THE CONVOLUTION $x^{-r} * x^s$

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1. Introduction. In a recent paper [1], Jones extended the definition of the convolution of distributions so that further convolutions could be defined. The convolution $w_1 * w_2$ of two distributions w_1 and w_2 was defined as the limit of the sequence $\{w_{1n} * w_{2n}\}$, provided the limit w exists in the sense that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi(x) w_{1n} * w_{2n} dx = \int_{-\infty}^{\infty} \phi(x) w(x) dx$$

for all fine functions ϕ in the terminology of Jones [2], where

$$w_{1n}(x) = w_1(x)\tau(x/n), \quad w_{2n}(x) = w_2(x)\tau(x/n)$$

and τ is an infinitely differentiable function satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leqslant \tau(x) \leqslant 1$,
(iii) $\tau(x) = 1$ for $|x| \leqslant 1/2$,
(iv) $\tau(x) = 0$ for $|x| \geqslant 1$.

2. The convolution $x^{-r} * x^s$. We prove that in the sense of Jones' definition of convolution for distributions that

$$x^{-r} * x^s = 0 \tag{1}$$

for $s = 0, 1, 2, \ldots, r-1$ and $r = 1, 2, \ldots$.

We write

$$(x^{-r})_n = x^{-r}\tau(x/n), \quad (x^s)_n = x^s\tau(x/n).$$

Formally we have

$$(x^{-r})_n * (x^s)_n = \int_{-\infty}^{\infty} t^{-r}\tau(t/n)(x-t)^s \tau\left(\frac{x-t}{n}\right) dt$$

but since t^{-r} is not a summable function we must interpret the integral in the distributional sense. Putting

$$\phi_s(t) = \tau(t/n)(x-t)^s \tau\left(\frac{x-t}{n}\right),$$

we note that $\phi_s(t)$ is a fine function so that we can write

$$(x^{-r})_n * (x^s)_n = (t^{-r}, \phi_s(t)).$$

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We must now distinguish between odd and even r. First of all we have

$$(x^{-2r-1})_n*(x^s)_n = (t^{-2r-1}, \phi_s(t))$$
$$= \int_0^\infty t^{-2r-1}\left\{\phi_s(t)-\phi_s(-t)-2\sum_{i=1}^{r}\frac{t^{2i-1}}{(2i-1)!}\phi_s^{(2i-1)}(0)\right\}dt$$
$$= \int_0^n t^{-2r-1}\left\{\phi_s(t)-\phi_s(-t)-2\sum_{i=1}^{r}\frac{t^{2i-1}}{(2i-1)!}\phi_s^{(2i-1)}(0)\right\}dt,$$

since $\phi_s(t)=0$ for $|t|\geqslant n$.

Now for arbitrary fixed $c>0$ choose N such that $4c<N$. Then if

$$0\leqslant t\leqslant n/4, \quad |x|\leqslant c, \quad N\leqslant n \tag{2}$$

it follows that

$$\left|\frac{x\pm t}{n}\right|\leqslant 1/2.$$

From Taylor's Theorem we have

$$\phi_s(t)-\phi_s(-t)-2\sum_{i=1}^{r}\frac{t^{2i-1}}{(2i-1)!}\phi_s^{(2i-1)}(0)=\frac{2\xi t^{2r+1}}{(2r)!}\phi_s^{(2r+1)}(t_0),$$

where $0\leqslant\xi\leqslant 1$ and $-\xi t\leqslant t_0\leqslant \xi t$. But since

$$\phi_s(t)=\tau(t/n)(x-t)^s\tau\left(\frac{x-t}{n}\right)$$

and

$$\tau(t/n)=1 \quad \text{for} \quad |t|\leqslant n/2,$$

it follows that if t, x and n are subject to the above inequalities (2), then

$$\phi_s^{(2r+1)}(t_0)=0.$$

Thus

$$(x^{-2r-1})_n*(x^s)_n=\int_{n/4}^{n} t^{-2r-1}\left\{\phi_s(t)-\phi_s(-t)-2\sum_{i=1}^{r}\frac{t^{2i-1}}{(2i-1)!}\phi_s^{(2i-1)}(0)\right\}dt$$
$$=n^{-2r}\int_{\frac{1}{4}}^{1} u^{-2r-1}\left\{\phi_s(nu)-\phi_s(-nu)-2\sum_{i=1}^{r}\frac{(nu)^{2i-1}}{(2i-1)!}\phi_s^{(2i-1)}(0)\right\}du,$$

where $t=nu$. Obviously

$$\lim_{n\to\infty} n^{-2r}\int_{\frac{1}{4}}^{1} u^{-2r-1}\sum_{i=1}^{r}\frac{(nu)^{2i-1}}{(2i-1)!}\phi_s^{(2i-1)}(0)du=0.$$

Since

$$\phi_s(nu)-\phi_s(-nu)=\tau(u)(x-nu)^s\tau\left(\frac{x}{n}-u\right)-\tau(u)(x+nu)^s\tau\left(\frac{x}{n}+u\right)$$

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it is obvious that, if $s<2r$,

$$\lim_{n\to\infty} n^{-2r}\int_{\frac{1}{2}}^{1} u^{-2r-1}\{\phi_s(nu)-\phi_s(-nu)\}du=0$$

and if $s=2r$

$$\lim_{n\to\infty} n^{-2r}\int_{\frac{1}{2}}^{1} u^{-2r-1}\{\phi_{2r}(nu)-\phi_{2r}(-nu)\}du$$

$$=\lim_{n\to\infty}\int_{\frac{1}{2}}^{1} u^{-1}\tau(u)\left\{\tau\left(\frac{x}{n}-u\right)-\tau\left(\frac{x}{n}+u\right)\right\}du$$

$$=0$$

since

$$\lim_{n\to\infty}\tau\left(\frac{x}{n}-u\right)=\lim_{n\to\infty}\tau\left(\frac{x}{n}+u\right)=\tau(u).$$

We have thus proved that

$$\lim_{n\to\infty}(x^{-2r-1})_n*(x^s)_n=0$$

for each x, the convergence obviously being uniform on every finite interval.

Thus for arbitrary fine function ϕ we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\phi(x)(x^{-2r-1})_n*(x^s)_n dx=0$$

since a fine function vanishes identically outside a finite interval. This completes the proof of equation (1) for odd τ.

Next we have

$$\begin{aligned}(x^{-2r})_n*(x^s)_n&=(t^{-2r},\phi_s(t))\\&=\int_0^{\infty} t^{-2r}\left\{\phi_s(t)+\phi_s(-t)-2\sum_{i=0}^{r-1}\frac{t^{2i}}{(2i)!}\phi_s^{(2i)}(0)\right\}dt\\&=\int_0^{n} t^{-2r}\left\{\phi_s(t)+\phi_s(-t)-2\sum_{i=0}^{r-1}\frac{t^{2i}}{(2i)!}\phi_s^{(2i)}(0)\right\}dt.\end{aligned}$$

From Taylor's Theorem we have

$$\phi_s(t)+\phi_s(-t)-2\sum_{i=0}^{r-1}\frac{t^{2i}}{(2i)!}\phi_s^{(2i)}(0)=\frac{2\xi t^{2r}}{(2r-1)!}\phi_s^{(2r)}(t_0),$$

where $0\leqslant\xi\leqslant 1$ and $-\xi t\leqslant t_0\leqslant \xi t$. It follows that if t, x and n again satisfy inequalities (2), then

$$\phi_s^{(2r)}(t_0)=0.$$

Thus

$$(x^{-2r})_n * (x^s)_n = \int_{n/4}^{n} t^{-2r}\left\{\phi_s(t)+\phi_s(-t)-2\sum_{i=0}^{r-1}\frac{t^{2i}}{(2i)!}\,\phi_s^{(2i)}(0)\right\}dt$$

and it follows, as in the previous case, that if $s \leqslant 2r-1$

$$\lim_{n\to\infty}(x^{-2r})_n * (x^s)_n = 0$$

for each x, the convergence being uniform on every finite interval.

Again, for arbitrary fine function ϕ, we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\phi(x)(x^{-2r})_n * (x^s)_n dx = 0,$$

which completes the proof of equation (1).

REFERENCES

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