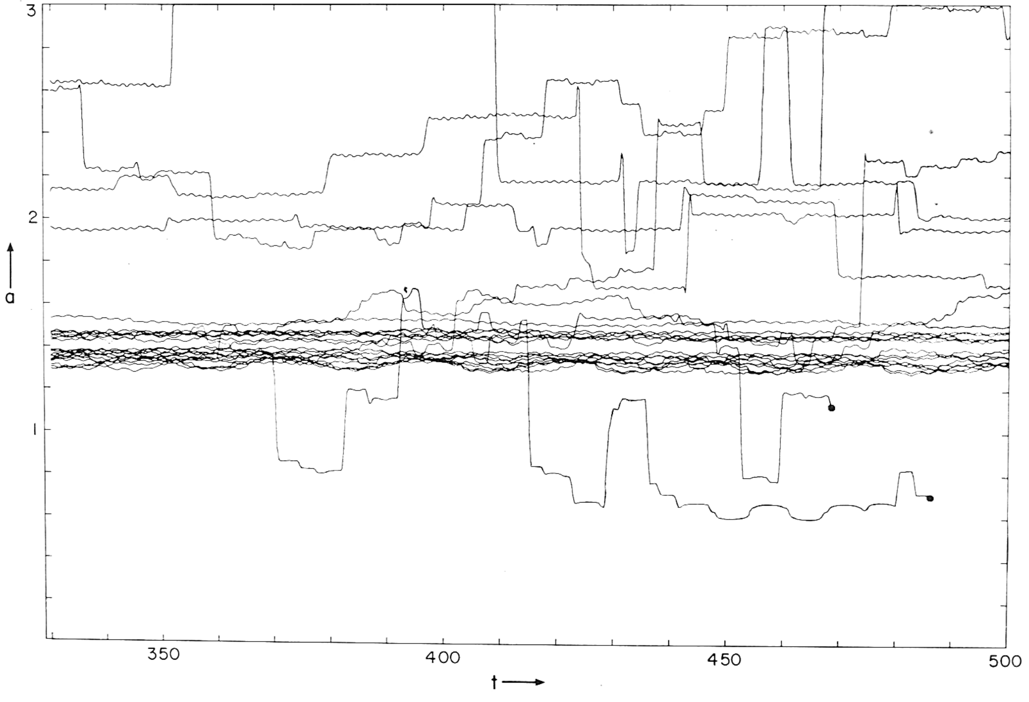


Fig. 19. Continuation of Figure 18 to 500 revolutions of Jupiter.

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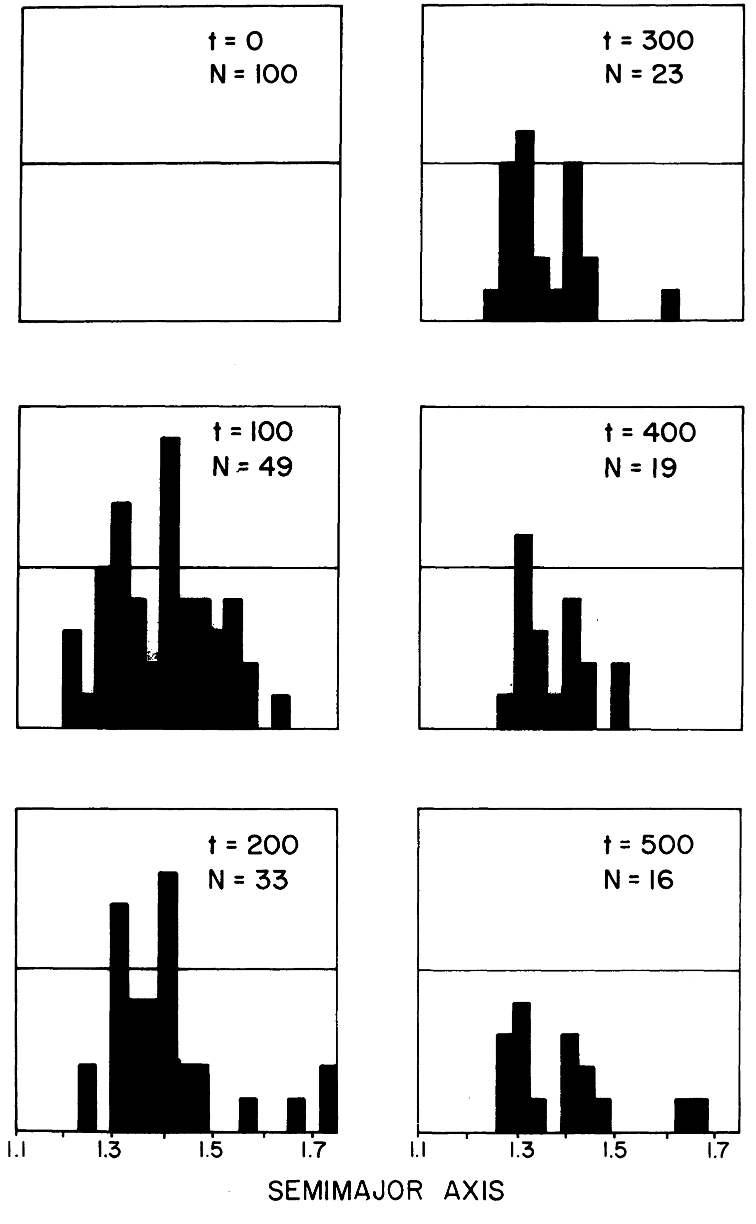


Fig. 20. Histogram of the number density of objects between Jupiter and Saturn as a function of time. Bodies near 1.3 and 1.45 may remain for some time; those near 1.65 are temporary.

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tween asteroids, or to some other mechanism (such as gas drag) to depopulate that region. Figure 22 is a version of Figure 7 with the escapers removed. We see that the oscillations in semimajor axis are quite pronounced exterior to the 1/2 resonance (0.63), indicating that collisions could have been quite effective. The activity drops off markedly interior to the 1/2 resonance.

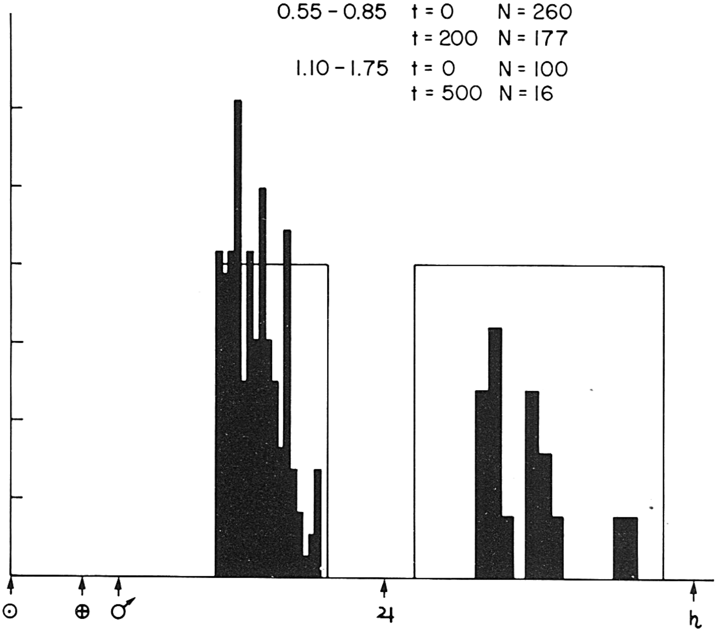


Fig. 21. Results of this survey. Number of remaining objects scaled for comparison with the initial distribution. $N$ is the total number of bodies remaining after the indicated times in periods of Jupiter.

However, the time derivative of the approximate Jacobian for the elliptic restricted three-body problem is proportional to $\mu e$ (the mass ratio of Jupiter to the Sun multiplied by the eccentricity of Jupiter), a quantity of the order of $6 \times 10^{-5}$, which indicates a time scale long in comparison to the interval studied in this survey but short with respect to the age of the solar system. Thus, we are compelled to attempt an investigation of the long-period or secular perturbations. Such a study is also suggested by the fact that, while it is known that there are no secular terms in the semimajor axis to order $\mu^2$ ($10^{-6}$), the situation for higher orders is uncertain.

Times of the order of $10^6$ Jupiter revolutions are not yet accessible by direct numerical integration such as was used in this study. Averaging techniques would bring such times within reach, but we strongly suspect that averaging stabilizes the equations of motion; an effect we are anxious to avoid. Thus we are led to a mixture of analytic and numerical techniques on which we will report in a subsequent communication.

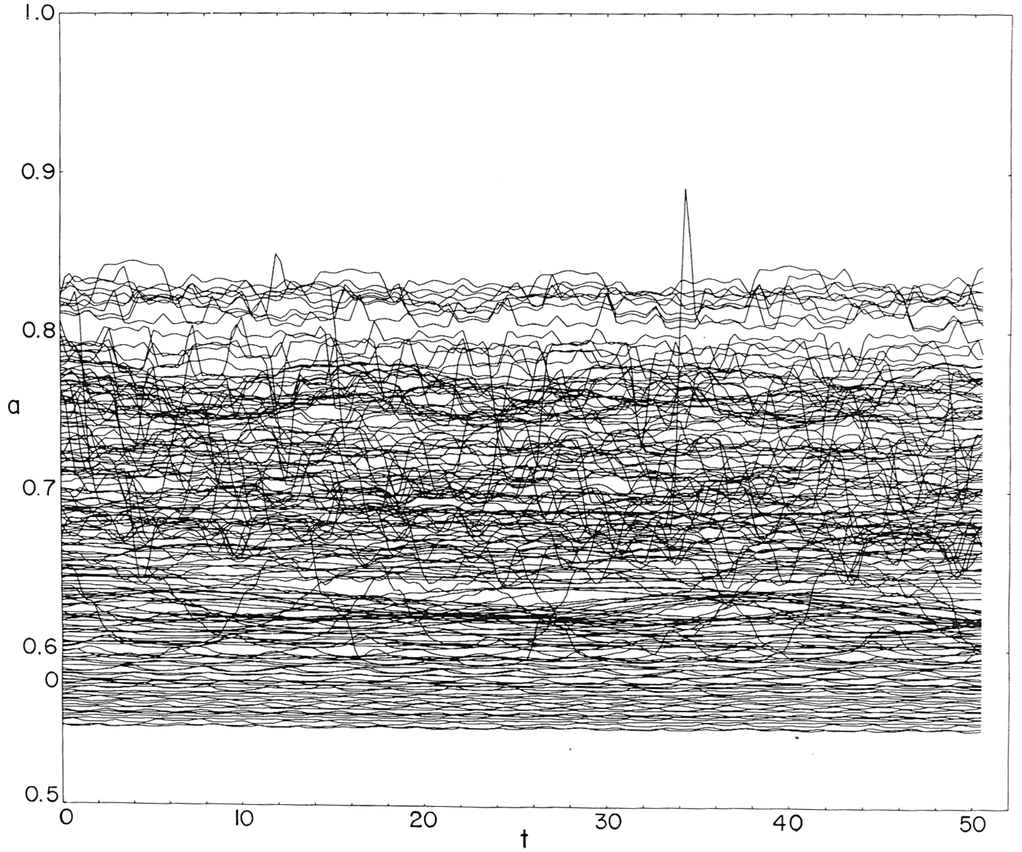


Fig. 22. Semimajor axis vs time in periods of Jupiter with escaping objects removed.

## Acknowledgements

We wish to express our gratitude and regard for the competent and imaginative computer programming of our colleague Mr Rudolf Loeser. Prof. Giuseppe Colombo provided valuable insights in our many conversations. We also acknowledge the encouragement and support of Prof. Fred L. Whipple.

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# DISCUSSION

*H. Scholl:* Which integration method did you use?

*M. Lecar:* A method which uses analytical forms for the first four time derivatives of the force.

*H. Scholl:* Is there any asteroid in the Hecuba gap which left the gap after 6000 yr?

*M. Lecar:* No.

*H. Scholl:* Did you use different values for the masses of Jupiter and Saturn in your computations?

*M. Lecar:* No.

*G. Hertz:* Did you integrate in rectangular coordinates?

*M. Lecar:* Yes.

*G. Hertz:* Is your original hypothesis correct?

*M. Lecar:* I am not sure. For the relatively short time intervals that we integrated asteroids still remained between 3.3 and 4.0. It is in conflict with observed distribution.

*L. Kresák:* Do you find any Apollo-type asteroids originating from your initial orbits?

*M. Lecar:* I don't know. We did not follow the asteroids after they left the asteroid belt.

*R. H. Miller:* You attributed the larger value of Jacobi constant (1.536 instead of 1.520) to the effect of Jupiter's eccentricity. Have you checked that a circular orbit for Jupiter yields a value of 1.520?

*M. Lecar:* In the circular problem the Jacobian is conserved to a part in $10^5$ for the time intervals we integrated in this investigation.