Glasgow Mathematical Journal (2024), 1–5
doi:10.1017/S0017089524000247

Glasgow Mathematical Journal

RESEARCH ARTICLE

Classification of multiplication modules over multiplication rings with finitely many minimal primes

Volodymyr Bavula

School of Mathematics and Statistics, University of Sheffield, Sheffield, S3 7RH, UK
Email: v.bavula@sheffield.ac.uk

Received: 13 April 2024; **Revised:** 14 July 2024; **Accepted:** 18 July 2024

Keywords: multiplication module; multiplication ring; Dedekind domain; Artinian local principal ideal ring

2020 Mathematics Subject Classification: *Primary* - 13C05, 13E05, 13F05, 13F10

Abstract

A classification of multiplication modules over multiplication rings with finitely many minimal primes is obtained. A characterization of multiplication rings with finitely many minimal primes is given via faithful, Noetherian, distributive modules. It is proven that for a multiplication ring with finitely many minimal primes every faithful, Noetherian, distributive module is a faithful multiplication module, and vice versa.

1. Introduction

In this paper, all rings are commutative with 1 and all modules are unital. A ring R is called a *multiplication ring* if I and J are ideals of R such that $J \subseteq I$ then $J = I'I$ for some ideal I' of R. An R-module M is called a *multiplication module* if each submodule of M is equal to IM for some ideal I of the ring R. The concept of multiplication ring was introduced by Krull in [5]. In [6], Mott proved that a multiplication ring has finitely many minimal prime ideals iff it is a Noetherian ring.

The next theorem is a description of multiplication rings with finitely many minimal primes.

Theorem 1.1. *([1, Theorem 1.1]) Let R be a ring with finitely many minimal prime ideals. Then the ring R is a multiplication ring iff $R \cong \prod_{i=1}^{n} D_i$ is a finite direct product of rings where D_i is either a Dedekind domain or an Artinian, local principal ideal ring.*

Classification of multiplication modules over multiplication rings with finitely many minimal primes. Using Theorem 1.1, a criterion for a direct sum of modules to be a multiplication module (Theorem 2.1) and some other results, a classification of multiplication modules over a multiplication ring with finitely many minimal primes is given, Theorem 1.2.

Theorem 1.2. *Let R be a multiplication ring with finitely many minimal primes, that is $R \cong \prod_{i=1}^{n} D_i$ is a finite direct product of rings where D_i is either a Dedekind domain or an Artinian, local principal ideal ring and $1 = e_1 + \cdots + e_n$ be the corresponding sum of orthogonal idempotents of the ring R. Let M be an R-modules and $M = \oplus_{i=1}^{n} M_i$ where $M_i := e_iM$. Then the R-module M is a multiplication R-module iff each D_i-module M_i is either isomorphic to D_i or to D_i/I_i where I_i is a nonzero ideal of D_i or to a nonzero ideal of the ring D_i in case when the ring D_i is a Dedekind domain.*

Classification of faithful multiplication modules over a multiplication ring with finitely many minimal primes.

Theorem 1.3. *Let R be a multiplication ring with finitely many minimal primes. We keep the notation of Theorem 1.2 ($R \cong \prod_{i=1}^{n} D_i$). Then an R-module $M = \oplus_{i=1}^{n} M_i$ (where $M_i = e_i M$) is a faithful multiplication R-module iff for each $i = 1, \ldots, n$, either ${}_R M_i \simeq D_i$ or ${}_R M_i \simeq I_i$ where I_i is a nonzero ideal of the ring D_i in case when D_i is a Dedekind domain.*

Proof. *The theorem follows at once from Theorem 1.2.* □

Characterization of multiplication rings with finitely many minimal primes via faithful, Noetherian, distributive modules. Let R be a ring and M be an R-module. A submodule N of M is called a *distributive submodule* if one of the following equivalent conditions holds: For any submodules M_1 and M_2 of M,

$$(M_1 + M_2) \cap N = M_1 \cap N + M_2 \cap N,$$
$$M_1 \cap M_2 + N = (M_1 + N) \cap (M_2 + N).$$

The R-module M is called a *distributive module* if all submodules of M are distributive submodules.

Theorem 1.4. *A commutative ring R is a multiplication ring with finitely many minimal primes iff there is a faithful, Noetherian, distributive R-module.*

Classification of faithful, Noetherian, distributive modules over a multiplication ring with finitely many minimal primes.

Theorem 1.5. *Let R be a multiplication ring with finitely many minimal primes. Then every faithful, Noetherian, distributive R-module is a faithful multiplication R-module, and vice versa.*

2. Proofs

In this section, we prove the results from the Introduction.

Definition 2.1. *We say that the **intersection condition** holds for a direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ of nonzero R-modules M_λ if for all submodules N of M, $N = \bigoplus_{\lambda\in\Lambda} (N \bigcap M_\lambda)$.*

Definition 2.2. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with $\mathrm{card}(\Lambda) \geqslant 2$, $\mathfrak{a}_\lambda = \mathrm{ann}_R(M_\lambda)$ and $\mathfrak{a}'_\lambda = \cap_{\mu\neq\lambda} \mathfrak{a}_\mu$. We say that the **orthogonality condition** holds for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ if $\mathfrak{a}'_\lambda M_\mu = \delta_{\lambda\mu} M_\mu$ for all $\lambda, \mu \in \Lambda$. Clearly, $\mathfrak{a}'_\lambda \neq 0$ for all $\lambda \in \Lambda$ (since all $M_\lambda \neq 0$). In particular, $\mathfrak{a}_\lambda \neq 0$ for all $\lambda \in \Lambda$.*

Definition 2.3. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with $\mathrm{card}(\Lambda) \geq 2$. We say that the **strong orthogonality condition** holds for M if for each set of R-modules $\{N_\lambda\}_{\lambda\in\Lambda}$ such that $N_\lambda \subseteq M_\lambda$, there is a set of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of R such that $I_\lambda M_\mu = \delta_{\lambda\mu} N_\lambda$ for all $\lambda, \mu \in \Lambda$ where $\delta_{\lambda\mu}$ is the Kronecker delta. The set of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ is called an **orthogonalizer** of $\{N_\lambda\}_{\lambda\in\Lambda}$.*

Theorem 2.1 is one of the criteria for a direct sum of modules to be a multiplication module that are obtained in [1]. It is given via the intersection and strong orthogonality conditions.

https://doi.org/10.1017/S0017089524000247 Published online by Cambridge University Press

Theorem 2.4. *([2]) Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with* $\mathrm{card}(\Lambda) \geq 2$. *Then M is a multiplication module iff the intersection and strong orthogonality conditions hold for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$.*

An R-module is called a *cyclic* if it is 1-generated. For an R-module M, let $\mathrm{Cyc}_R(M)$ be the set of its *cyclic* submodules. For an R-module M, we denote by $\mathrm{ann}_R(M)$ its annihilator. An R-module M is called *faithful* if $\mathrm{ann}_R(M) = 0$. For a submodule N of M, the set $[N:M] := \mathrm{ann}_R(M/N) = \{r \in R \mid rM \subseteq N\}$ is an ideal of the ring R that contains the *annihilator* $\mathrm{ann}_R(M) = [0:M]$ of the module M. The set $\theta(M) := \sum_{C\in\mathrm{Cyc}_R(M)} [C:M]$ is an ideal of R. Clearly, $\mathrm{ann}_R(M) \subseteq \theta(M)$. If M is an ideal of R then $M \subseteq \theta(M)$.

Proof of Theorem 1.2. ($\Leftarrow$) *All the D_i-modules M_i of the theorem are multiplication D_i-modules. Hence, the direct sum $\oplus_{i=1}^n M_i$ is a multiplication module over the direct product rings $R = \prod_{i=1}^n D_i$.*

($\Rightarrow$) *Suppose that the R-module $M = \oplus_{i=1}^n M_i$ is a multiplication R-module where $M_i = e_iM$ for $i = 1, \dots, n$. We have the following claims.*

(i) The D_i-module M_i is a multiplication D_i-module: The statement is obvious since $R = \prod_{i=1}^n D_i$.

(ii) The D_i-module M_i is a finitely generated D_i-module: Since M_i is a multiplication D_i-module,

$$M_i = \sum_{C\in\mathrm{Cyc}_{D_i}(M_i)} C = \sum_{C\in\mathrm{Cyc}_{D_i}(M_i)} [C:M_i]M_i = \Big(\sum_{C\in\mathrm{Cyc}_{D_i}(M_i)} [C:M_i]\Big)M_i = \theta(M_i)M_i.$$

The ideal $\theta(M_i) = \sum_{C\in\mathrm{Cyc}_{D_i}(M_i)} [C:M_i]$ of the Noetherian ring D_i is a finitely generated D_i-module, that is, $\theta(M_i) = \sum_{i=1}^{n_i} D_i\theta_i$ for some elements $\theta_i \in \theta(M_i)$. Then

$$M_i = \theta(M_i)M_i = \sum_{i=1}^{n_i} D_i\theta_iM_i \subseteq \sum_{i=1}^{n_i} C_i \subseteq M_i,$$

and so the D_i-module $M_i = \sum_{i=1}^{n_i} C_i$ is finitely generated.

(iii) Suppose that the ring D_i is a Dedekind domain. Then the D_i-module M_i is isomorphic either to D_i or to D_i/I_i or to J_i where I_i and J_i are ideals of the ring D_i: It is well-known that a nonzero finitely generated module $\mathcal{M}$ over a Dedekind domain D is a direct sum $\mathcal{M} = \mathcal{F} \oplus \mathcal{T}$ of a torsion-free D-module $\mathcal{F}$ and a torsion D-module $\mathcal{T}$; $\mathcal{F} = I \oplus D^m$ for some ideal I of D and $m \geq 0$; and $\mathcal{T} = \oplus_{i=1}^{t_i} D/\mathfrak{p}_i^{m_i}$ where $\mathfrak{p}_i$ are maximal ideals of the ring D and $m_i \in \mathbb{N}$. Suppose that the D-module $\mathcal{M}$ is a multiplication D-module. By Theorem 2.1, the direct sum of D-modules

$$\mathcal{M} = I \oplus D^m \oplus \bigoplus_{i=1}^{t_i} D/\mathfrak{p}_i^{m_i}$$

must satisfy the strong orthogonality conditions. Hence, either $\mathcal{M} = I$ of $\mathcal{M} = D$ or $\mathcal{M} = \oplus_{i=1}^{t_i} D/\mathfrak{p}_i^{m_i}$ where $\mathfrak{p}_1, \dots, \mathfrak{p}_{t_i}$ are distinct maximal ideals of the ring D, and so $\mathcal{M} = \oplus_{i=1}^{t_i} D/\mathfrak{p}_i^{m_i} \simeq D/\prod_{i=1}^{t_i} \mathfrak{p}_i^{m_i}$.

(iv) Suppose that D_i is an Artinian, local, principal ideal ring. Then the D_i-module M_i is isomorphic either to D_i or to D_i/I_i where I_i is a nonzero ideal of D_i: Let $D = D_i$ and $\mathfrak{m}$ be the maximal ideal of the local ring D_i and $\mathfrak{m}^\nu \neq 0$ and $\mathfrak{m}^{\nu+1} = 0$ for some natural number ν. Then

$$\{D, \mathfrak{m}, \mathfrak{m}^2, \dots, \mathfrak{m}^\nu, \mathfrak{m}^{\nu+1} = 0\}$$

is the set of all the ideals of the ring D. The D-module M_i is a nonzero finitely generated multiplication D-module. Hence, $\{M_i, \mathfrak{m}M_i, \mathfrak{m}^2M_i, \dots, \mathfrak{m}^\mu M_i, \mathfrak{m}^{\mu+1}M_i = 0\}$ is the set of all D-submodules of M_i for some natural number μ such that $\mu \leq \nu$. In particular, the D-module M_i is a uniserial D-module since

$$M_i \supset \mathfrak{m}M_i \supset \mathfrak{m}^2M_i \supset \cdots \supset \mathfrak{m}^\mu M_i \supset \mathfrak{m}^{\mu+1}M_i = 0.$$

Since the D-module M_i is a uniserial, we have that

$$\dim_{k_\mathfrak{m}} (M_i/\mathfrak{m}M_i) = 1$$

where $k_\mathfrak{m} := D/\mathfrak{m}$, and so $M_i = Dm_i + \mathfrak{m}M_i$ for some element $m_i \in M_i \backslash \mathfrak{m}M_i$. By the Nakayama Lemma, $M_i = Dm_i$, and the statement (iv) follows. □

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Corollary 2.5. *Let R be an Artinian multiplication ring. Then every multiplication R-module is an epimorphic image of the R-module R.*

Proof. *The corollary follows at once from Theorem 1.2.* □

Corollary 2.6. *Let R be a multiplication ring with finitely many minimal primes and M be a multiplication R-module. Then*

1. *The endomorphism ring* $\mathrm{End}_R(M)$ *is also a multiplication ring.*
2. $\mathrm{End}_R(M) \simeq R/\mathrm{ann}_R(M)$.
3. *The* $\mathrm{End}_R(M)$*-module M is a faithful multiplication* $\mathrm{End}_R(M)$*-module.*

Proof. *The corollary follows at once from Theorem 1.2.* □

In the proof of Theorem 1.4, we will use the following results.

Theorem 2.7. *Let R be a commutative ring.*

1. *([3, Corollary, p. 177]) Let M be a Noetherian distributive R-module. Then every submodule of M which is locally nonzero at every maximal ideal of R, is of the form IM where I is a unique product of maximal ideals of R.*
2. *([3, Lemma 2.(ii)]) A finitely generated R-module M is a multiplication module iff the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is a multiplication module for all prime/maximal ideals $\mathfrak{p}$ of R.*
3. *([4, Theorem 1.3.(ii)]) (Cancellation Law) If M is a finitely generated, faithful multiplication R-module then for any two ideals A and B of R, $AM \subseteq BM$ iff $A \subseteq B$.*

Proof of Theorem 1.4. *($\Rightarrow$) By Theorem 1.2, the R-module R is a faithful, Noetherian, distributive R-module.*

($\Leftarrow$) Let M be faithful, Noetherian, distributive R-module.

(i) The ring R is a Noetherian ring: The R-module M is Noetherian, hence finitely generated, $M = \sum_{i=1}^n Rm_i$ for some elements $m_1, \dots, m_n \in M$. The R-module M is a faithful module. Hence, the map $R \to \oplus_{i=1}^n Rm_i$, $r \mapsto (rm_1, \dots, rm_n)$ is an R-monomorphism. The direct sum is a Noetherian R-module (as a finite direct sum of Noetherian modules), and the statement (i) follows.

(ii) The ring R has only finitely many minimal primes: The statement (ii) follows from the statement (i).

(iii) For all maximal ideals $\mathfrak{m}$ of the ring R, the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is faithful, Noetherian and distributive: The R-module M is finitely generated. Hence, $\mathrm{ann}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \mathrm{ann}_R(M)_{\mathfrak{m}} = 0$ since $\mathrm{ann}_R(M) = 0$. Clearly, the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is Noetherian and distributive (since the R-module M is so and localizations respect finite intersections).

(iv) The $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is a multiplication $R_{\mathfrak{m}}$-module:

The statement (iv) follows from the statement (iii) and Theorem 2.7.(1).

(v) The R-module M is a multiplication module: The R-module M is finitely generated. By the statement (iv) and Theorem 2.7.(2), the R-module M is a multiplication R-module.

Let $(\mathcal{I}(R), \subseteq)$ be the lattice of ideals of the ring R and $(\mathrm{Sub}_R(M), \subseteq)$ be the lattice of R-submodules of the R-module M.

(vi) The map $\mathcal{I}(R) \to \mathrm{Sub}_R(M)$, $I \mapsto IM$ is an isomorphism of latices: The R-module M is a finitely generated, faithful multiplication module (the statement (v)), and the statement (vi) follows from Theorem 2.7.(3).

(vii) The ring R is a multiplication ring: The statement (vii) follows from the statements (v) and (vi).

Now, the theorem follows from the statements (ii) and (vii). □

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Proof of Theorem 1.5. ($\Rightarrow$) *See the statement (vi) in the proof of Theorem 1.4.*
($\Leftarrow$) *This implication follows at once from Theorem 1.3.* □

Conflicts of interest. No potential conflict of interest was reported by the author.

Data availability statement. Data sharing not applicable—no new data generated.

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