C^*-Algebras of Irreversible Dynamical Systems

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Abstract. We show that certain C^*-algebras which have been studied by, among others, Arzumanian, Vershik, Deaconu, and Renault, in connection with a measure-preserving transformation of a measure space or a covering map of a compact space, are special cases of the endomorphism crossed product construction recently introduced by the first named author. As a consequence these algebras are given presentations in terms of generators and relations. These results come as a consequence of a general theorem on faithfulness of representations which are covariant with respect to certain circle actions. For the case of topologically free covering maps we prove a stronger result on faithfulness of representations which needs no covariance. We also give a necessary and sufficient condition for simplicity.

1   Introduction

Virtually all of the rich interplay between the theory of operator algebras and dynamical systems occurs by means of a construction usually referred to as the *crossed product*. The basic idea, which was first employed by von Neumann [18] in 1936 in the context of automorphisms of measure spaces, consists in attaching to a given dynamical system an operator algebra (a W^*-algebra in the case of von Neumann's original construction) whose algebraic structure is expected to reflect dynamical properties of the given system. The analogue of this construction for the case of C^*-algebras is due to I. Gelfand, with co-authors M. Naimark and S. Fomin, who used it later for certain special dynamical systems.

By far the greatest advances in this enterprise have been achieved for *reversible* systems, *i.e.,* when the dynamics is implemented by a group of invertible transformations. This theory (sometimes called the algebraic theory of dynamical systems) now has many important results and has become widespread and popular. Nevertheless a lot of effort has been put into extending these advances to *irreversible* systems or *dynamics of semigroups* and thus an important theory emerged attempting to parallel the theory for reversible systems.

The theory of C^*-algebras for irreversible systems breaks down into two somewhat disjoint areas, the first one consisting of the construction and study of C^*-algebras associated to classical dynamical systems, *i.e.,* transformations of measure or topological spaces (see [3, 4, 8, 22]), while the second deals with the study of crossed products of C^*-algebras by endomorphisms or semigroups thereof (see [1, 6, 16, 17, 23]).

As it turned out, these two areas have had little intersection, partly because the latter, which has perhaps received the greatest share of attention in recent times, boasts

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its greatest successes when the endomorphisms involved have a hereditary range, a hypothesis which is absent when one deals with endomorphisms arising from classical systems.

The difference between the C^*- and W^*-algebra approach is more serious in the case of irreversible dynamics than in the reversible one. The point is that the involution in the C^*-algebra for the irreversible case must be defined separately, while for the group case, the involution is defined more or less uniquely. At the same time if we consider measure-preserving dynamics and L^2-theory, the involution exists automatically and this also makes it possible to define C^*-algebras. This was the way of the papers [3, 4] — to consider a measure-preserving action of the endomorphism and the C^*-hull in L^2 of the multiplicators and endomorphism. After this we can define another representation of this C^*-algebra which leads to von Neumann factors. Actually we do not need the invariant measure itself but only the conditional expectation (*e.g.,* conditional measure on the partition into pre-images of the points). In the purely topological situation we can first consider a very large C^*-algebra and then factorize it using a certain conditional expectation (see below).

In this paper we consider the oldest, and perhaps not so intensively studied, relationship between classical irreversible systems and C^*-algebras. Our focus is on two classes of C^*-algebras, the first one being defined in terms of a measure-preserving transformation T of a measure space (X, μ), and closely related to the Arzumanian–Vershik algebras (see [3]). These algebras, which are defined below, will be denoted here by $\widetilde{AV}(X, T, \mu)$.

The second class of C^*-algebras that we study here arises from the consideration of self covering maps $T\colon X \to X$, where X is a compact space (see [4, 8, 22]). The algebras in this second class will be denoted by $C^*(X, T)$.

It is the main goal of this work to give a unified treatment of these algebras by first showing (Theorems 6.1 and 9.1 below) that they are special cases of the endomorphism crossed product construction recently introduced by the first named author (incidentally that construction also includes crossed products by endomorphisms with hereditary range [11, §4]).

In addition we show how to use this unified perspective to prove new results about $\widetilde{AV}(X, T, \mu)$ and $C^*(X, T)$ along the lines of [1, 6, 14, 16, 23]. First we prove a version of what has been called the "gauge invariant uniqueness Theorem" [15, 2.3] by showing (Theorem 4.2) that any representation of the crossed product which is faithful on the core algebra and which is covariant relative to certain circle actions must be faithful on all of the crossed product algebra. The application of this to either $AV(X, T, \mu)$ or $C^*(X, T)$ leads to theorems on faithfulness of representations which are perhaps hitherto unknown.

As another application we show (Theorem 5.3) how the proposal outlined in the closing paragraph of [3] to extend the definition of the Arzumanian–Vershik algebra to a non-commutative setting can be modified to achieve a down-to-earth elementary construction of the crossed product algebra (assuming the existence of certain faithful states).

We then specialize to the case that the range of the endomorphism considered admits a conditional expectation of *index-finite type* according to Watatani [26]. We

first show (Corollary 7.2) that the process of "dividing out by the redundancies" in Definition (2.2) below may be achieved by introducing a single relation and hence we obtain a straightforward presentation of the crossed product algebra in terms of generators and relations.

The situation leading up to $C^*(X, T)$, namely that of a covering map, is shown to fall under the hypothesis of finiteness of the index and hence we obtain (Theorem 9.2) a very concise set of relations defining $C^*(X, T)$.

Still in the case of a covering map we obtain another result on faithfulness of representations, this time without regard for circle actions but under the hypothesis of *topological freeness*: a continuous map $T\colon X \to X$ is said to be topologically free[1] (see also [25, 2.1] [2], [8], and [14, 2.1]) if for every pair of nonnegative integers (n, m) with $n \neq m$, one has that the set $\{x \in X : T^n(x) = T^m(x)\}$ has empty interior. Precisely, we show (Theorem 10.3) that under this hypothesis any representation of the crossed product which is faithful on $C(X)$ must itself be faithful. We moreover show (Theorem 11.2) that the crossed product algebra is simple if and only if T is irreducible in the sense that there are no closed nontrivial sets $F \subseteq X$ such that $T^{-1}(F) = F$. This result improves on Deaconu's characterization of simplicity [8] since the above notion of irreducibility is much weaker than Deaconu's notion of minimality.

The hypothesis that T is a covering map is directly related to the finiteness of the index which in turn is responsible for the existence of a conditional expectation from the crossed product to $C(X)$ [12, 8.9]. Since that conditional expectation turns out to be one of our main tools in proving the characterization of simplicity, our methods completely break down in the infinite index case. The characterization of simplicity in the infinite index case therefore stands out as one of the main questions left unresolved.

Last, but not least, we should mention that among the algebras which can be described as $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ are the Cuntz–Krieger algebras [11, 6.2] and the Bunce–Deddens algebras [8], which therefore provide examples of the algebras that we discuss here.

2   The Algebras

Perhaps the first attempt at constructing an operator algebra from an irreversible dynamical system is Arzumanian and Vershik's 1978 paper [3]. Given a (not necessarily invertible) measure-preserving transformation T of a measure space (X, μ) they consider the C^*-algebra of operators on $L^2(X, \mu)$ generated by the *multiplication operators*

$$M_f\colon \xi \in L^2(X, \mu) \longmapsto f\xi \in L^2(X, \mu),$$

for all $f \in L^\infty(X, \mu)$, in addition to the isometry S defined by

$$S(\xi)|_x = \xi(T(x)), \quad \forall \xi \in L^2(X, \mu), \quad \forall x \in X.$$

We will refer to this algebra as the *Arzumanian–Vershik* algebra and we will denote it by $\mathrm{AV}(X, T, \mu)$.

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[1] Also called essentially free in [8].

While it would be desirable that the algebraic structure of $AV(X, T, \mu)$ depend more on the dynamical properties of T than on the invariant measure μ chosen, it is evident that it does depend heavily on μ. In fact there is a specific way in which μ influences the algebraic structure of $AV(X, T, \mu)$ which we would like to describe. To this end, assume that we are under the hypothesis of [5, §3, No. 1, Théorème 1] so that μ may be disintegrated along the fibers of T. This means that there exists a collection $\{\mu^x\}_{x \in X}$, where each μ^x is a probability measure on $T^{-1}(x)$, and for every μ-integrable function f on X one has that

$$(2.1) \qquad \int_X f(x) \, d\mu(x) = \int_X \left(\int_{T^{-1}(x)} f(y) \, d\mu^x(y) \right) \, d\nu(x),$$

where $\nu = T^*(\mu)$ (observe that $\nu = \mu$ since μ is invariant under T). For every $f \in L^\infty(X, \mu)$ define the function $\mathcal{L}(f)$ on X by

$$(2.2) \qquad \mathcal{L}(f)|_x = \int_{T^{-1}(x)} f(y) \, d\mu^x(y).$$

One may then verify that

$$(2.3) \qquad S^* M_f S = M_{\mathcal{L}(f)},$$

for every $f \in L^\infty(X, \mu)$. One therefore sees that the measure μ, or at least the collection formed by the μ^x, does indeed influence the algebraic structure of $AV(X, T, \mu)$.

Observe that in case T is the identity map on X, one has that S is the identity operator on $L^2(X, \mu)$. Because this is not in accordance with the classical case of crossed products by automorphisms (where the unitary element implementing the given automorphism has full spectrum), it is sometimes useful to change the definition of $AV(X, T, \mu)$ as follows: letting U be the bilateral shift on $\ell^2(\mathbb{Z})$, define the operators $\tilde{S}$ and $\tilde{M}_f$ on the Hilbert space $L^2(X, \mu) \otimes \ell^2(\mathbb{Z})$ by

$$\tilde{S} = S \otimes U, \quad \text{and} \quad \tilde{M}_f = M_f \otimes 1,$$

for all $f \in L^\infty(X, \mu)$. We then let $\widetilde{AV}(X, T, \mu)$ be the C^*-algebra of operators on $L^2(X, \mu) \otimes \ell^2(\mathbb{Z})$ generated by $\tilde{S}$ and all the $\tilde{M}_f$. The advantage of this alternative construction is that when T is an automorphism of X one has that

$$\widetilde{AV}(X, T, \mu) \simeq L^\infty(X, \mu) \rtimes_T \mathbb{Z},$$

where the right-hand side is the classical crossed product [19] of $L^\infty(X, \mu)$ under the automorphism induced by T.

There is another important construction which has been proposed as a replacement to classical crossed products in the case that X is a compact space and $T \colon X \to X$ is a covering map. Building on [4], Deaconu [8] has shown how to construct an r-discrete groupoid from the pair (X, T) to which one may apply Renault's well-known construction [21] leading up to a C^*-algebra which we will denote by $C^*(X, T)$.

Still under the assumption that T is a covering map each inverse image $T^{-1}(x)$ is a finite set which in turn admits a canonical probability measure μ^x, namely the normalized counting measure. Equation (2.2) may then be employed and this time it defines a bounded positive operator

$$\mathcal{L} \colon C(X) \to C(X).$$

On considering $C(X)$ as a subalgebra of $C^*(X, T)$ and letting S denote the isometry in $C^*(X, T)$ described shortly before the statement of Theorem 2 in [8], it is easy to see that

$$S^* f S = \mathcal{L}(f), \quad \forall f \in C(X),$$

which is seen to be another manifestation of 2.3.

In a recent article [11], the first named author has proposed yet another construction of crossed product in the irreversible setting (see also [12, 13]), the ingredients of which are a unital C^*-algebra A, an injective $*$-endomorphism

$$\alpha \colon A \to A$$

such that $\alpha(1) = 1$, and a *transfer operator*, namely a continuous positive linear map

$$\mathcal{L} \colon A \to A$$

such that

$$(2.4) \qquad\qquad \mathcal{L}\left(a\alpha(b)\right) = \mathcal{L}(a)b, \quad \forall\, a, b \in A,$$

which we will moreover suppose satisfies $\mathcal{L}(1) = 1$.

The main examples we have in mind are of course related to the above situations, where A is either $L^\infty(X, \mu)$ or $C(X)$ and α is given in both cases by the expression

$$(2.5) \qquad\qquad \alpha \colon f \in A \mapsto f \circ T \in A.$$

As for a transfer operator, one may use (2.2) after the appropriate choice of probability measures μ^x is made.

Let us fix for the time being a C^*-algebra A, an injective $*$-endomorphism α of A preserving the unit, and a transfer operator $\mathcal{L}$ such that $\mathcal{L}(1) = 1$.

Definition 2.1 ([11, 3.1])   We will let $\mathcal{T}(A, \alpha, \mathcal{L})$ denote the universal unital C^*-algebra generated by a copy of A and an element $\hat{S}$ subject to the relations

(i) $\hat{S}a = \alpha(a)\hat{S}$, and
(ii) $\hat{S}^* a \hat{S} = \mathcal{L}(a)$,

for every $a \in A$.

As proved in [11, 3.5], the canonical map from A to $\mathcal{T}(A, \alpha, \mathcal{L})$ is injective so we may view A as a subalgebra of $\mathcal{T}(A, \alpha, \mathcal{L})$. Since $\mathcal{L}(1) = 1$, we have that

$$\hat{S}^*\hat{S} = \hat{S}^*1\hat{S} = \mathcal{L}(1) = 1,$$

and hence we see that $\hat{S}$ is an isometry.

Following [11, 3.6] a *redundancy* is a pair (a, k) of elements in $\mathcal{T}(A, \alpha, \mathcal{L})$ such that k is in the closure of $A\hat{S}\hat{S}^*A$, a is in A, and

$$ab\hat{S} = kb\hat{S}, \quad \forall\, b \in A.$$

Definition 2.2 ([11, 3.7])   The crossed product of A by α relative to $\mathcal{L}$, denoted by $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$, is defined to be the quotient of $\mathcal{T}(A, \alpha, \mathcal{L})$ by the closed two-sided ideal generated by the set of differences $a - k$, for all[2] redundancies (a, k).

As a consequence of our assumption that $\mathcal{L}(1) = 1$, we have by [11, 2.3] that the composition $E = \alpha \circ \mathcal{L}$ is a conditional expectation from A onto the range of α. In addition $\mathcal{L}$ may be given in terms of E by $\mathcal{L} = \alpha^{-1} \circ E$. It is easy to see that E is faithful (meaning that $E(a^*a) = 0 \Rightarrow a = 0$) if and only if $\mathcal{L}$ is faithful (in the sense that $\mathcal{L}(a^*a) = 0 \Rightarrow a = 0$). We thank Iain Raeburn for pointing out the importance of considering faithful transfer operators.

We will assume from now on that $\mathcal{L}$ is faithful. It then follows from [12, 4.12] that the canonical map from A to $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is injective and we use it to think of A as a subalgebra of $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$. We will denote by S the image of $\hat{S}$ in $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.

3   Preliminaries

For ease of reference let us now list the assumptions which will be in force for the largest part of this work.

Hypotheses 3.1 (Standing Hypotheses)

(i)   *A is a unital C^*-algebra;*
(ii)   *α is an injective *-endomorphism of A such that $\alpha(1) = 1$;*
(iii)   *$\mathcal{L}$ is a faithful transfer operator such that $\mathcal{L}(1) = 1$.*

We now wish to review some results from [12] in a form which is suitable for our purposes. Recall from [12, 2.3] that $\mathcal{T}(A, \alpha, \mathcal{L})$ (resp. $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$) coincides with the closed linear span of the set of elements of the form $a\hat{S}^n\hat{S}^{*m}b$ (resp. $aS^nS^{*m}b$), for $a, b \in A$.

Proposition 3.2 (See also [11, 2.3])   *For each $n \in \mathbb{N}$ the map $\mathcal{E}_n$ defined by*

$$\mathcal{E}_n = \alpha^n \circ \mathcal{L}^n$$

is a conditional expectation from A onto the range of α^n.

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[2]We should remark that in [11, Definition 3.7], one uses only the redundancies (a, k) such that a belongs to the ideal of A generated by the range of α. But, under the present hypothesis that α preserves the unit, this ideal coincides with A.

Proof   It is clear that the range of $\mathcal{E}_n$ is contained in the range of α^n. Moreover if $a \in A$ and $b \in \alpha^n(A)$, say $b = \alpha^n(c)$ with $c \in A$, we have that

$$\mathcal{E}_n(ab) = \alpha^n(\mathcal{L}^n(a\alpha^n(c))) = \alpha^n(\mathcal{L}^n(a))\alpha^n(c) = \mathcal{E}_n(a)b.$$

Plugging $a = 1$ above, and noticing that $\mathcal{E}_n(1) = 1$, we see that $\mathcal{E}_n$ is the identity on the range of α^n. The remaining details are now elementary. ∎

Definition 3.3   We will denote by $\mathcal{M}_n$ and $\mathcal{K}_n$ the subsets of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ given by

$$\mathcal{M}_n = \overline{AS^n}, \quad \text{and} \quad \mathcal{K}_n = \overline{\mathcal{M}_n \mathcal{M}_n^*} = \overline{AS^n S^{*n} A}.$$

Proposition 3.4   *For every $n, m \in \mathbb{N}$ with $n \leq m$ we have that*

(i) $\mathcal{K}_n \mathcal{M}_m \subseteq \mathcal{M}_m$, *and*
(ii) $\mathcal{K}_n \mathcal{K}_m \subseteq \mathcal{K}_m$.

Proof   Given $a, b, c \in A$ we have

$$(aS^n S^{n*} b)(cS^m) = aS^n S^{n*} bcS^n S^{m-n} = a\alpha^n(\mathcal{L}^n(bc))S^m,$$

whence (i) follows. As for (ii), we have

$$\mathcal{K}_n \mathcal{K}_m \subseteq \overline{\mathcal{K}_n \mathcal{M}_m \mathcal{M}_m^*} \subseteq \overline{\mathcal{M}_m \mathcal{M}_m^*} = \mathcal{K}_m.$$ ∎

It follows that each $\mathcal{K}_n$ is a C^*-subalgebra of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$. In some cases (see below), the $\mathcal{K}_n$ form an increasing sequence. In any event the closure of the sum of all $\mathcal{K}_n$'s is a C^*-subalgebra which we denote by

$$\mathcal{U} = \overline{\sum_{n=0}^{\infty} \mathcal{K}_n}.$$

Obviously $\mathcal{U}$ is the linear span of the set of elements of the form $aS^n S^{*n} b$, where $n \in \mathbb{N}$ and $a, b \in A$.

There is (see [12, 3.3]) a canonical action γ of the circle group $\mathbb{T}$ on $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ given by

$$\gamma_z(S) = zS, \quad \text{and} \quad \gamma_z(a) = a, \quad \forall a \in A, \quad \forall z \in \mathbb{T},$$

which we shall call the *gauge action*. For any circle action the expression

$$(3.1) \qquad\qquad F(a) = \int_{z \in \mathbb{T}} \gamma_z(a)\, dz, \quad \forall a \in A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$$

gives a faithful conditional expectation onto the algebra of fixed points for γ. By [12, 3.5] we have that this fixed point algebra is precisely $\mathcal{U}$.

4 Faithfulness of Covariant Representations

Our goal in this section is to establish sufficient conditions for a representation of $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ to be faithful. We therefore fix throughout this section a $*$-homomorphism

$$\pi \colon A \rtimes_{\alpha,\mathcal{L}} \mathbb{N} \to B,$$

where B is any C^*-algebra, quite often chosen to be the algebra of all bounded operators on a Hilbert space, in which case we will be talking of a representation of $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ as indicated above.

Lemma 4.1   *If π is faithful on A then π is faithful on all of $\mathcal{U}$.*

Proof   We begin by claiming that π is isometric when restricted to $\mathcal{M}_n$ for all $n \in \mathbb{N}$. In order to see this let $a \in A$ and notice that

$$\|\pi(aS^n)\|^2 = \|\pi(S^{*n}a^*aS^n)\| = \|\pi(\mathcal{L}^n(a^*a))\|$$
$$= \|\mathcal{L}^n(a^*a)\| = \|S^{*n}a^*aS^n\| = \|aS^n\|^2,$$

hence proving the claim.

Let $k \in \mathcal{U}$ and suppose that $k = \sum_{i=0}^{n} k_i$, where each $k_i \in \mathcal{K}_i$. By Proposition 3.4(i) we have that $kbS^n \in \mathcal{M}_n$ for every $b \in A$. Supposing that $\pi(k) = 0$ we have that $\pi(kbS^n) = 0$ for every b in A and hence $kbS^n = 0$ by the above claim. It follows that $kk' = 0$ for all $k' \in \mathcal{K}_n$ and hence that $(k_0, k_1, \dots, k_n)$ is a redundancy of order n as defined in [12, 6.2]. By [12, 6.3] we have that $k = 0$ in $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ and hence that π is injective on $\sum_{i=0}^{n} \mathcal{K}_i$. Since the latter is a C^*-algebra by Proposition 3.4(ii) and [19, 1.5.8] we see that π is isometric there and hence also on the inductive limit of these algebras as $n \to \infty$ which coincides with $\mathcal{U}$.   ∎

This allows us to prove a version of the "gauge invariant uniqueness theorem" as in [15, 2.3].

Theorem 4.2   *Under the assumptions of Hypotheses 3.1 let B be a C^*-algebra and $\pi \colon A \rtimes_{\alpha,\mathcal{L}} \mathbb{N} \to B$ be a $*$-homomorphism which is faithful on A. Suppose that B admits an action of the circle group such that π is covariant relative to the gauge action on $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ and the given action on B. Then π is faithful on all of $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.*

Proof   By [10, 2.9] it suffices to prove that π is faithful on the fixed point algebra for the gauge action, which we know is $\mathcal{U}$. But this obviously follows from Lemma 4.1.   ∎

5 States

In this section we will develop the necessary tools in order to show that the generalized Arzumanian–Vershik algebra $\widetilde{\mathrm{AV}}(X, T, \mu)$ fits our crossed product construction. Since we will be working in the non-commutative setting, what we do here, especially

Theorem 5.3 below, is related to the situation discussed in the last paragraph of [3]. As before we keep Hypotheses 3.1 in force.

Throughout this section we will fix a state ϕ on A which is invariant under both α and $\mathcal{L}$, *i.e.,* such that

$$\phi(\alpha(a)) = \phi(a) = \phi(\mathcal{L}(a)), \quad \forall\, a \in A.$$

Observe that ϕ is then necessarily invariant under the conditional expectations $\mathcal{E}_n$ of Proposition 3.2.

Let ρ denote the GNS representation associated to ϕ and denote by H the corresponding Hilbert space and by ξ the cyclic vector. Because ϕ is α-invariant one has that the correspondence

$$\rho(a)\xi \mapsto \rho(\alpha(a))\xi, \quad \forall\, a \in A,$$

preserves norm and hence extends to H giving an isometry $\check{S} \in \mathcal{B}(H)$ such that

$$(5.1) \qquad \check{S}(\rho(a)\xi) = \rho(\alpha(a))\xi, \quad \forall\, a \in A.$$

We next claim that for every $a \in A$ one has that

$$(5.2) \qquad \check{S}^*(\rho(a)\xi) = \rho(\mathcal{L}(a))\xi.$$

In fact, given $b \in A$ we have

$$\langle \check{S}^*(\rho(a)\xi), \rho(b)\xi \rangle = \langle \rho(a)\xi, \check{S}(\rho(b)\xi) \rangle = \langle \rho(a)\xi, \rho(\alpha(b))\xi \rangle$$
$$= \phi(\alpha(b^*)a)\$ = \phi(\mathcal{L}(\alpha(b^*)a)) = \phi(b^*\mathcal{L}(a))$$
$$= \langle \rho(\mathcal{L}(a))\xi, \rho(b)\xi \rangle,$$

from which (5.2) follows. With this it is easy to show that for every $a \in A$ one has that

$$\check{S}\rho(a) = \rho(\alpha(a))\check{S}, \quad \text{and} \quad \check{S}^*\rho(a)\check{S} = \rho(\mathcal{L}(a)).$$

By the universal property it follows that there exists a representation Π of $\mathcal{T}(A, \alpha, \mathcal{L})$ on H such that

$$\Pi(\hat{S}) = \check{S}, \quad \text{and} \quad \Pi(a) = \rho(a),$$

for all $a \in A$.

Proposition 5.1 *For any redundancy (a, k) one has that $\Pi(a - k) = 0$. Therefore Π factors through a representation π of $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ on H such that $\pi(S) = \check{S}$ and $\pi(a) = \rho(a)$ for all $a \in A$.*

Proof Substituting $a = 1$ in (5.1) we see that $\check{S}(\xi) = \xi$ and hence for any $b \in A$ we have

$$\Pi(b\hat{S})\xi = \rho(b)\check{S}\xi = \rho(b)\xi.$$

Therefore,

$$\rho(a)\rho(b)\xi = \rho(ab)\xi = \Pi(ab\hat{S})\xi = \Pi(kb\hat{S})\xi = \Pi(k)\Pi(b\hat{S})\xi = \Pi(k)\rho(b)\xi,$$

which implies that $\rho(a) = \Pi(k)$. ∎

There is another representation of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ obtained from ϕ which we would like to describe next.

Proposition 5.2   *Consider the above GNS representation ρ of A and let U denote the bilateral shift on $\ell^2(\mathbb{Z})$. Then there exists a representation $\tilde{\pi}$ of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ on $H \otimes \ell^2(\mathbb{Z})$ such that*

$$\tilde{\pi}(S) = \check{S} \otimes U, \quad and \quad \tilde{\pi}(a) = \rho(a) \otimes 1,$$

for all $a \in A$.

Proof   It is elementary to check that

$$(\check{S} \otimes U)(\rho(a) \otimes 1) = (\rho(\alpha(a)) \otimes 1)(\check{S} \otimes U)$$

and

$$(\check{S} \otimes U)^*(\rho(a) \otimes 1)(\check{S} \otimes U) = \rho(\mathcal{L}(a)) \otimes 1,$$

from which one concludes that there exists a representation $\tilde{\Pi}$ of $\mathcal{T}(A, \alpha, \mathcal{L})$ on $H \otimes \ell^2(\mathbb{Z})$ such that

$$\tilde{\Pi}(\hat{S}) = \check{S} \otimes U, \quad \text{and} \quad \tilde{\Pi}(a) = \rho(a) \otimes 1.$$

As before, we claim that $\tilde{\Pi}(a - k) = 0$ for any redundancy (a, k). In order to see this observe that given $b, c \in A$ we have

$$\tilde{\Pi}(b\hat{S}\hat{S}^*c) = (\rho(b) \otimes 1)(\check{S} \otimes U)(\check{S}^* \otimes U^*)(\rho(c) \otimes 1) = \Pi(b\hat{S}\hat{S}^*c) \otimes 1.$$

This implies that $\tilde{\Pi}(k) = \Pi(k) \otimes 1$ for all $k \in \overline{A\hat{S}\hat{S}^*A}$ and hence the claim follows from Proposition 5.1. It therefore follows that $\tilde{\Pi}$ factors through $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$, yielding the desired representation.   ∎

The main result of this section is in order:

Theorem 5.3   *Under Hypotheses 3.1 let ϕ be a state on A which is invariant under both α and $\mathcal{L}$. Suppose that the GNS representation of ϕ, denoted by ρ, is faithful. Then the representation $\tilde{\pi}$ of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ constructed above is faithful. Therefore, $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ is canonically isomorphic to the algebra of operators on $H \otimes \ell^2(\mathbb{Z})$ generated by the set*

$$\{\rho(a) \otimes 1 : a \in A\} \cup \{\check{S} \otimes U\},$$

where $\check{S}$ is given by (5.1) and U is the bilateral shift on $\ell^2(\mathbb{Z})$.

Proof   For each $z \in \mathbb{T}$ let V_z be the unitary operator on $\ell^2(\mathbb{Z})$ given on the canonical basis $\{e_n\}_{n \in \mathbb{Z}}$ by $V_z(e_n) = z^n e_n$. We then have that $V_z U V_z^* = z U$ which implies that

$$(1 \otimes V_z)\tilde{\pi}(S)(1 \otimes V_z^*) = z\,\tilde{\pi}(S),$$

while

$$(1 \otimes V_z)\tilde{\pi}(a)(1 \otimes V_z^*) = \tilde{\pi}(a),$$

for all $a \in A$. Let B be the C^*-algebra formed by the operators T on $H \otimes \ell^2(\mathbb{Z})$ for which the map $z \mapsto (1 \otimes V_z)T(1 \otimes V_z^*)$ is norm continuous. The above calculations show that $\tilde{\pi}$ maps $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ into B. Moreover it is clear that $\tilde{\pi}$ is covariant for the dual action on $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ and the action by conjugation by $1 \otimes V_z$ on B. The conclusion then follows from Theorem 4.2. ∎

6 The Arzumanian–Vershik Algebra

Let (X, μ) be a probability space and $T : X \to X$ be a measure-preserving transformation. It is easy to see that the correspondence

$$\alpha \colon f \in L^\infty(X, \mu) \mapsto f \circ T \in L^\infty(X, \mu)$$

gives an injective $*$-endomorphism of $L^\infty(X, \mu)$. We will assume that μ may be disintegrated along the fibers of T as in (2.1). Letting $\{\mu^x\}_{x \in X}$ be such a disintegration, it is straightforward to verify that the operator $\mathcal{L}$ defined by (2.2) is a normalized transfer operator. By (2.1) it is obvious that $\mathcal{L}$ is faithful. We therefore have all of the ingredients needed to form the crossed product algebra $L^\infty(X, \mu) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.

Denote by ϕ the state on $L^\infty(X, \mu)$ given by integration against μ. It is clear that ϕ is invariant under α. On the other hand, observe that (2.1) together with the fact mentioned there that $\nu = \mu$ says precisely that ϕ is invariant under $\mathcal{L}$ as well.

It is easy to see that $\mathrm{AV}(X, T, \mu)$ is precisely the range of the representation π of Proposition 5.1 while $\widetilde{\mathrm{AV}}(X, T, \mu)$ is the range of the representation $\tilde{\pi}$ of Proposition 5.2 . Since it is obvious that the GNS representation associated to ϕ is faithful on $L^\infty(X, \mu)$ we conclude from Theorem 5.3 that:

Theorem 6.1   *The generalized Arzumanian–Vershik algebra* $\widetilde{\mathrm{AV}}(X, T, \mu)$ *is canonically isomorphic to the crossed product* $L^\infty(X, \mu) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.

The above result does not rule out the possibility that $\mathrm{AV}(X, T, \mu)$ might be isomorphic to $L^\infty(X, \mu) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ as well. Since the former algebra is a quotient of the latter by Proposition 5.1, these algebras would be isomorphic in case $L^\infty(X, \mu) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is simple, for example.

Unfortunately we do not have a precise characterization of simplicity, but see below for a similar characterization when T is a covering map of a compact space X.

Theorem 6.1 also helps explain to what extent $\widetilde{\mathrm{AV}}(X, T, \mu)$ depends on μ. While there is no question that the probability measures μ^x influence the outcome, we see that this is as far as the dependence goes. In particular, if one takes another measure ν such that the ν^x coincide with the μ^x, then the algebras $\widetilde{\mathrm{AV}}(X, T, \mu)$ and $\widetilde{\mathrm{AV}}(X, T, \nu)$ will be isomorphic.

7 Finite Index Endomorphisms

In this section we return to the general case described in Hypotheses 3.1. We will denote the conditional expectation $\mathcal{E}_1$ of Proposition 3.2 simply by E. The main assumption to be added from here on is that E is of *index-finite type* according to

[26, 1.2.2 and 2.1.6]. That is, we will assume the existence of a *quasi-basis* for E, namely, a finite sequence $\{u_1, \ldots, u_m\} \subseteq A$ such that

$$(7.1) \qquad a = \sum_{i=1}^{m} u_i E(u_i^* a), \quad \forall\, a \in A.$$

In this case one defines the *index* of E by

$$\mathrm{ind}(E) = \sum_{i=1}^{m} u_i u_i^*.$$

It is well known that $\mathrm{ind}(E)$ does not depend on the choice of the u_i's, that it belongs to the center of A [26, 1.2.8] and is invertible [26, 2.3.1].

In particular, A is a finitely generated right B-module with $\{u_1, \ldots, u_m\}$ being a generating set. See also Proposition 8.2 below for a sufficient condition for E to be of index-finite type in the case that A is a commutative C^*-algebra.

We would first like to show that, under the present hypothesis, the process of dividing out by the redundancies in the definition of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ (Definition 2.2) can be achieved by adding a single relation.

Proposition 7.1 *Let $\{u_1, \ldots, u_m\}$ be a quasi-basis for E and denote by k_0 the element of $\mathcal{T}(A, \alpha, \mathcal{L})$ given by $k_0 = \sum_{i=1}^{m} u_i \hat{S}\hat{S}^* u_i^*$. Then the pair $(1, k_0)$ is a redundancy and hence*

$$1 = \sum_{i=1}^{m} u_i SS^* u_i^*$$

in $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$. Moreover the kernel of the natural quotient map $q \colon \mathcal{T}(A, \alpha, \mathcal{L}) \to A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ coincides with the closed two-sided ideal generated by $1 - k_0$.

Proof Given $b \in A$ we have

$$k_0 b \hat{S} = \sum_{i=1}^{m} u_i \hat{S}\hat{S}^* u_i^* b \hat{S} = \sum_{i=1}^{m} u_i \alpha(\mathcal{L}(u_i^* b))\hat{S} = \sum_{i=1}^{m} u_i E(u_i^* b)\hat{S} = b\hat{S},$$

proving the first statement. Next let (a, k) be any other redundancy. Then

$$k k_0 = \sum_{i=1}^{m} k u_i \hat{S}\hat{S}^* u_i^* = \sum_{i=1}^{m} a u_i \hat{S}\hat{S}^* u_i^* = a k_0.$$

It follows that

$$a - k = (a - k)(1 - k_0)$$

and hence that $a - k$ belongs to the ideal generated by $1 - k_0$. ∎

The description of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ in terms of generators and relations therefore becomes:

Corollary 7.2 *Under Hypotheses 3.1 suppose that the conditional expectation* $E = \alpha \circ \mathcal{L}$ *is of index-finite type and let* $\{u_1, \ldots, u_m\}$ *be a quasi-basis for E. Then* $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ *is the universal* C^**-algebra generated by a copy of A and an isometry S subject to the relations:*

(i) $Sa = \alpha(a)S$,
(ii) $S^* a S = \mathcal{L}(a)$, *and*
(iii) $1 = \sum_{i=1}^{m} u_i SS^* u_i^*$,

for all a in A.

Proof   Follows immediately from the theorem above.   ∎

We will take the remainder of this section to develop a few technical consequences of the fact that E is of index-finite type, to be used in later sections. Some of these are interesting in their own right. The first one is essentially the combined contents of [12, Propositions 8.2 and 8.3]:

Proposition 7.3   *If* $\{u_1, \ldots, u_m\}$ *is a quasi-basis for E and* $n \in \mathbb{N}$ *then*

(i) $\sum_{i=1}^{m} \alpha^n(u_i) S^{n+1} S^{*n+1} \alpha^n(u_i^*) = S^n S^{*n}$,
(ii) $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$.

Proof   We have

$$\sum_{i=1}^{m} \alpha^n(u_i) S^{n+1} S^{*n+1} \alpha^n(u_i^*) = \sum_{i=1}^{m} S^n u_i SS^* u_i^* S^{*n} = S^n S^{*n},$$

proving (i). The second point then follows immediately from (i).   ∎

It follows that the fixed point algebra for the gauge action, namely $\mathcal{U}$, is the inductive limit of the $\mathcal{K}_n$. Recall from [12, 8.9] that there exists a conditional expectation $G \colon A \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \to A$ such that

$$G(aS^n S^{*m} b) = \delta_{nm} a I_n^{-1} b, \quad \forall a, b \in A, \quad \forall n, m \in \mathbb{N},$$

where δ is the Kronecker symbol and

$$(7.2) \qquad\qquad I_n = \mathrm{ind}(E) \alpha(\mathrm{ind}(E)) \cdots \alpha^{n-1}(\mathrm{ind}(E)).$$

Proposition 7.4   *Let* $\{u_1, \ldots, u_m\}$ *be a quasi-basis for E and put* $Z = \{1, \ldots, m\}$. *For each* $n \in \mathbb{N} \setminus \{0\}$ *and for each multi-index* $i = (i_0, i_1, \ldots, i_{n-1}) \in Z^n$ *let*

$$u_{(i)} = u_{i_0} \alpha(u_{i_1}) \alpha^2(u_{i_2}) \cdots \alpha^{n-1}(u_{i_{n-1}}).$$

Then

(i) $\sum_{i \in Z^n} u_{(i)} S^n S^{*n} u_{(i)}^* = 1$.
(ii)   *If A is commutative then* $\sum_{i \in Z^n} u_{(i)} u_{(i)}^* = I_n$.

Proof For $n = 1$ the conclusion follows from Proposition 7.1 and the definition of $\mathrm{ind}(E)$. So assume that $n > 1$ and observe that

$$\sum_{i \in Z^n} u_{(i)} S^n S^{*n} u_{(i)}^* = \sum_{i \in Z^{n-1}} \sum_{j=1}^m u_{(i)} \alpha^{n-1}(u_j) S^n S^{*n} \alpha^{n-1}(u_j^*) u_{(i)}^*)$$

$$= \sum_{i \in Z^{n-1}} u_{(i)} S^{n-1} S^{*n-1} u_{(i)}^* \qquad \text{Proposition 7.3(i)}$$

$$= 1,$$

where the last step is by induction. As for (ii) observe that

$$\sum_{i \in Z^n} u_{(i)} u_{(i)}^* = \sum_{i \in Z^{n-1}} \sum_{j=1}^m u_{(i)} \alpha^{n-1}(u_j) \alpha^{n-1}(u_j^*) u_{(i)}^*$$

$$= \sum_{i \in Z^{n-1}} u_{(i)} \alpha^{n-1}(\mathrm{ind}(E)) u_{(i)}^* = \alpha^{n-1}(\mathrm{ind}(E)) \sum_{i \in Z^{n-1}} u_{(i)} u_{(i)}^*,$$

and the conclusion again follows by induction. ∎

As a consequence we will see that the conditional expectation G may be described by an algebraic expression on each K_n when A is abelian.

Corollary 7.5 *Suppose that E is of index-finite type and A is a commutative C^*-algebra. Then for each $n \in \mathbb{N}$ there exists a finite set $\{v_1, \dots, v_p\} \subseteq A$ such that*

(i)   *$G(a) = \sum_{i=1}^p v_i a v_i^*$, for all $a \in \mathcal{K}_n$,*
(ii) *$\sum_{i=1}^p v_i v_i^* = 1$.*

Proof Let $\{u_{(i)}\}_{i \in Z^n}$ be as in Proposition 7.4 and for $i \in Z^n$ set $v_{(i)} = I_n^{-1/2} u_{(i)}$, where I_n is defined in (7.2). Then for any $a, b \in A$ we have

$$\sum_{i \in Z^n} v_{(i)} a S^n S^{*n} b v_{(i)}^* = a I_n^{-1/2} \Big(\sum_{i \in Z^n} u_{(i)} S^n S^{*n} u_{(i)}^* \Big) I_n^{-1/2} b = a I_n^{-1} b = G(a S^n S^{*n} b).$$

Finally observe that plugging $a = 1$ in (i) gives (ii). ∎

Recall that $\mathcal{E}_n$ was defined to be the composition $\mathcal{E}_n = \alpha^n \circ \mathcal{L}^n$. If we observe that $\mathcal{L} = \alpha^{-1} \circ E$ we may write $\mathcal{E}_n$ as

$$\mathcal{E}_n = \alpha^n \underbrace{(\alpha^{-1} E) \cdots (\alpha^{-1} E)}_{n \text{ times}} = \big(\alpha^{n-1} E \alpha^{-(n-1)} \big) \cdots \big(\alpha^2 E \alpha^{-2} \big) \big(\alpha E \alpha^{-1} \big) E.$$

Viewing each $\alpha^k E \alpha^{-k}$ as a conditional expectation from $\alpha^k(A)$ to $\alpha^{k+1}(A)$, it is obvious that it is of index-finite type. By [26, 1.7.1] $\mathcal{E}_n$ is of index-finite type as well and hence by [26, 2.1.5] there are constants $\lambda_n > 0$ such that $\|\mathcal{E}_n(a^* a)\|^{1/2} \geq \lambda_n \|a\|$, for all a in A.

Lemma 7.6   Suppose that E is of index-finite type. Then for each $n \in \mathbb{N}$ one has that AS^n is closed in $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ so that $\mathcal{M}_n = AS^n$ (without closure).

Proof   For each $a \in A$ we have

$$\|aS^n\| = \|S^{*n}a^*aS^n\|^{1/2} = \|\mathcal{L}^n(a^*a)\|^{1/2} = \|\alpha^n(\mathcal{L}^n(a^*a))\|^{1/2}$$
$$= \|\mathcal{E}_n(a^*a)\|^{1/2} \geq \lambda_n\|a\|.$$

Therefore the map $a \in A \mapsto aS^n \in \mathcal{M}_n$ is a Banach space isomorphism onto its range which in turn is a complete normed space, hence closed. ∎

Proposition 7.7   Suppose that E is of index-finite type and let $n \in \mathbb{N}$. Then any element $k \in \mathcal{K}_n$ may be written as a finite sum of the form $\sum_{i=1}^m a_i S^n S^{*n} b_i^*$, where $m \in \mathbb{N}$ and $a_i, b_i \in A$. If $k \geq 0$ we may take $b_i = a_i$.

Proof   Let $\{u_{(i)}\}_{i \in \mathbb{Z}^n}$ be as in Proposition 7.4. Then for each $i \in \mathbb{Z}^n$ we have by Proposition 3.4(i) that $ku_{(i)}S^n \in \mathcal{M}_n = AS^n$. So there is $a_i \in A$ such that $ku_{(i)}S^n = a_iS^n$ and by Proposition 7.4(i)

$$k = k\Big(\sum_{i \in \mathbb{Z}^n} u_{(i)} S^n S^{*n} u_{(i)}^*\Big) = \sum_{i \in \mathbb{Z}^n} a_i S^n S^{*n} u_{(i)}^*.$$

If $k \geq 0$ write $k = l^*l$, with $l = \sum_{i=1}^m x_i S^n S^{*n} y_i$, and $x_i, y_i \in A$. Then

$$k = \sum_{i,j=1}^m x_i S^n S^{*n} y_i y_j^* S^n S^{*n} x_j^* = \sum_{i,j=1}^m x_i \mathcal{E}_n(y_i y_j^*) S^n S^{*n} x_j^*.$$

Since $(\mathcal{E}_n(y_i y_j^*))_{i,j}$ is a positive $m \times m$ matrix over $\alpha^n(A)$ by [24, IV.3.4] there exists another such matrix, say $(c_{i,j})_{i,j}$ such that $\mathcal{E}_n(y_i y_j^*) = \sum_{k=1}^m c_{ik} c_{jk}^*$ for all i and j. Therefore

$$k = \sum_{i,j,k=1}^m x_i c_{ik} c_{jk}^* S^n S^{*n} x_j^* = \sum_{i,j,k=1}^m x_i c_{ik} S^n S^{*n} c_{jk}^* x_j^*$$
$$= \sum_{k=1}^m \Big(\sum_{i=1}^m x_i c_{ik}\Big) S^n S^{*n} \Big(\sum_{j=1}^m c_{jk}^* x_j^*\Big) = \sum_{k=1}^m a_k S^n S^{*n} a_k^*,$$

where $a_k = \sum_{i=1}^m x_i c_{ik}$. ∎

Proposition 7.8   If E is of index-finite type the restriction of the conditional expectation G above to each $\mathcal{K}_n$ is faithful.

Proof Let $k \geq 0$ in $\mathcal{K}_n$ be such that $G(k) = 0$. Write $k = \sum_{i=1}^{m} a_i S^n S^{*n} a_i^*$ as in Proposition 7.7. Then

$$0 = G\left(\sum_{i=1}^{m} a_i S^n S^{*n} a_i^* \right) = \sum_{i=1}^{m} a_i I_n a_i^*,$$

which implies that $a_i I_n a_i^* = 0$, for all i, and hence also that $a_i I_n^{1/2} = 0$. Since I_n is invertible we have that $a_i = 0$ and so $k = 0$. ∎

We will now see that G is also faithful on $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ in the following weak sense.

Proposition 7.9 *Suppose that E is of index-finite type and let $a \geq 0$ in $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ be such that $G(xax^*) = 0$ for all $x \in \mathcal{U}$. Then $a = 0$.*

Proof Using the conditional expectation F of (3.1) we have

$$0 = G(xax^*) = G(F(xax^*)) = G(xF(a)x^*),$$

for all $x \in \mathcal{U}$. Observe that this allows us to suppose that $a \in \mathcal{U}$. In fact, should the result be proved in this special case, we would have that $F(a) = 0$ in which case $a = 0$ since F is faithful. We therefore suppose that $a \in \mathcal{U}$.

By the polarization identity we have that $G(xay^*) = 0$ for all $x, y \in \mathcal{U}$. Supposing by way of contradiction that $a \neq 0$ observe that the closed two-sided ideal J of $\mathcal{U}$ given by

$$J = \{b \in \mathcal{U} : G(xby^*) = 0, \ \forall x, y \in \mathcal{U}\}.$$

is nontrivial. Since $\mathcal{U} = \overline{\bigcup_{n \in \mathbb{N}} \mathcal{K}_n}$, it follows from [1, 1.3] (see also [7, 3.1]) that there exists a nonzero element $b \in J \cap \mathcal{K}_n$ for some n. But since $G(b^*b) = 0$ we have by Proposition 7.8 that $b = 0$, which is a contradiction. ∎

8 Covering Maps

From this section on we will let X be a compact topological space and $T : X \to X$ be a continuous surjective map. We will then consider the C^*-algebra $A = C(X)$ equipped with the endomorphism α given by

$$(8.1) \qquad\qquad \alpha : f \in C(X) \mapsto f \circ T \in C(X).$$

We will assume another important hypothesis, namely that T is a covering map. As we will see this will allow us to define a somewhat canonical transfer operator.

Proposition 8.1 *If $T \colon X \to X$ is a covering map, then for each $f \in A$ the function $\mathbf{L}(f)$ given by*

$$\mathbf{L}(f)|_x = \sum_{t \in T^{-1}(x)} f(t), \quad \forall x \in X,$$

is continuous on X.

Proof   Given $x_0 \in X$ let W be an open neighborhood of x_0 such that $T^{-1}(W)$ is the disjoint union of open sets $U_1, \ldots, U_n$ such that T is a homeomorphism from each U_i onto W. For each $i = 1, \ldots, n$ let $\phi_i \colon W \to U_i$ be the inverse of the restriction of T to U_i. Then for all $x \in W$ one has that $T^{-1}(x) = \{\phi_1(x), \ldots, \phi_n(x)\}$ and hence

$$\mathbf{L}(f)|_x = \sum_{i=1}^{n} f(\phi_i(x))$$

so that $\mathbf{L}(f)$ is seen to be continuous at x_0.   ∎

Setting

$$(8.2) \qquad \mathcal{L}(f) = \mathbf{L}(1)^{-1}\mathbf{L}(f) \quad \text{and} \quad E = \alpha \circ \mathcal{L},$$

it is easy to see that $\mathcal{L}$ is a faithful transfer operator for α, that E is a conditional expectation from A onto the range of α, and that $\mathcal{L} = \alpha^{-1} \circ E$. Adopting the notation $\Lambda = \alpha(\mathbf{L}(1))$ it is easy to see that for all $x \in X$

$$(8.3) \qquad \Lambda(x) = \#\{t \in X : T(t) = T(x)\}$$

and

$$(8.4) \qquad E(f)|_x = \frac{1}{\Lambda(x)} \sum_{\substack{t \in X \\ T(t)=T(x)}} f(t).$$

The next result is taken from [12, §11] and is reproduced here for the convenience of the reader:

Proposition 8.2   *Let $\{V_i\}_{i=1}^{m}$ be a finite open covering of X such that the restriction of T to each V_i is one-to-one and let $\{v_i\}_{i=1}^{m}$ be a partition of unity subordinate to that covering. Setting $u_i = (\Lambda v_i)^{1/2}$ one has that $\{u_i\}_{i=1}^{m}$ is a quasi-basis for E and hence E is of index-finite type. Moreover $\mathrm{ind}(E) = \Lambda$.*

Proof   Observe that for all $f \in A$ and $x \in X$ one has that

$$\sum_{i=1}^{m} u_i E(u_i f)|_x = \sum_{i=1}^{m} u_i(x) \frac{1}{\Lambda(x)} \sum_{\substack{t \in X \\ T(t)=T(x)}} u_i(t) f(t)$$

$$= \sum_{i=1}^{m} u_i(x) \frac{1}{\Lambda(x)} u_i(x) f(x) = \sum_{i=1}^{m} v_i(x) f(x) = f(x).$$

Therefore $\{u_1, \ldots, u_m\}$ is a quasi-basis for E and

$$\mathrm{ind}(E) = \sum_{i=1}^{m} u_i^2 = \sum_{i=1}^{m} \Lambda v_i = \Lambda.$$ ∎

In the final result of this section we denote by $\mathrm{supp}(f)$ the support of the function f, namely the closure of the set $\{x : f(x) \neq 0\}$.

Lemma 8.3   *Let $\mathcal{L}$ be any transfer operator for α (such as the one defined in (8.2) but not necessarily). Then for $f \in C(X)$ one has*

(i) $\operatorname{supp}(\alpha(f)) \subseteq T^{-1}(\operatorname{supp}(f))$, *and*
(ii) $\operatorname{supp}(\mathcal{L}(f)) \subseteq T(\operatorname{supp}(f))$.

Proof   The first point is trivial. As for (ii), let x be such that $\mathcal{L}(f)|_x \neq 0$. We claim that $x \in T(\operatorname{supp}(f))$. Otherwise let g be a continuous function on X which vanishes on $T(\operatorname{supp}(f))$ and such that $g(x) = 1$. Then $f\alpha(g) = 0$ because for all $y \in X$

$$f(y)\alpha(g)(y) = f(y)g(T(y)),$$

and if $f(y) \neq 0$ then $y \in \operatorname{supp}(f)$ in which case $g(T(y)) = 0$. We then have

$$\mathcal{L}(f)|_x = \mathcal{L}(f)|_x g(x) = \mathcal{L}(f)g|_x = \mathcal{L}\big(f\alpha(g)\big)|_x = 0,$$

contradicting our assumption. This proves our claim that $x \in T(\operatorname{supp}(f))$. Since the latter is closed the conclusion follows.   ∎

9   The Algebra $C^*(X, T)$

As before let X be a compact topological space and let $T : X \to X$ be a covering map. We will suppose, as in [8], that each point of X has exactly p inverse images under T.

Motivated by earlier work of Renault on the Cuntz groupoid [21] and on Arzumanian and Vershik's paper [4], Deaconu [8] has considered the set

$$\Gamma = \{(x, n, y) \in X \times \mathbb{Z} \times X : \exists\, k, l \in \mathbb{N},\ n = l - k,\ T^k(x) = T^l(y)\}$$

with the following groupoid structure: the product of (x, n, y) and (w, m, z) is defined if (and only if) $y = w$ in which case it is given by $(x, n + m, z)$. The inverse of (x, n, y) is set to be $(y, -n, x)$. Deaconu has proven that Γ admits the structure of an r-discrete locally compact groupoid with Haar system given by counting measures (see [8, Theorem 1]). We will denote the corresponding groupoid C^*-algebra [21] by $C^*(X, T)$.

We will view $C(X)$ as a subalgebra of $C^*(X, T)$ by identifying a function $f \in C(X)$ with the function (also denoted) f on Γ given as

$$f(x, n, y) = [n = 0\ \wedge\ x = y]f(x),$$

where the brackets correspond to the obvious boolean valued function. Let v be the element of $C^*(X, T)$ defined shortly before the statement of Theorem 2 in [8] by

$$v(x, n, y) = [n = -1\ \wedge\ y = T(x)]/\sqrt{p}$$

where p is the index of the covering.

Theorem 9.1 *Let X be a compact topological space and $T\colon X \to X$ be a covering map. Consider the endomorphism α of $C(X)$ given by (8.1) and the transfer operator $\mathcal{L}$ defined in (8.2). Then there exists a *-isomorphism ϕ from $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ onto $C^*(X, T)$ which is the identity on $C(X)$ and such that $\phi(S) = v$.*

Proof By [8] we have that

$$(vv^*)(x, n, y) = [n = 0 \ \wedge \ T(x) = T(y)]/p.$$

Let v_i, u_i, and Λ be as in Proposition 8.2 observing that in the present case Λ is the constant function equal to p. Since T is injective when restricted to the set where each u_i does not vanish, we have

$$
\begin{aligned}
(u_i vv^* u_i)(x, n, y) &= u_i(x) \, (vv^*)(x, n, y) \, u_i(y) \\
&= [n = 0 \ \wedge \ T(x) = T(y)]u_i(x)u_i(y)/p \\
&= [n = 0 \ \wedge \ x = y]u_i(x)^2/p = [n = 0 \ \wedge \ x = y]v_i(x).
\end{aligned}
$$

Since the v_i form a partition of unity, we have that

$$\sum_{i=1}^{m} u_i vv^* u_i = 1.$$

By direct computation one can prove that $vf = \alpha(f)v$ and $v^* fv = \mathcal{L}(f)$ for each $f \in C(X)$, so by Corollary 7.2 there exists a *-homomorphism

$$\phi\colon C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \to C^*(X, T)$$

such that $\phi(f) = f$, for all $f \in C(X)$ and $\phi(S) = v$. It is clear that ϕ is injective on $C(X)$ and covariant for the gauge action on $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ and the circle action on $C^*(X, T)$ described in [8] in terms of the cocycle

$$c(x, n, y) = -n.$$

That ϕ is injective then follows from Theorem 4.2. On the other hand we have that $C(X) \cup \{v\}$ generates $C^*(X, T)$ as a C^*-algebra and hence ϕ is surjective as well. ∎

The description of the $C^*(X, T)$ in terms of generators and relations therefore becomes:

Theorem 9.2 *Let X be a compact topological space and let $T\colon X \to X$ be a covering map. Then $C^*(X, T)$ is the universal C^*-algebra generated by a copy of $C(X)$ and an isometry S subject to the relations*
(i) $Sf = \alpha(f)S$,
(ii) $S^* fS = \mathcal{L}(f)$,
(iii) $1 = \sum_{i=1}^{m} u_i SS^* u_i^*$,

for all f in C(X), where $\mathcal{L}$ is given by (8.2) *and the u_i are as in Proposition 8.2.*

Proof   This follows immediately from the result above and Corollary 7.2.   ■

Before closing this section we should remark that an example of the situation treated above is given by the Markov one-sided subshift associated to a zero-one matrix A with no zero rows, in which case $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_A$ by [11, Theorem 6.2].

10   Topological Freeness

In this section we will obtain another result about faithfulness of representations. This time, instead of requiring gauge-covariance as in Theorem 4.2, we will make assumptions on the dynamical properties of the transformation T.

We keep the standing assumptions of Section 8, namely that T is a covering map of the compact space X and the endomorphism α of $C(X)$ is given by (8.1). The transfer operator $\mathcal{L}$ will be given, as before, by (8.2).

We begin by describing the main hypothesis to be used below (see also [25, 2.1], [2], [8], and [14, 2.1]).

Definition 10.1   We will say that the dynamical system (X, T) is *topologically free* if for every pair of nonnegative integers (n, m) with $n \neq m$, the set

$$\{x \in X : T^n(x) = T^m(x)\}$$

has empty interior.

The next result is motivated by [14, Lemma 2.3].

Lemma 10.2   *Let $x_0 \in X$ and $n, m \in \mathbb{N}$ be such that $T^n(x_0) \neq T^m(x_0)$. Then there exists $h \in C(X)$ with*

(i) $0 \leq h \leq 1$,
(ii) $h(x_0) = 1$,
(iii) $hS^n S^{*m} h = 0$.

Proof   We begin by claiming that there exists an open set $U \subseteq X$ with $x_0 \in U$ such that $U \cap T^{-n}(T^m(U)) = \varnothing$. In fact let $A, B \subseteq X$ be disjoint open sets such that $T^n(x_0) \in A$ and $T^m(x_0) \in B$ and set $U = T^{-n}(A) \cap T^{-m}(B)$. Then obviously $x_0 \in U$ and $T^n(U) \cap T^m(U) = \varnothing$. In addition

$$\varnothing = T^{-n}\big(T^n(U) \cap T^m(U)\big) = T^{-n}\big(T^n(U)\big) \cap T^{-n}\big(T^m(U)\big) \supseteq U \cap T^{-n}\big(T^m(U)\big),$$

thus proving our claim. Pick $h \in C(X)$ satisfying (i) and (ii) above and such that supp$(h) \subseteq U$. Then

$$\|hS^n S^{*m} h\|^2 = \|hS^n S^{*m} h^2 S^m S^{*n} h\| = \|h\alpha^n(\mathcal{L}^m(h^2))S^n S^{*n} h\|.$$

So it suffices to prove that $h\alpha^n(\mathcal{L}^m(h^2)) = 0$. For this purpose observe that by [11, 3.2] we have that $\mathcal{L}^m$ is a transfer operator for α^m and hence by Lemma 8.3 we have

$$\operatorname{supp}\bigl(\alpha^n(\mathcal{L}^m(h^2))\bigr) \subseteq T^{-n}\bigl(\operatorname{supp}(\mathcal{L}^m(h^2))\bigr) \subseteq T^{-n}\bigl(T^m(\operatorname{supp}(h^2))\bigr)$$
$$\subseteq T^{-n}(T^m(U)).$$

Since the latter set is disjoint from U, and hence also disjoint from $\operatorname{supp}(h)$, we have that $h\alpha^n(\mathcal{L}^m(h^2))$ is indeed zero. ∎

We now come to the main result of this section (for similar results see [1, 1.2], [6, 2.1], [14, 2.6], [16, 3.7], [23, 2.1]).

Theorem 10.3 *Let T be a topologically free covering map of the compact space X, let α be the endomorphism of $C(X)$ given by (8.1), and $\mathcal{L}$ be the transfer operator given by (8.2). Then any nontrivial ideal of $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ must have a nontrivial intersection with $C(X)$. Given a C^*-algebra B and a *-homomorphism $\pi\colon C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N} \to B$ which is faithful on $C(X)$ one has that π is faithful on all of $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.*

Proof Let $\pi\colon C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N} \to B$ be faithful on $C(X)$. We begin by claiming that

$$(\dagger) \qquad\qquad\qquad \|G(a)\| \leq \|\pi(a)\|$$

for all $a \geq 0$ in $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$. In order to prove this observe that the set of nonnegative elements of the form

$$(\ddagger) \qquad\qquad\qquad a = \sum_{i=1}^{t} a_i S^{n_i} S^{*m_i} b_i,$$

where $n_i, m_i \in \mathbb{N}$ and $a_i, b_i \in C(X)$, is dense in the positive cone of $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$. In fact, given any $a \geq 0$ in $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ one may write $a = b^*b$ and then approximate b by elements of the above form by [12, 2.3]. Using [12, 2.2] we have that b^*b again has the above form and clearly b^*b can be made to be as close as necessary to a. In order to prove $(\dagger)$ we may therefore assume that a has the form of $(\ddagger)$. Let us also suppose that the sum in $(\ddagger)$ is arranged in such a way that $n_i = m_i$ for all $i = 1, \dots, s$, while $n_i \neq m_i$ for $i = s+1, \dots, t$.

For each $i = s+1, \dots, t$ we have by hypothesis that $\{x \in X : T^{n_i}(x) \neq T^{m_i}(x)\}$ is an open dense set and hence so is their intersection. Consequently, fixing a real number ρ with $0 < \rho < 1$, there exists $x_0 \in X$ such that $G(a)|_{x_0} > \rho\|G(a)\|$ and $T^{n_i}(x_0) \neq T^{m_i}(x_0)$ for all $i = s+1, \dots, t$.

Using Lemma 10.2 choose for each $i = s+1, \dots, t$ an $h_i \in C(X)$ with $0 \leq h_i \leq 1$, $h_i(x_0) = 1$, and $h_i S^{n_i} S^{*m_i} h_i = 0$. Setting $h = h_{s+1} \cdots h_t$ we therefore have that

$$hah = \sum_{i=1}^{t} a_i h S^{n_i} S^{*m_i} h b_i = \sum_{i=1}^{s} a_i h S^{n_i} S^{*n_i} h b_i = hF(a)h,$$

where F is given by (3.1). Since $F(a) \in \mathcal{K}_n$, where $n = \max\{n_1, \ldots, n_s\}$, by Proposition 7.3(ii) we may choose by Corollary 7.5 a finite set $\{v_1, \ldots, v_p\} \subseteq C(X)$ such that $\sum_{i=1}^{p} v_i v_i^* = 1$ and

$$G(a) = G(F(a)) = \sum_{i=1}^{p} v_i F(a) v_i^*.$$

It follows that

$$\|hG(a)h\| = \|\pi(hG(a)h)\| = \left\| \sum_{i=1}^{p} \pi(hv_i F(a) v_i^* h) \right\|$$

$$= \left\| \sum_{i=1}^{p} \pi(v_i hah v_i^*) \right\| \leq \|\pi(a)\|.$$

Since

$$\|hG(a)h\| \geq h(x_0) G(a)|_{x_0} h(x_0) = G(a)|_{x_0} > \rho \|G(a)\|$$

and ρ is arbitrary we conclude that $\|G(a)\| \leq \|\pi(a)\|$ as claimed.

Let $a \in C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ be such that $\pi(a) = 0$. Then for any $b \in C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ we have

$$\|G(b^* a^* ab)\| \leq \|\pi(b^* a^* ab)\| = 0,$$

and hence $G(b^* a^* ab) = 0$. Applying Proposition 7.9 we conclude that $a^* a = 0$ and hence also that $a = 0$.

The assertion about ideals in the statement is proved as follows. Given an ideal J of $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ which is not the trivial ideal let π be the quotient map. Then π cannot be faithful on $C(X)$ or else it would be faithful on all of $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ by what we have already proved. It follows that $J \cap C(X) \neq \{0\}$. ∎

11 Simplicity

As before, we fix a compact space X and a covering map T of X, and we let α and $\mathcal{L}$ be given respectively by (8.1) and (8.2). Our main goal here will be to find a necessary and sufficient condition for $C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ to be simple.

Recall that two points $x, y \in X$ are said to be *trajectory-equivalent* (see *e.g.,* [3]) when there are $n, m \in \mathbb{N}$ such that $T^n(x) = T^m(y)$. We will denote this by $x \sim y$. A subset $Y \subseteq X$ is said to be *invariant* if $x \sim y \in Y$ implies that $x \in Y$. It is easy to see that Y is invariant if and only if $T^{-1}(Y) = Y$. We will say that T is *irreducible* when there is no closed (equivalently open) invariant set other than $\varnothing$ and X. Notice that irreducibility is weaker than the condition of minimality defined in [8].

Proposition 11.1   *If T is irreducible then either*

(i)   *T is topologically free, or*
(ii)   *X is finite and T is a cyclic permutation of X.*

Proof Suppose that T is irreducible and not topologically free. Then there is a nonempty open set $U \subseteq X$ and $n, m \in \mathbb{N}$ with $n \neq m$ such that $T^n(u) = T^m(u)$ for all u in U. Clearly $\bigcup_{k,l \in \mathbb{N}} T^{-k}(T^l(U))$ is an open invariant set so, being nonempty, it must coincide with X. In this case $\{T^{-k}(T^l(U))\}_{k,l \in \mathbb{N}}$ is an open cover of X and hence admits a finite subcover, say $\{T^{-k_i}(T^{l_i}(U))\}_{i=1,\dots,p}$. Given any $x \in X$ choose i such that $x \in T^{-k_i}(T^{l_i}(U))$. Therefore there exists $u \in U$ such that $T^{k_i}(x) = T^{l_i}(u)$. It follows that

$$T^{n+k_i}(x) = T^n(T^{l_i}(u)) = T^{l_i}(T^n(u)) = T^{l_i}(T^m(u)) = T^m(T^{l_i}(u)) = T^{m+k_i}(x).$$

Setting $k = \max\{k_1, \dots, k_p\}$ we therefore see that

$$T^{n+k}(x) = T^{m+k}(x), \quad \forall\, x \in X.$$

Assuming without loss of generality that $n > m$, and setting $r = m+k$ and $s = n-m$, this translates into $T^{s+r}(x) = T^r(x)$. Since T^r is a surjective map we conclude that $T^s = 1$. The reader may now easily prove that we must be in the situation described by (ii). ∎

The next result improves on Deaconu's characterization of simplicity [8]. It is also a generalization of the corresponding result for crossed products by automorphisms (see [9, 20, 25]).

Theorem 11.2 *Let T be a covering map of an infinite compact space X to itself, let α be the endomorphism of $C(X)$ given by* (8.1)*, and let $\mathcal{L}$ be the transfer operator given by* (8.2)*. Then $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is simple if and only if T is irreducible.*

Proof Supposing that T is irreducible, let J be an ideal in $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$. Obviously $J \cap C(X)$ is an ideal in $C(X)$ and hence coincides with $C_0(U)$ for some open set $U \subseteq X$. We claim that U is invariant. For this we need to prove that $x \sim y \in U$ implies that $x \in U$.

Let f be a nonnegative function in $C_0(U)$ such that $f(y) = 1$. Given that $x \sim y$, let $n, m \in \mathbb{N}$ be such that $T^n(x) = T^m(y)$. Taking $\{u_{(i)}\}_{i \in Z^n}$ as in Proposition 7.4 we therefore have that

$$J \ni \sum_{i \in Z^n} u_{(i)} S^n S^{*m} f S^m S^{*n} u_{(i)}^* = \sum_{i \in Z^n} u_{(i)} \alpha^n(\mathcal{L}^m(f)) S^n S^{*n} u_{(i)}^*$$
$$= \alpha^n(\mathcal{L}^m(f)) \sum_{i \in Z^n} u_{(i)} S^n S^{*n} u_{(i)}^* = \alpha^n(\mathcal{L}^m(f)),$$

by Proposition 7.4(i). It follows that $\alpha^n\big(\mathcal{L}^m(f)\big) \in C_0(U)$. On the other hand, observe that

$$\alpha^n\big(\mathcal{L}^m(f)\big)|_x = \mathcal{L}^m(f)|_{T^n(x)} = \mathcal{L}^m(f)|_{T^m(y)} = \alpha^m(\mathcal{L}^m(f))|_y$$
$$= \mathcal{E}_m(f)|_y \geq c f(y) > 0,$$

where the constant c appearing above is obtained from [26, 2.1.5]. One must then have that $x \in U$, which concludes the proof of the claim that U is invariant. Since T is irreducible we have that U is either empty or equal to X. In case $U = \varnothing$ we have that $J \cap C(X) = \{0\}$, but since T is topologically free by Proposition 11.1 we have that $J = \{0\}$ by Theorem 10.3. On the other hand if $U = X$ we have that $1 \in C_0(U) \subseteq J$ in which case $J = C(X) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$.

Let us now suppose that T is not irreducible, that is, that there exists a nontrivial open invariant subset $U \subseteq X$. Let $I = C_0(U)$. It is then clear that $\alpha(I) \subseteq I$ and $\mathcal{L}(I) \subseteq I$. The conclusion then follows from the following result which actually holds under more general hypotheses. ∎

Proposition 11.3 *Let α be an injective *-endomorphism of a unital C^*-algebra A with $\alpha(1) = 1$ and let $\mathcal{L}$ be a transfer operator for α such that $\mathcal{L}(1) = 1$. If there exists a closed two-sided ideal I of A such that $\alpha(I) \subseteq I$ and $\mathcal{L}(I) \subseteq I$ then there exists a closed two-sided ideal J of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ such that $J \cap A = I$.*

Proof Consider the operators $\tilde{\alpha}$ and $\tilde{\mathcal{L}}$ on A/I obtained by passing α and $\mathcal{L}$ to the quotient, respectively. It is then obvious that $\tilde{\alpha}$ is an endomorphism of A/I and $\tilde{\mathcal{L}}$ is a transfer operator for $\tilde{\alpha}$. Denote by $\tilde{S}$ the standard isometry of $(A/I) \rtimes_{\tilde{\alpha}, \tilde{\mathcal{L}}} \mathbb{N}$ and view the quotient map of A modulo I as a *-homomorphism

$$q \colon A \to A/I \subseteq (A/I) \rtimes_{\tilde{\alpha}, \tilde{\mathcal{L}}} \mathbb{N}.$$

It is elementary to check that

$$\tilde{S}q(a) = q(\alpha(a))\tilde{S}, \quad \text{and} \quad \tilde{S}^*q(a)\tilde{S} = q(\mathcal{L}(a)),$$

for all $a \in A$, which implies that there exists a *-homomorphism

$$\phi \colon \mathcal{T}(A, \alpha, \mathcal{L}) \to (A/I) \rtimes_{\tilde{\alpha}, \tilde{\mathcal{L}}} N$$

such that $\phi(a) = q(a)$, for all $a \in A$, and $\phi(\hat{S}) = \tilde{S}$. If $(a, k) \in A \times \overline{A\hat{S}\hat{S}^*A}$ is a redundancy it is easy to see that $\phi(a) = \phi(k)$ and hence that ϕ drops to the quotient, providing a *-homomorphism

$$\psi \colon A \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \to (A/I) \rtimes_{\tilde{\alpha}, \tilde{\mathcal{L}}} N$$

which coincides with ϕ, and hence also with q, on A. The kernel of ψ is then the desired ideal. ∎

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