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HOMOLOGY OF ABELIAN COVERINGS OF LINKS AND SPATIAL GRAPHS

MAKOTO SAKUMA

ABSTRACT. We give (1) a formula of the first Betti numbers of abelian coverings of links in terms of the Alexander ideals, (2) certain estimates of the orders of the torsion parts of their first homology groups in terms of the Alexander polynomials, and (3) a structure theorem of the first homology groups of $\mathbb{Z}_2^d$-coverings of spatial graphs. As an application, we generalize a result of E. Hironaka on polynomial periodicity of the first Betti numbers in certain towers of abelian coverings of complex surfaces.

Introduction. Classical results of Goeritz [Ge] and Fox [F] show that the rank and the order of the first homology group of a cyclic covering of the 3-sphere S^3 branched over a knot can be expressed in terms of its Alexander invariants. (Closely related results had been obtained by Zariski [Z], where he had studied the first Betti numbers of cyclic branched coverings of the complex projective plane.) Since then, the homology groups of abelian coverings have been studied extensively. (See [GS, Gr, GL, He, Hl1, 2, HK, MM, P, Ri, Sk1, 2, SS, Su1, 2, VW, W]; for related studies from the view point of complex surfaces, see [Hr1, L1, 2, Sr] and references therein.) In particular, in the crucial article [MM], Mayberry and Murasugi obtained a formula which expresses the order of the first homology group of an arbitrary abelian covering of S^3 branched over a link L in terms of the Alexander polynomials of the sublinks of L. They also gave a formula on the orders of the torsion parts of the first homology groups of unbranched coverings under certain conditions. However we don't have a general formula for the first Betti numbers of abelian coverings except for the formula for "strictly cyclic" coverings of links (see [HK]) and the formula for those of unbranched coverings corresponding to "product representations" (see [L2, Hr1, Sr]). Further, the formulae in [MM] are proved only for links in S^3, and give no information on the torsion parts of the homology groups in case they have nontrivial ranks.

The purpose of this paper is to consider these remaining problems. In fact, we give a precise formula of the first Betti numbers of (branched or unbranched) abelian coverings of links in homology 3-spheres (Theorem 1.1), and certain estimates of the orders of the torsion parts of their first homology groups (Theorem 8.1). As an application of the proof of Theorem 1.1, we give a refinement of a certain weak version of Torres' second condition (Proposition 6.3). Our formula of the Betti numbers are actually valid under more general situations (Theorem 7.3); and as an application, we give a generalization of the result of Hironaka [Hr1, Theorem 1.7] on "polynomial periodicity" of the first

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Betti numbers in certain towers of abelian coverings of complex surfaces (Theorem 7.5). After having completed this work, I was informed that she had also obtained a similar generalization of her result [Hr2].

Our basic idea is to decompose the homology group of an abelian covering into the direct sum of the "pure parts" of the homology groups of smaller cyclic coverings. Though this idea is very natural from the view point of the elementary representation theory of finite groups, I came to the idea through the article of Nakao [Nk1], where he proved the following theorem by using Reidemeister-Schreier method:

THEOREM [NK1]. *Let θ be a θ-curve (i.e., a graph with two vertices and three edges, where each edge joins the two vertices) embedded in S^3, and let K_i $(i = 1,2,3)$ be its constituent knots. Then*

$$H_1\big(M_{2\oplus 2}(\theta);\mathbb{Z}\big) \cong \bigoplus_{i=1}^{3} H_1\big(M_2(K_i);\mathbb{Z}\big),$$

where $M_{2\oplus 2}(\theta)$ [resp. $M_2(K_i)$] is the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ [resp. $\mathbb{Z}_2$] covering of S^3 branched over θ [resp. K_i].

In [Nk2], he generalized this result to the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ covering of S^3 branched over a complete graph with 4 vertices. In this paper, we also give a generalization of these results to $\mathbb{Z}_2^d$ coverings of homology 3-spheres branched over graphs (Theorems 14.1 and 14.2).

This paper consists of three chapters. Sections 1, 2, and 3, respectively, are focused on the first Betti numbers of abelian coverings of links in homology 3-spheres, the torsion parts of their homology groups, and the structure of the first homology groups of $\mathbb{Z}_2^d$-coverings of spatial graphs in homology 3-spheres. In Section 1, homology groups are considered to be with coefficients $\mathbb{C}$, the complex number field, unless otherwise stated; in Sections 2 and 3, homology groups are considered to be with coefficients $\mathbb{Z}$, the ring of the integers, unless otherwise stated.

1. **Betti numbers of abelian coverings of links.**

1. Statement of results. Let $L = K_1 \cup K_2 \cup \cdots \cup K_\mu$ be an oriented link in a homology 3-sphere M and let $E(L) = M - \operatorname{int} N(L)$ be its exterior, where $N(L)$ denotes a regular neighbourhood of L in M. Put $G = H_1\big(E(L);\mathbb{Z}\big)$, and let t_i be the element of G represented by a meridian of K_i for each i $(1 \leq i \leq \mu)$. We recall the definition of the elementary ideals of L. Let $\tilde{E}(L)$ be the universal abelian covering of $E(L)$, and let $\tilde{*}$ be the inverse image in $\tilde{E}(L)$ of a base point $* \in E(L)$. Then $H_1\big(\tilde{E}(L),\tilde{*};\mathbb{Z}\big)$ has a structure of a module over $\mathbb{Z}[G]$, the integral group ring of G. This module is called the *Alexander module* of L, and the *d-th elementary ideal* $\mathfrak{E}_d(L)$ of L is defined to be the d-th elementary ideal, or the d-th determinantal ideal (*cf.* [Bo, p. 101]), of the Alexander module of L. To be more precise, consider a finite presentation of the Alexander module

$$\mathbb{Z}[G]^p \xrightarrow{Q} \mathbb{Z}[G]^q \longrightarrow H_1\big(\tilde{E}(L),\tilde{*};\mathbb{Z}\big) \longrightarrow 0.$$

Then $\mathfrak{E}_d(L)$ is the ideal in $\mathbb{Z}[G]$ generated by the $(q-d)\times(q-d)$ sub-determinants of the matrix Q. If $d<0$ [resp. $d\geq q$], $\mathfrak{E}_d(L)$ is defined to be 0 [resp. $\mathbb{Z}[G]$]. The elementary ideals of a link can be calculated from the link group. In fact, if $\langle x_1,\ldots,x_q \mid r_1,\ldots,r_p\rangle$ is a presentation of the link group $\pi_1\big(E(L)\big)$, then the matrix $(\frac{\partial r_i}{\partial x_j})^{\gamma}_{1\leq i\leq p,1\leq j\leq q}$ is a presentation matrix of the $\mathbb{Z}[G]$-module $H_1\big(\tilde{E}(L),\tilde{*};\mathbb{Z}\big)$. Here $\frac{\partial}{\partial x_j}$ is Fox's free derivation, and γ is (the ring homomorphism between group rings induced by) the Hurewicz homomorphism $\pi_1\big(E(L)\big)\longrightarrow G$ (*cf.* [BZ, Chapter 9]). For a group homomorphism φ from $H_1\big(E(L);\mathbb{Z}\big)$ to S^1, the multiplicative group of complex numbers of modulus 1, we define null $(L;\varphi)$, the φ-*nullity* of L, by

$$\operatorname{null}(L;\varphi)=\max\big\{d \mid \varphi\big(\mathfrak{E}_d(L)\big)=0\big\}.$$

Here the symbol φ in the parenthesis denotes the ring homomorphism $\mathbb{Z}[G]\longrightarrow\mathbb{C}$ induced by φ. It should be noted that (1) null $(L;\varphi)$ is defined even if $L=\emptyset$, and (2) rank $Q^{\varphi}=q-\operatorname{null}(L;\varphi)-1$ for any presentation matrix Q with size (p,q) of the Alexander module of L.

Let A be a finite abelian group, and let $\pi\colon G=H_1\big(E(L);\mathbb{Z}\big)\longrightarrow A$ be an epimorphism. Then $E_\pi(L)$ and $M_\pi(L)$ are, respectively, the unbranched covering of $E(L)$ and the branched covering of M branched over L associated with π. Let $\mathcal{Z}_A$ denote the set of the irreducible representations of A over $\mathbb{C}$, *i.e.*, $\mathcal{Z}_A$ is the set of the group homomorphisms $\zeta\colon A\longrightarrow S^1$ $(\subset\mathbb{C})$. $\mathcal{Z}_A^*$ denotes the set of the nontrivial irreducible representations of A over $\mathbb{C}$, *i.e.*, $\mathcal{Z}_A^*=\mathcal{Z}_A-\{1\}$. For an element $\zeta\in\mathcal{Z}_A$, let $L_{\zeta\pi}$ be the sublink of L consisting of those components K_i such that $\zeta\pi(t_i)\neq 1$. Note that

$$\begin{aligned} H_1\big(E(L_{\zeta\pi});\mathbb{Z}\big) &\cong H_1\big(E(L);\mathbb{Z}\big)/\langle t_i \mid K_i\not\subset L_{\zeta\pi}\rangle\\ &\cong H_1\big(E(L);\mathbb{Z}\big)/\langle t_i \mid \zeta\pi(t_i)=1\rangle, \end{aligned}$$

and therefore, $\zeta\pi$ induces a homomorphism $H_1\big(E(L_{\zeta\pi});\mathbb{Z}\big)\longrightarrow S^1$. We denote this homomorphism by the same symbol. In this chapter, we prove the following theorem:

THEOREM 1.1. *The first Betti numbers of $E_\pi(L)$ and $M_\pi(L)$ are determined as follows:*

(1) $\beta_1\big(E_\pi(L)\big)=1+\sum_{\zeta\in\mathcal{Z}_A}\operatorname{null}(L;\zeta\pi)=\mu+\sum_{\zeta\in\mathcal{Z}_A^*}\operatorname{null}(L;\zeta\pi)$.

(2) $\beta_1\big(M_\pi(L)\big)=1+\sum_{\zeta\in\mathcal{Z}_A}\operatorname{null}(L_{\zeta\pi};\zeta\pi)=\sum_{\zeta\in\mathcal{Z}_A^*}\operatorname{null}(L_{\zeta\pi};\zeta\pi)$.

2. *Outline of the proof of Theorem 1.1.* First, we quickly review some elementary facts in the representation theory of finite groups (*cf.* [NT]). For each $\zeta\in\mathcal{Z}_A$, put

$$e_\zeta=\frac{1}{|A|}\sum_{a\in A}\overline{\zeta(a)}a,$$

where $|A|$ denotes the order of A (*cf.* [NT, p. 197]). Then e_ζ is an element of the group ring $\mathbb{C}[A]$ of A over $\mathbb{C}$, and the following holds:

LEMMA 2.1. *(1)* $ae_\zeta = \zeta(a)e_\zeta$ *for any* $a \in A$.

(2) $e_\zeta e_{\zeta'} = \begin{cases} e_\zeta & \text{if } \zeta = \zeta', \\ 0 & \text{if } \zeta \neq \zeta'. \end{cases}$

(3) Let B *be a subgroup of* A. *Then*

$$\sum_{\eta \in \mathcal{Z}_A(B)} e_\eta = \frac{1}{|B|} \operatorname{Tr} B,$$

where $\operatorname{Tr} B = \sum_{b \in B} b$ *and* $\mathcal{Z}_A(B) = \{\eta \in \mathcal{Z}_A \mid \operatorname{Ker}(\eta) \supset B\}$. *In particular,*

$$\sum_{\zeta \in \mathcal{Z}_A} e_\zeta = 1.$$

For $\zeta \in \mathcal{Z}_A$, let (ζ) be the irreducible $\mathbb{C}[A]$-module determined by ζ, *i.e.*, the 1-dimensional complex vector space $\mathbb{C}$ where the action of A is defined by $az = \zeta(a)z$ ($a \in A, z \in \mathbb{C}$). Let H be a finitely generated $\mathbb{C}[A]$-module. For each $\zeta \in \mathcal{Z}_A$, put $[H]_\zeta = e_\zeta H$. We call it the *ζ-component* of H. Then by Lemma 2.1, we have the following (*cf.* [NT, pp. 16-17]):

LEMMA 2.2. *(1)* $[H]_\zeta$ *is a* $\mathbb{C}[A]$*-submodule of* H*, which is isomorphic to the direct sum* $(\zeta)^p$ *for some non-negative integer* p.

(2) $H \cong \bigoplus_{\zeta \in \mathcal{Z}_A} [H]_\zeta$.

Schur's lemma asserts the following (*cf.* [NT, p. 23]):

LEMMA 2.3. *Let* ζ *and* ζ' *be mutually different elements of* $\mathcal{Z}_A$. *Then the* 0*-map is the only* $\mathbb{C}[A]$*-homomorphism from* $(\zeta)^p$ *to* $(\zeta')^q$ *for any positive integers* p *and* q.

The proof of Theorem 1.1 is divided into the following steps. Throughout this chapter, homology groups are considered to be with coefficients $\mathbb{C}$ unless otherwise stated.

STEP 0. $H_1\big(E_\pi(L)\big)$ and $H_1\big(M_\pi(L)\big)$ are finitely generated $\mathbb{C}[A]$-modules, and hence, by Lemma 2.2,

$$H_1\big(E_\pi(L)\big) \cong \bigoplus_{\zeta \in \mathcal{Z}_A} \big[H_1\big(E_\pi(L)\big)\big]_\zeta,$$
$$H_1\big(M_\pi(L)\big) \cong \bigoplus_{\zeta \in \mathcal{Z}_A} \big[H_1\big(M_\pi(L)\big)\big]_\zeta.$$

STEP 1. For each $\zeta \in \mathcal{Z}_A$, the ζ-components of the first homology groups of the abelian coverings are isomorphic to those of certain cyclic coverings, *i.e.*,

$$\big[H_1\big(E_\pi(L)\big)\big]_\zeta \cong \big[H_1\big(E_{\zeta\pi}(L)\big)\big]_\zeta,$$
$$\big[H_1\big(M_\pi(L)\big)\big]_\zeta \cong \big[H_1\big(M_{\zeta\pi}(L)\big)\big]_\zeta.$$

Here $E_{\zeta\pi}$ and $M_{\zeta\pi}$, respectively, denote the unbranched cyclic covering and the branched cyclic covering corresponding to the homomorphism $\zeta\pi$ with the covering transformation group $\operatorname{Im}(\zeta) = \operatorname{Im}(\zeta\pi)$.

STEP 2. $\left[H_1\big(M_{\zeta\pi}(L)\big)\right]_\zeta \cong \left[H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)\right]_\zeta$.

STEP 3. The dimensions of the above modules over $\mathbb{C}$ are determined by the nullities of L as follows:

$$\dim\left[H_1\big(E_{\zeta\pi}(L)\big)\right]_\zeta = \begin{cases} \operatorname{null}(L;\zeta\pi) & \text{if } \zeta \neq 1, \\ \operatorname{null}(L;\zeta\pi)+1 & \text{if } \zeta = 1, \end{cases}$$

$$\dim\left[H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)\right]_\zeta = \begin{cases} \operatorname{null}(L_{\zeta\pi};\zeta\pi) & \text{if } \zeta \neq 1, \\ \operatorname{null}(L_{\zeta\pi};\zeta\pi)+1 & \text{if } \zeta = 1. \end{cases}$$

We can now obtain Theorem 1.1. [Note that if $\zeta = 1$, then $\left[H_1\big(E_{\zeta\pi}(L)\big)\right]_{\zeta\pi} \cong H_1\big(E(L)\big) \cong \mathbb{C}^\mu$ and $\left[H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)\right]_{\zeta\pi} \cong H_1(M) \cong 0$.]

3. *Proof of Step 1.* Put $B = \operatorname{Ker}(\zeta) \subset A$. Then $M_{\zeta\pi}(L) \cong M_\pi(L)/B$. Hence, by an argument using transfer (see [Br, pp. 118–120]), we obtain

$$H_1\big(M_{\zeta\pi}(L)\big) \cong H_1\big(M_\pi(L)/B\big) \cong (\operatorname{Tr} B)H_1\big(M_\pi(L)\big).$$

Further,

$$\begin{aligned}(\operatorname{Tr} B)H_1\big(M_\pi(L)\big) &\cong \Big(\sum_{\eta\in \mathcal{Z}_A(B)} e_\eta\Big)H_1\big(M_\pi(L)\big) && \text{by Lemma 2.1(3)} \\ &\cong \bigoplus_{\eta\in \mathcal{Z}_A(B)} \left[H_1\big(M_\pi(L)\big)\right]_\eta && \text{by Lemmas 2.1(2) and 2.2(2).}\end{aligned}$$

Hence, by using Lemma 2.1(2) and the fact that $\zeta \in \mathcal{Z}_A(B)$, we see

$$\left[H_1\big(M_{\zeta\pi}(L)\big)\right]_\zeta \cong \Big[\bigoplus_{\eta\in \mathcal{Z}_A(B)} \left[H_1\big(M_\pi(L)\big)\right]_\eta\Big]_\zeta \cong \left[H_1(M_\pi(L))\right]_\zeta.$$

This proves the second part of Step 1. The first part of Step 1 is proved similarly.

4. *Proof of Step 2.* Put $n = |\operatorname{Im}(\zeta)|$ and identify $\operatorname{Im}(\zeta)$ with the abstract group $\langle t \mid t^n = 1\rangle$. Let $\bar{\zeta}$ be the homomorphism $A \longrightarrow \operatorname{Im}(\zeta) \cong \langle t \mid t^n = 1\rangle$ determined by ζ. Then any $\mathbb{C}\langle t \mid t^n = 1\rangle$-module is regarded as a $\mathbb{C}[A]$-module via $\bar{\zeta}$. Note that $M_{\zeta\pi}(L) \cong M_{\zeta\pi}(L_{\zeta\pi})$. If $\zeta = 1$, then $M_{\zeta\pi}(L_{\zeta\pi}) = M = E(L_{\zeta\pi})$, and the assertion holds trivially. Suppose $\zeta \neq 1$, *i.e.*, $n > 1$. For each componenet K_i of $L_{\zeta\pi}$, let b_i be the element of $H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)$ represented by a lift of a meridian of K_i in $E_{\zeta\pi}(L_{\zeta\pi})$. Then $H_1\big(M_{\zeta\pi}(L_{\zeta\pi})\big) \cong H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)/\mathcal{B}$, where $\mathcal{B}$ denotes the $\mathbb{C}\langle t \mid t^n = 1\rangle$-submodule generated by $\{b_i \mid K_i \subset L_{\zeta\pi}\}$. For each component K_i of $L_{\zeta\pi}$, there is an integer d_i such that $(t^{d_i}-1)b_i = 0$ and $0 < d_i < n$. [In fact, let n_i be the order of $\zeta\pi(t_i) \in \langle t \mid t^n = 1\rangle$; then n_i is a nontrivial divisor of n, and $d_i = n/n_i$ satisfies the condition.] Hence, $\mathcal{B}$ is contained in $\oplus_{|\operatorname{Im}(\eta)|<n}\left[H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)\right]_\eta$. So, $\mathcal{B} \cap \left[H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)\right]_\zeta = 0$, and we see

$$\begin{aligned}\left[H_1\big(M_{\zeta\pi}(L)\big)\right]_\zeta &\cong \left[H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)/\mathcal{B}\right]_\zeta \\ &\cong \left[H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)\right]_\zeta \Big/ \Big(\mathcal{B} \cap \left[H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)\right]_\zeta\Big) \\ &\cong \left[H_1\big(E_{\zeta\pi}(L_{\zeta\pi})\big)\right]_\zeta.\end{aligned}$$

5. *Proof of Step 3.* Let Q be a finite presentation matrix of the $\mathbb{C}[G]$-module $H_1\big(\tilde{E}(L),\tilde{*}\big)$, *i.e.*,

$$\mathbb{C}[G]^p \xrightarrow{Q} \mathbb{C}[G]^q \longrightarrow H_1\big(\tilde{E}(L),\tilde{*}\big) \longrightarrow 0.$$

Then we obtain the following exact sequence of $\mathbb{C}[A]$-modules (*cf.* [BZ, Chapter 9.C]);

$$\mathbb{C}\langle t \mid t^n = 1\rangle^p \xrightarrow{Q^{\zeta\pi}} \mathbb{C}\langle t \mid t^n = 1\rangle^q \longrightarrow H_1\big(E_{\zeta\pi}(L),\tilde{*}\big) \longrightarrow 0.$$

Here $\tilde{*}$ denotes the inverse image in $E_{\zeta\pi}(L)$ of a point $* \in E(L)$. By Lemma 2.3, this exact sequence induces the following exact sequence;

$$(\zeta)^p \xrightarrow{Q^{\zeta\pi}} (\zeta)^q \longrightarrow \big[H_1\big(E_{\zeta\pi}(L),\tilde{*}\big)\big]_\zeta \longrightarrow 0.$$

Hence,

$$\dim\big[H_1\big(E_{\zeta\pi}(L),\tilde{*}\big)\big]_\zeta = q - \operatorname{rank} Q^{\zeta\pi} = \operatorname{null}(L;\zeta\pi) + 1.$$

On the other hand, from the homology exact sequence of the pair $\big(E_{\zeta\pi}(L),\tilde{*}\big)$ and Lemma 2.3, we see

$$\big[H_1\big(E_{\zeta\pi}(L),\tilde{*}\big)\big]_\zeta \cong \begin{cases} \big[H_1\big(E_{\zeta\pi}(L)\big)\big]_\zeta \oplus (\zeta), & \text{if } \zeta \neq 1 \\ H_1\big(E(L)\big) & \text{if } \zeta = 1. \end{cases}$$

This completes the proof of the second part of Step 3. The same argument also proves the second part of Step 3. [Note that the above argument works even if $L = \emptyset$.] Now the proof of Theorem 1.1. is completed.

6. *Relation to Torres' second condition.* For each i $(1 \leq i \leq \mu)$, let $\mathcal{P}_i$ be the image of $H_1\big(\partial N(K_i);\mathbb{Z}\big)$ in $H_1\big(E(L);\mathbb{Z}\big)$. Then Torres' second condition [T] (*cf.* [Hl1, p. 83]) implies the following:

PROPOSITION 6.1. *Suppose* $\mu = |L| \geq 2$. *Let* $\varphi: H_1\big(E(L);\mathbb{Z}\big) \longrightarrow S^1 \subset \mathbb{C}$ *be a homomorphism such that* $\mathcal{P}_i \subset \operatorname{Ker}(\varphi)$ *for some i. Then* $\operatorname{null}(L;\varphi) \geq 1$.

PROOF. Suppose $\mathcal{P}_\mu \subset \operatorname{Ker}(\varphi)$. Then $\varphi(t_\mu) = 1$ and $\varphi(t_1^{\lambda_1}\cdots t_{\mu-1}^{\lambda_{\mu-1}}) = 1$, where $\lambda_i = \operatorname{lk}(K_i,K_\mu)$. Hence $\Delta_L\big(\varphi(t_1),\ldots,\varphi(t_\mu)\big) = 0$ by Torres' second condition, except when $\mu = 2$ and $\varphi = 1$. Since $\mathfrak{E}_1(L) = I_\mu\langle\Delta_L\rangle$, where $I_\mu = \langle t_1 - 1,\ldots,t_\mu - 1\rangle$ (see [Hl1, p. 86]), we obtain the desired result.

In this section, we improve this proposition by using the previous arguments. To do this, consider the homomorphism $j: H_1\big(\partial E_\pi(L)\big) \longrightarrow H_1\big(E_\pi(L)\big)$ induced by the inclusion. The following proposition gives the structure of $\operatorname{Im}(j)$ as a $\mathbb{C}[A]$-module.

PROPOSITION 6.2. $\operatorname{Im}(j) \cong \oplus_{i=1}^{\mu}\,\mathbb{C}[A/\pi(\mathcal{P}_i)]$.

PROOF. Note that $H_1\big(\partial E_\pi(L)\big) \cong \oplus_{i=1}^{\mu}\{\mathbb{C}[A/\pi(\mathcal{P}_i)] \oplus \mathbb{C}[A/\pi(\mathcal{P}_i)]\}$. So, we have only to show that the following identity holds for any $\zeta \in Z_A$;

$$\dim \operatorname{Im}(j_\zeta) = \frac{1}{2}\dim\big[H_1\big(\partial E_\pi(L)\big)\big]_\zeta.$$

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By the homology exact sequence and Lemma 2.3, we obtain the following exact sequence for each $\zeta \in \mathcal{Z}_A$;

$$\left[H_2\left(E_\pi(L), \partial E_\pi(L)\right)\right]_\zeta \xrightarrow{\partial_\zeta} \left[H_1\left(\partial E_\pi(L)\right)\right]_\zeta \xrightarrow{j_\zeta} \left[H_1\left(E_\pi(L)\right)\right]_\zeta.$$

Here ∂_ζ and j_ζ, respectively, are the restrictions of the boundary homomorphism ∂ and the homomorphism j to the ζ-components. On the other hand, by considering restrictions of intersection forms, we obtain the following commutative diagram, where ϕ_1 and ϕ_2 are A-equivariant and non-singular:

$$\begin{array}{ccc}
\left[H_1\left(\partial E_\pi(L)\right)\right]_\zeta \times \left[H_1\left(\partial E_\pi(L)\right)\right]_\zeta & \xrightarrow{\phi_1} & \mathbb{C} \\
\uparrow \partial_\zeta \qquad j_\zeta \downarrow & & \nearrow \phi_2 \\
\left[H_2\left(E_\pi(L), \partial E_\pi(L)\right)\right]_\zeta \times \left[H_1\left(E_\pi(L)\right)\right]_\zeta & &
\end{array}$$

DIAGRAM 6.1

Thus we see;

$$\begin{aligned}
x \in \operatorname{Ker}(j_\zeta) &\Leftrightarrow \phi_2\left(y, j_\zeta(x)\right) = 0 \quad \text{for any } y \in \left[H_2\left(E_\pi(L), \partial E_\pi(L)\right)\right]_\zeta, \\
&\Leftrightarrow \phi_1\left(\partial_\zeta(y), x\right) = 0 \quad \text{for any } y \in \left[H_2\left(E_\pi(L), \partial E_\pi(L)\right)\right]_\zeta, \\
&\Leftrightarrow x \in \left(\operatorname{Im}(\partial_\zeta)\right)^\perp.
\end{aligned}$$

Here $\perp$ denotes the the orthogonal complement with respect to ϕ_1. Hence,

$$\operatorname{Ker}(j_\zeta) = \left(\operatorname{Im}(\partial_\zeta)\right)^\perp = \left(\operatorname{Ker}(j_\zeta)\right)^\perp.$$

This implies that $\dim \operatorname{Ker}(j_\zeta) = \frac{1}{2} \dim\left[H_1\left(\partial E_\pi(L)\right)\right]_\zeta$, and hence, we obtain the desired result.

For each $\zeta \in \mathcal{Z}_A$, put $m(\zeta) = \#\{i \mid \mathcal{P}_i \subset \operatorname{Ker}(\zeta\pi)\}$. Then

$$\bigoplus_{i=1}^{\mu} \mathbb{C}[A/\pi(\mathcal{P}_i)] \cong \bigoplus_{i=1}^{\mu} \Big\{ \bigoplus_{\mathcal{P}_i \subset \operatorname{Ker}(\zeta\pi)} (\zeta) \Big\} \cong \bigoplus_{\zeta \in \mathcal{Z}_A} (\zeta)^{m(\zeta)}.$$

Hence we have $[\operatorname{Im}(j)]_\zeta \cong (\zeta)^{m(\zeta)}$. So, by Step 3 and Proposition 6.2,

$$\operatorname{null}(L; \zeta\pi) = \begin{cases} \dim\left[H_1\left(E_\pi(L)\right)\right]_\zeta \geq \dim[\operatorname{Im}(j)]_\zeta = m(\zeta) & \text{if } \zeta \neq 1, \\ \dim H_1\left(E(L)\right) - 1 = \mu - 1 & \text{if } \zeta = 1. \end{cases}$$

Thus we have the following proposition, which refines Proposition 6.1.

PROPOSITION 6.3. *Let $\varphi: H_1\big(E(L);\mathbb{Z}\big) \longrightarrow S^1 \subset \mathbb{C}$ be a nontrivial homomorphism with a finite image. Then*

$$\mathrm{null}\,(L;\varphi) \geq \#\{i \mid \mathcal{T}_i \subset \mathrm{Ker}(\varphi)\}.$$

7. *Polynomial periodicity.* A sequence $\{\beta(n)\}_{n\in\mathbb{N}}$ of integers is said to be of *polynomial periodic* if there is an integer N and a finite sequence of polynomials $p_0(x), p_1(x), \ldots, p_{N-1}(x)$ such that if $n \equiv i \pmod N$ and $0 \leq i \leq N-1$ then $\beta(n) = p_i(n)$. Sarnak [Sr] and Hironaka [Hr1] proved polynomial periodicity of the Betti numbers in certain towers of abelian coverings. In this section we give a generalization of their results. To do this, we generalize Theorem 1.1. In this section, (M, L) denotes a pair which satisfies one of the following conditions:

(7.1) M is a compact n-manifold (possibly with boundary), and L is a (possibly empty) union of mutually disjoint codimension 2 locally flat proper submanifolds $K_1, \ldots, K_\mu$ of M.

(7.2) M is a compact smooth complex surface, and L is a (possibly empty) union of (possibly singular) complex curves $K_1, \ldots, K_\mu$ in M.

Let $E(L)$, $G = H_1(E(L);\mathbb{Z})$, $t_i \in G$ $(1 \leq i \leq \mu)$, $\mathfrak{E}_d(L)$ $(\subset \mathbb{Z}[G])$, $\mathrm{null}\,(L;\varphi)$, A, $\pi: G \longrightarrow A$, $E_\pi(L)$ and $M_\pi(L)$ be as in Section 1, except that in case (M, L) is as in (7.2), $M_\pi(L)$ does not denote the branched covering itself but denotes a desingularization of the branched covering. (For branched coverings of complex manifolds, see [Nm].) In this case, we denote the branched covering by the symbol $\hat{M}_\pi(L)$. Note that all desingularizations of a given complex surface are mutually birationally equivalent, and hence, they have the same first Betti numbers. Theorem 1.1 is generalized as follows:

THEOREM 7.3. *Suppose (M, L) is as in (7.1) or (7.2). Then*
(1) $\beta_1\big(E_\pi(L)\big) = 1 + \sum_{\zeta\in Z_A} \mathrm{null}\,(L;\zeta\pi) = \beta_1\big(E(L)\big) + \sum_{\zeta\in Z_A^*} \mathrm{null}\,(L;\zeta\pi)$.
(2) $\beta_1\big(M_\pi(L)\big) = 1 + \sum_{\zeta\in Z_A} \mathrm{null}\,(L_{\zeta\pi};\zeta\pi) = \beta_1(M) + \sum_{\zeta\in Z_A^*} \mathrm{null}\,(L_{\zeta\pi};\zeta\pi)$.

PROOF. If (M, L) is as in (7.1), then the proof of this theorem is the same as that of Theorem 1.1 except that $\beta_1(M) = \mathrm{null}\,(\emptyset; 1) + 1$ does not vanish in general.

Suppose (M, L) is as in (7.2). Let S be the set of the singularities of L, and let $N(S)$ be a regular neighbourhood of S in M. Let $\hat{N}(S)$ be the inverse image of $N(S)$ in $\hat{M}_\pi(L)$. Then, by [L1, (3.2)], $H_1\big(M_\pi(L)\big) \cong H_1\big(\hat{M}_\pi(L) - \mathrm{int}\,\hat{N}(S)\big)$. [Here, we essentially use the fact that M is 2-dimensional over $\mathbb{C}$, because the arguments in [L1, (3.2)] uses the fact that the intersection matrix of the exceptional curves of a desingularization of a complex surface is negative definite.] On the other hand, the pair $\big(M - \mathrm{int}\,N(S), L - \mathrm{int}\,N(S)\big)$ satisfies (7.1), and $\hat{M}_\pi(L) - \mathrm{int}\,\hat{N}(S)$ is a branched covering of $M - \mathrm{int}\,N(L)$ branched over $L - \mathrm{int}\,N(S)$. Hence we obtain the desired result from the corresponding result for the case where (7.1) is satisfied.

REMARK 7.4. Suppose (M, L) is as in (7.2) and L consists of smooth curves with normal crossings. Then $\hat{M}_\pi(L)$ has only rational singularities, and we see $\beta_1\big(\hat{M}_\pi(L)\big) = \beta_1\big(M_\pi(L)\big)$.

By using Theorem 7.3, we prove the following theorem, which generalizes the result of Hironaka [Hr1, Theorem 1.7] (*cf.* [Sr]):

THEOREM 7.5. *Let $\tilde{A}$ be an abelian group, and let $\phi: G = H_1\big(E(L); \mathbb{Z}\big) \longrightarrow \tilde{A}$ be an epimorphism. Put $A(n) = \tilde{A} \otimes Z_n$, and let $\pi_n: G \longrightarrow A(n)$ be the composite of ϕ and the projection $\tilde{A} \longrightarrow A(n)$. Then the first Betti numbers $\beta_1\big(E_{\pi_n}(L)\big)$ and $\beta_1\big(M_{\pi_n}(L)\big)$ are polynomial periodic with respect to n.*

REMARK 7.6. Hironaka [Hr1] proved the above theorem in case (M, L) is as in (7.2) and satisfies certain homological conditions. She also observed in [Hr1, Remark 2.3] that in case $M = S^3$, the above theorem immediately follows from Sarnak's result [Sr, Corollary 1.4] by using the special presentation matrix of $H_1\big(M_\pi(L)\big)$ given by Mayberry and Murasugi [MM]. This is a generalization of the result of Gordon [Gr, Theorem 4.1(ii)] that the first Betti numbers of the branched cyclic coverings of a knot in a homology 3-sphere is periodic.

The following lemma is a reformulation of [Sr, Proposition 1.7] (see also [Hr1, Propositions 1.9 and 2.5]), and it is a key for the proof of the above theorem:

LEMMA 7.7. *Let F be a finitely generated free abelian group, and let $\mathfrak{E}$ be an ideal in $\mathbb{Z}[F]$. For each positive integer n, put $F(n) = F \otimes \mathbb{Z}_n$, and let $\pi_n: F \longrightarrow F(n)$ be the projection. Let $V(\mathfrak{E})_n = \{\zeta \in Z_{F(n)} \mid \zeta\pi_n(\mathfrak{E}) = 0\}$. Then the number $|V(\mathfrak{E})_n|$ is polynomial periodic.*

COROLLARY 7.8. *Let G, $\tilde{A}$, $A(n)$, $\pi_n: G \longrightarrow A(n)$ be as in Theorem 7.5, and let $\mathfrak{E}$ be an ideal in $\mathbb{Z}[G]$. Let $V(\mathfrak{E}; \pi_n) = \{\zeta \in Z_{A(n)} \mid \zeta\pi_n(\mathfrak{E}) = 0\}$. Then $|V(\mathfrak{E}; \pi_n)|$ is polynomial periodic.*

PROOF. Since G is finitely generated, there is a finitely generated free abelian group F and an epimorphism $\psi: F \longrightarrow G$. Let $\bar{\mathfrak{E}}$ be the ideal in $\mathbb{Z}[F]$ generated by $\psi^{-1}(\mathfrak{E})$ and $\mathrm{Ker}\big[\phi\psi: \mathbb{Z}[F] \longrightarrow \mathbb{Z}[\tilde{A}]\big]$. Then we can see that there is a bijection between $V(\mathfrak{E}; \pi_n)$ and $V(\bar{\mathfrak{E}})_n$. Hence, the desired result follows from Lemma 7.7.

Keeping the above corollary in mind, we reformulate Theorem 7.3. Let L' be a "sublink" of L, *i.e.*, a (possibly empty) union of comoponents of L. [Here a component of L means a connected component or an irreducible component of L according as (M, L) is as in (7.1) or (7.2).] Let $p: \mathbb{Z}[G] = \mathbb{Z}\big[H_1\big(E(L); \mathbb{Z}\big)\big] \longrightarrow \mathbb{Z}\big[H_1\big(E(L'); \mathbb{Z}\big)\big]$ be the natural projection. For each non-negative integer d, put $\tilde{\mathfrak{E}}_d(L') = p^{-1}\big(\mathfrak{E}_d(L')\big)$. Let L'' be a sublink of L'. Then $\tilde{\mathfrak{E}}_d(L', L'')$ denotes the ideal in $\mathbb{Z}[G]$ generated by $\tilde{\mathfrak{E}}_d(L')$ and $\{t_i - 1 \mid K_i \not\subset L''\}$. Note that $\tilde{\mathfrak{E}}_d(L') = \tilde{\mathfrak{E}}_d(L', L')$. Let

$$V_d(L'; \pi) = \Big\{\zeta \in Z_A \mid \zeta\pi\big(\tilde{\mathfrak{E}}_d(L')\big) = 0\Big\},$$
$$V_d^*(L'; \pi) = \{\zeta \in V_d(L'; \pi) \mid \zeta\pi(t_i - 1) \neq 0 \text{ if } K_i \subset L'\},$$

$$V_d(L',L'';\pi) = \{\zeta \in V_d(L';\pi) \mid \zeta\pi(t_i - 1) = 0 \text{ if } K_i \not\subseteq L''\},$$
$$= \left\{\zeta \in \mathcal{Z}_A \mid \zeta\pi\left(\tilde{\mathfrak{F}}_d(L',L'')\right) = 0\right\}.$$

Then we can see that Theorem 7.3 is equivalent to the following:

THEOREM 7.9. *Suppose (M,L) is as in (7.1) or (7.2). Then*
(1) $\beta_1\left(E_\pi(L)\right) = 1 + \sum_d |V_d(L;\pi)|$.
(2) $\beta_1\left(M_\pi(L)\right) = 1 + \sum_{L' \subset L} \sum_d |V_d^*(L';\pi)|$.

Further, we have the following:

LEMMA 7.10. $\left|V_d^*(L';\pi)\right| = \sum_{L'' \subset L'} (-1)^{|L'-L''|} \left|V_d(L',L'';\pi)\right|$.

PROOF. We prove this lemma when $L' = L$. The other cases can be proved similarly. For each sublink L' of L, put $[L'] = V_d(L,L';\pi)$ and $[L']^* = \{\zeta \in [L] \mid \zeta\pi(t_i - 1) = 0$ if and only if $K_i \not\subseteq L'\}$. Then, $[L'] = \coprod_{L'' \subset L'} [L'']^*$ and $[L]^* = V_d^*(L;\pi)$. We show that

$$\left|[L']^*\right| = \sum_{L'' \subset L'} (-1)^{|L'-L''|} \left|[L'']\right|$$

for any sublink L' of L, by using induction on $|L'|$. For $L' = \emptyset$, we see $[\emptyset]^* = [\emptyset]$, and the desired identity clearly holds. Next, we show that if the identity holds for any proper sublink of L', then the identity holds for L'. We show this when $L' = L$. (The same argument works for any L'.) By using the fact that $[L] = \coprod_{L' \subset L} [L']^*$, we see

$$\begin{aligned}
\left|[L]^*\right| &= \left|[L]\right| - \sum_{\substack{L' \subset L \\ \neq}} \left|[L']^*\right| \\
&= \left|[L]\right| - \sum_{\substack{L' \subset L \\ \neq}} \left\{ \sum_{L'' \subset L'} (-1)^{|L'-L''|} \left|[L'']\right| \right\} \\
&= \left|[L]\right| - \sum_{\substack{L'' \subset L \\ \neq}} \left\{ \sum_{\substack{L'' \subset L' \subset L \\ \neq}} (-1)^{|L'-L''|} \right\} \left|[L'']\right| \\
&= \left|[L]\right| - \sum_{\substack{L'' \subset L \\ \neq}} \left\{ \sum_{r=0}^{|L-L''|-1} (-1)^r \binom{|L-L''|}{r} \right\} \left|[L'']\right| \\
&= \left|[L]\right| - \sum_{\substack{L'' \subset L \\ \neq}} (-1)^{|L-L''|-1} \left|[L'']\right| \\
&= \sum_{L' \subset L} (-1)^{|L-L'|} \left|[L']\right|.
\end{aligned}$$

This completes the proof of Lemma 7.10.

We now prove Theorem 7.5. By definition,

$$V_d(L;\pi_n) = V\left(\tilde{\mathfrak{F}}_d(L);\pi_n\right),$$
$$V_d(L',L'';\pi_n) = V\left(\tilde{\mathfrak{F}}_d(L',L'');\pi_n\right).$$

Hence, by Theorem 7.9 and Lemma 7.10,

$$\beta_1\big(E_{\pi_n}(L)\big) = 1 + \sum_d \Big|V\big(\tilde{\mathfrak{C}}_d(L);\pi_n\big)\Big|,$$

$$\beta_1\big(M_{\pi_n}(L)\big) = 1 + \sum_d \sum_{L'\subset L} \sum_{L''\subset L'} (-1)^{|L'-L''|}\Big|V\big(\tilde{\mathfrak{C}}_d(L',L'');\pi_n\big)\Big|.$$

Hence, by Corollary 7.8, $\beta_1\big(E_{\pi_n}(L)\big)$ and $\beta_1\big(M_{\pi_n}(L)\big)$ are polynomial periodic.

2. Torsion numbers of abelian coverings of links.

8. *Statement of results.* In this chapter, we study the $\mathbb{Z}$-torsion parts of the integral homology groups of abelian coverings of links in homology 3-spheres. Throughout this chapter, homology groups are considered to be with coefficients $\mathbb{Z}$ unless otherwise stated, and we use the following notation. If H is a finitely generated abelian group and p is a prime number, then $H^{(p)}$ denotes the p-torsion part of H, *i.e.*, $H^{(p)} = \{x \in H \mid p^e x = 0$ for some $e \geq 0\}$. Then $\operatorname{Tor} H$, the $\mathbb{Z}$-torsion part of H, is the direct sum $\bigoplus_p H^{(p)}$, where p runs over the set of the prime numbers. If $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ is a set of complex numbers indexed by a finite set Λ, then $\hat{\prod}_{\lambda\in\Lambda}\alpha_\lambda$ denotes the product of all non-zero α_λ's. If $\alpha_\lambda = 0$ for any $\lambda \in \Lambda$, it is defined to be 1. For an integer n and a prime number p, $n^{(p)}$ denotes the p-component of n. Thus $|\operatorname{Tor} H|^{(p)}$, the p-component of the order of $\operatorname{Tor} H$, is equal to $|H^{(p)}|$, the order of the p-component $H^{(p)}$ of H.

Let M, L, A, π, $E_\pi(L)$, and $M_\pi(L)$ be as in Section 1. Then we have the following:

THEOREM 8.1. *Let p be a prime number which does not divide $|A|$.*
(1) $|H_1\big(E_\pi(L)\big)|^{(p)}$ *is divisible by* $\Big|\hat{\prod}_{\zeta\in Z_A^*}\Delta_L\big(\zeta\pi(t_1),\ldots,\zeta\pi(t_\mu)\big)\Big|^{(p)}$.
(2) $\Big|H_1\big(M_\pi(L)\big)\Big|^{(p)}$ *is divisible by* $\Big|\hat{\prod}_{\zeta\in Z_A^*}\Delta_{L_{\zeta\pi}}\big(\zeta\pi(t_1),\ldots,\zeta\pi(t_\mu)\big)\Big|^{(p)}$.

REMARK 8.2. (1) Unfortunately, the above estimates are not so good as explained in Section 13. However, we have the following:

(i) The two numbers in Theorem 8.1(1) are equal, provided that either (a) $\beta_1\big(M_\pi(L)\big) = 0$ and π is "indivisible" in the sense of [*MM*], *i.e.*, $\pi(\mathcal{P}_i) = A$ for any i $(1 \leq i \leq \mu)$, or (b) $\pi_1\big(E(L)\big)$ is generated by two elements.

(ii) The two numbers in Theorem 8.1(2) are equal, provided that either (a) $\beta_1\big(M_\pi(L)\big) = 0$, or (b) $\pi_1\big(E(L)\big)$ is generated by two elements.

(2) In case M is S^3 and Condition (a) (in (i) or (ii)) holds, each of the above identities follows from the results of Mayberry and Murasugi [MM]. In fact, they gave precise formulae for the orders of $\operatorname{Tor} H_1\big(E_\pi(L)\big)$ and $H_1\big(M_\pi(L)\big)$ under these conditions.

9. *Outline of the proof of Theorem 8.1.* The proof of Theorem 8.1 is parallel to the proof of Theorem 1.1. Let $\mathcal{C}_A$ denote the set of the subgroups B of A such that A/B is cyclic. For each $B \in \mathcal{C}_A$, put

$$\mathcal{E}_B = |A| \sum_{\operatorname{Ker}\zeta=B} e_\zeta = \sum_{g\in G}\Big\{\sum_{\operatorname{Ker}\zeta=B}\overline{\zeta(g)}\Big\}g.$$

Then $\mathcal{E}_B$ is an element of the integral group ring $\mathbb{Z}[A]$ of A, and we can easily prove the following by using Lemma 2.1.

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LEMMA 9.1. *(1)* $b\mathcal{E}_B = \mathcal{E}_B$ *for any* $b \in B$.

(2) $\mathcal{E}_B \cdot \mathcal{E}_{B'} = \begin{cases} |A|\mathcal{E}_B & \text{if } B = B', \\ 0 & \text{if } B \neq B' \end{cases}$.

(3) Let B *be a subgroup of* A*, and put* $\mathcal{C}_A(B) = \{B' \in \mathcal{C}_A \mid B' \supset B\}$. *Then*

$$\sum_{B' \in \mathcal{C}_A(B)} \mathcal{E}_{B'} = [A;B]\,\mathrm{Tr}\,B.$$

In particular,

$$\sum_{B \in \mathcal{C}_A} \mathcal{E}_B = |A|.$$

Let H be a finitely generated $\mathbb{Z}[A]$-module. For each $B \in \mathcal{C}_A$, put $[H]_B = \mathcal{E}_B H$. This is a $\mathbb{Z}[A]$-submodule of H, and it can also be considered as a $\mathbb{Z}[A/B]$-module, since the action of B on $[H]_B$ is trivial by Lemma 9.1(1).

LEMMA 9.2. *Let* Ψ *be the* $\mathbb{Z}[A]$*-homomorphism* $\oplus_{B \in \mathcal{C}_A}[H]_B \longrightarrow H$ *induced by the inclusions. Then we have the following:*

(1) $|A|\,\mathrm{Ker}(\Phi) = 0$.

(2) $\mathrm{Coker}(\Phi)$ *is a quotient of* $H/|A|H$.

PROOF. (1) Let $(\mathcal{E}_B x_B)_{B \in \mathcal{C}_A}$ be an element of $\mathrm{Ker}\,\Phi$, *i.e.*, $\sum_{B \in \mathcal{C}_A} \mathcal{E}_B x_B = 0$ in H. Then for any $B_0 \in \mathcal{C}_A$, we have the following by Lemma 9.1(2), which proves (1);

$$0 = \mathcal{E}_{B_0}\Big(\sum_{B \in \mathcal{C}_A} \mathcal{E}_B x_B\Big) = |A|\mathcal{E}_{B_0} x_{B_0}.$$

(2) By Lemma 9.1(3), we see $\mathrm{Im}(\Phi) \supset \mathrm{Im}(\sum_{B \in \mathcal{C}_A} \mathcal{E}_B) = |A|H$. This implies the desired result.

In the remainder of this chapter, p denotes a prime number which does not divide $|A|$.

LEMMA 9.3. $\mathcal{E}_B\big(H^{(p)}\big) = (\mathcal{E}_B H)^{(p)}$.

PROOF. Since $H^{(p)} \subset H$, $\mathcal{E}_B(H^{(p)})$ is contained in $(\mathcal{E}_B H)^{(p)}$. Conversely, let $x = \mathcal{E}_B y$ be an element of $(\mathcal{E}_B H)^{(p)}$. Since p does not divide $|A|$, there is an element $z \in H^{(p)}$ such that $x = |A|z$. Then,

$$|A|\mathcal{E}_B z = \mathcal{E}_B x = \mathcal{E}_B(\mathcal{E}_B y) = |A|\mathcal{E}_B y = |A|x.$$

Since x and $\mathcal{E}_B z$ are elements of $H^{(p)}$, we see $x = \mathcal{E}_B z \in \mathcal{E}_B(H^{(p)})$.

Let $[H]_B^{(p)}$ denote the submodule $\mathcal{E}_B(H^{(p)}) = (\mathcal{E}_B H)^{(p)}$. Then by Lemma 9.2, we have the following:

LEMMA 9.4. $H^{(p)} \cong \oplus_{B \in \mathcal{C}_A}[H]_B^{(p)}$.

The following lemmas can be proved easily:

LEMMA 9.5. *Suppose the $\mathbb{Z}[A]$-module structure of H comes from a $\mathbb{Z}[A/B]$-module structure for some $B \in \mathcal{C}_A$. Then $[H]_B^{(p)}$ (of the $Z[A]$-module H) is $\mathbb{Z}[A/B]$-isomorphic to $[H]_{\{1\}}^{(p)}$ (of the $\mathbb{Z}[A/B]$-module H).*

LEMMA 9.6. *Let B and B' be two mutually distinct elements of $\mathcal{C}_A$. Then for any pair of finitely generated $\mathbb{Z}[A]$-modules H and H', the 0-map is the only $\mathbb{Z}[A]$-homomorphism from $[H]_B^{(p)}$ to $[H']_{B'}^{(p)}$.*

We now give an outline of the proof of Theorem 8.1.

STEP 0. $H_1\big(E_\pi(L)\big)$ and $H_1\big(M_\pi(L)\big)$ are finitely generated $\mathbb{Z}[A]$-modules. So, by Lemma 9.4,

$$H_1\big(E_\pi(L)\big)^{(p)} \cong \bigoplus_{B\in\mathcal{C}_A} \big[H_1\big(E_\pi(L)\big)\big]_B^{(p)},$$

$$H_1\big(M_\pi(L)\big)^{(p)} \cong \bigoplus_{B\in\mathcal{C}_A} \big[H_1\big(M_\pi(L)\big)\big]_B^{(p)}.$$

STEP 1. Let B be an element of $\mathcal{C}_A$, and ρ be the projection $A \longrightarrow A/B$. Then

$$\big[H_1\big(E_\pi(L)\big)\big]_B^{(p)} \cong \big[H_1(E_{\rho\pi}(L)\big)\big]_B^{(p)},$$

$$\big[H_1\big(M_\pi(L)\big)\big]_B^{(p)} \cong [H_1(M_{\rho\pi}(L)\big)\big]_B^{(p)}.$$

STEP 2. Let $L_{\rho\pi}$ be the sublink of L consisting of those components K_i such that $\rho\pi(t_i) \neq 1$. Then

$$\big[H_1\big(M_{\rho\pi}(L)\big)\big]_B^{(p)} \cong \big[H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)\big]_B^{(p)}.$$

STEP 3.a. Let n be the order of the cyclic group $\operatorname{Im}(\rho)$. We can identify $\operatorname{Im}(\rho)$ with $\langle t \mid t^n = 1\rangle$ and find an epimorphism $\tilde\rho$ from $H_1\big(E(L)\big)$ to the infinite cyclic group $\langle t\rangle$, such that the following diagram is commutative:

$$\begin{tikzcd}
 & & & \langle t\rangle \arrow[d] \\
 & & & \langle t \mid t^n = 1\rangle \arrow[d, equal] \\
H_1\big(E(L_{\rho\pi})\big) \arrow[uurrr, "\tilde\rho"] & H_1\big(E(L)\big) \arrow[l] \arrow[r, "\pi"] \arrow[uurr, "\tilde\rho"'] & A \arrow[r, "\rho"] & A/B
\end{tikzcd}$$

DIAGRAM 9.1

Here the vertical map $\langle t\rangle \longrightarrow \langle t \mid t^n = 1\rangle$ is the natural projection, and the homomorphism $H_1\big(E(L_{\rho\pi})\big) \longrightarrow \langle t\rangle$ induced by $\tilde\rho$ is denoted by the same symbol. Then we have

$$\big[H_1\big(E_{\rho\pi}(L)\big)\big]_B^{(p)} \cong \left[\frac{H_1\big(E_{\tilde\rho}(L)\big)}{(t^n-1)H_1\big(E_{\tilde\rho}(L)\big)}\right]_{\{1\}}^{(p)},$$

$$\big[H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)\big]_B^{(p)} \cong \left[\frac{H_1\big(E_{\tilde\rho}(L_{\rho\pi})\big)}{(t^n-1)H_1\big(E_{\tilde\rho}(L_{\rho\pi})\big)}\right]_{\{1\}}^{(p)}.$$

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Here $E_{\tilde{\rho}}$ denotes the infinite cyclic covering corresponding to $\tilde{\rho}$, and the modules in the parentheses on the right-hand sides are regarded as $\mathbb{Z}\langle t \mid t^n = 1\rangle$-modules.

STEP 3.b. By using the results in Step 3.a, we prove the following:

(1) $\left|\left[H_1\big(E_{\rho\pi}(L)\big)\right]_B^{(p)}\right| = \left|\hat{\prod}_{\operatorname{Ker}\zeta=B}\Delta_L\big(\zeta\pi(t_1),\ldots,\zeta\pi(t_\mu)\big)\right|^{(p)}$, provided that the right-hand side is not 0.

(2) $\left|\left[H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)\right]_B^{(p)}\right| = \left|\hat{\prod}_{\operatorname{Ker}\zeta=B}\Delta_{L_{\zeta\pi}}\big(\zeta\pi(t_1),\ldots,\zeta\pi(t_\mu)\big)\right|^{(p)}$, provided that the right-hand side is not 0. We can obtain Theorem 8.1 through these steps. [Note that, if $B = A$, then $E_{\rho\pi}(L) = E(L)$ and $E_{\rho\pi}(L_{\rho\pi}) = M$.]

10. Proof of Step 1. As in Section 3, we see

$$\begin{aligned} H_1\big(M_{\rho\pi}(L)\big)^{(p)} &\cong H_1\big(M_\pi(L)/B\big)^{(p)} \\ &\cong (\operatorname{Tr} B)H_1\big(M_\pi(L)\big)^{(p)} \quad \text{by [Br, pp. 118-120]} \\ &\cong ([A;B]\operatorname{Tr} B)H_1\big(M_\pi(L)\big)^{(p)} \quad \text{since } p \text{ does not divide } |A| \\ &\cong \Big(\sum_{B'\in\mathcal{C}_A(B)} \mathcal{E}_{B'}\Big)H_1\big(M_\pi(L)\big)^{(p)} \quad \text{by Lemma 9.1(3)} \\ &\cong \bigoplus_{B'\in\mathcal{C}_A(B)} \left[H_1\big(M_\pi(L)\big)\right]_{B'}^{(p)} \quad \text{by Lemmas 9.1 (2) and 9.4.} \end{aligned}$$

Hence, by using Lemma 9.1(2) and the fact that $B \in \mathcal{C}_A(B)$, we see

$$\left[H_1\big(M_{\rho\pi}(L)\big)\right]_B^{(p)} \cong \Big[\bigoplus_{B'\in\mathcal{C}_A(B)} \left[H_1\big(M_\pi(L)\big)\right]_{B'}^{(p)}\Big]_B \cong \left[H_1\big(M_\pi(L)\big)\right]_B^{(p)}.$$

This completes the proof of the second part of Step 1. The first part can be proved similarly.

11. Proof of Step 2. Put $\Lambda_{(n)} = \mathbb{Z}\langle t \mid t^n = 1\rangle$ ($\cong \mathbb{Z}[A/B]$), and regard $H_1\big(M_{\rho\pi}(L)\big)$ and $H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)$ as $\Lambda_{(n)}$-modules. Then, by Lemma 9.5, the assertion in Step 2 is equivalent to the assertion that $\left[H_1\big(M_{\rho\pi}(L)\big)\right]_{\{1\}}^{(p)} \cong \left[H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)\right]_{\{1\}}^{(p)}$ as $\Lambda_{(n)}$-modules. As in Section 4, for each component K_i of $L_{\rho\pi}$, let b_i be the element of $H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)$ represented by a lift of a meridian of K_i in $E_{\rho\pi}(L_{\rho\pi})$. Since $M_{\rho\pi}(L) = M_{\rho\pi}(L_{\rho\pi})$, $H_1\big(M_{\rho\pi}(L)\big) \cong H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)/\mathcal{B}$, where $\mathcal{B}$ is the $\Lambda_{(n)}$-submodule of $H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)$ generated by $\{b_i \mid K_i \subset L_{\rho\pi}\}$. Put $\mathcal{E}_1 = \mathcal{E}_{\{1\}} \in \Lambda_{(n)}$.

LEMMA 11.1. *For any divisor d of n with $0 < d < n$, $t^d - 1$ divides $\mathcal{E}_1$ in $\Lambda_{(n)}$.*

PROOF. Let $\tilde{\mathcal{E}}_1(t)$ be the integral polynomial defined by $\tilde{\mathcal{E}}_1(t) = \sum_\omega\{\sum_{i=0}^{n-1}\bar{\omega}^i t^i\}$ where ω runs over all primitive n-th roots of 1. Then $\tilde{\mathcal{E}}_1(t) = \mathcal{E}_1$ in $\Lambda_{(n)}$. For any d-th root ω' of 1, we can easily see $\tilde{\mathcal{E}}_1(\omega') = 0$. Thus $t^d - 1$ divides $\tilde{\mathcal{E}}_1(t)$ in $\mathbb{Z}\langle t\rangle$, and we obtain the desired result.

LEMMA 11.2. $\mathcal{E}_1\mathcal{B} = 0$.

PROOF. For each $K_i \subset L_{\rho\pi}$, there is a divisor d_i of n such that $(t^{d_i} - 1)b_i = 0$ and $0 < d_i < n$ (see Section 4). Thus we see $\mathcal{E}_1 b_i = 0$ by Lemma 11.1.

LEMMA 11.3. *For any* $x \in \mathcal{B} \cap \mathcal{E}_1 H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)$, *we have* $nx = 0$.

PROOF. Since $x \in \mathcal{B}$, we have $\mathcal{E}_1 x = 0$ by Lemma 11.2. On the other hand, $x = \mathcal{E}_1 y$ for some $y \in H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)$. Hence $\mathcal{E}_1 x = \mathcal{E}_1^2 y = n\mathcal{E}_1 y = nx$ by Lemma 9.1 (2). Thus we have $nx = 0$.

Since p does not divide n, we see the following:

$$\begin{aligned}
\big[H_1\big(M_{\rho\pi}(L)\big)\big]_{\{1\}}^{(p)} &\cong \mathcal{E}_1\left(\frac{H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)}{\mathcal{B}}\right)^{(p)} \\
&\cong \left(\frac{\mathcal{E}_1 H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)}{\mathcal{B} \cap \mathcal{E}_1 H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)}\right)^{(p)} \\
&\cong \Big(\mathcal{E}_1 H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)\Big)^{(p)} \\
&\cong \big[H_1\big(E_{\rho\pi}(L_{\rho\pi})\big)\big]_{\{1\}}^{(p)}.
\end{aligned}$$

12. Proof of Step 3.a. First, we prove the existence of the epimorphism $\tilde{\rho}$. Let s be a generator of $\mathrm{Im}(\rho)$, and let n_i be an integer such that $\eta\pi(t_i) = s^{n_i}$ $(1 \le i \le \mu)$. Put $d = g.c.d.\{n_1, \dots . n_\mu\}$, then s^d is also a generator of $\mathrm{Im}(\rho)$. Put $t = s^d \in \mathrm{Im}(\rho)$. Then $\mathrm{Im}(\rho)$ is identified with $\langle t \mid t^n = 1\rangle$. Consider the infinite cyclic group $\langle t\rangle$ generated by the symbol t, and let $\tilde{\rho}$ be the homomorphism from $H_1\big(E(L)\big) = \langle t_1, \dots, t_\mu\rangle$ to $\langle t\rangle$ defined by $\tilde{\rho}(t_i) = t^{n_i/d}$. Then $\tilde{\rho}$ satisfies the required conditions.

The first isomorphism follows from Lemma 9.6 and the exact sequence induced by the following short exact sequence of chain complexes;

$$0 \longrightarrow C_*\big(E_{\tilde{\rho}}(L)\big) \xrightarrow{t^n-1} C_*\big(E_{\tilde{\rho}}(L)\big) \longrightarrow C_*\big(E_\rho(L)\big) \longrightarrow 0.$$

The second isomorphism can be proved similarly.

13. Proofs of Step 3.b and Remark 8.2.

LEMMA 13.1. *Let H be a finitely generated $\mathbb{Z}\langle t\rangle$-module and p a prime number which does not divide n. Then*

$$\left[\frac{H}{(t^n-1)H}\right]_{\{1\}}^{(p)} \cong \left[\frac{H}{\phi_n(t)H}\right]_{\{1\}}^{(p)},$$

where $\phi_n(t)$ is the n-th cyclotomic polynomial.

Suppose further that H has a square presentation matrix whose determinant is $\Delta(t)$, and that $\Delta(\omega) \neq 0$ for any primitive n-th root of 1. Then $\big|[H/(t^n-1)H]_{\{1\}}^{(p)}\big| = \big|R\big(\Delta(t), \phi_n(t)\big)\big|^{(p)}$. Here $R\big(\Delta(t), \phi_n(t)\big)$ denotes the resultant of $\Delta(t)$ and $\phi_n(t)$; so it is equal to $\prod_\omega \Delta(\omega)$, where ω runs over all primitive n-th roots of 1.

This lemma will be proven later. Now the first part of Step 3.b follows from the following facts:

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(1) the $\mathbb{Z}\langle t\rangle$-module $H_1\big(E_{\tilde\rho}(L)\big)$ has a square presentation matrix with determinant $(t-1)\Delta_L\big(\tilde\rho(t_1),\ldots,\tilde\rho(t_\mu)\big)$ or $\Delta_L\big(\tilde\rho(t_1),\ldots,\tilde\rho(t_\mu)\big)$ accoring as $\mu \geq 2$ or $\mu = 1$ (*cf.* [Sa1, Theorem 1(i)]).

(2) Since $\big|R\big(t-1,\phi_n(t)\big)\big| = |\phi_n(1)|$ is a divisor of n, and p is not a divisor of n,

$$\Big|R\Big((t-1)\Delta_L\big(\tilde\rho(t_1),\ldots,\tilde\rho(t_\mu)\big),\phi_n(t)\Big)\Big|^{(p)} = \Big|R\big(\Delta_L(\tilde\rho(t_1),\ldots,\tilde\rho(t_\mu)),\phi_n(t)\big)\Big|^{(p)}$$
$$= \Big|\prod_{\operatorname{Ker}\zeta=B}\Delta_L\big(\zeta\pi(t_1),\ldots,\zeta\pi(t_\mu)\big)\Big|^{(p)}.$$

The second part of Step 3.b is proved similarly.

We now discuss the effectiveness of Theorem 8.1. Let B, ρ, and $\tilde\rho$ be as in the above. Then we can see that, for any $\zeta \in \mathcal{Z}_A$ with $\operatorname{Ker}(\zeta) = B$, the following holds (see Section 1):

$$\big[H_1\big(E_{\rho\pi}(L);\mathbb{C}\big)\big]_\zeta \cong \left[\frac{H_1(E_{\tilde\rho}(L);\mathbb{C})}{(t^n-1)H_1(E_{\tilde\rho}(L);\mathbb{C})}\right]_\zeta \cong \left[\frac{H_1(E_{\tilde\rho}(L);\mathbb{C})}{\phi_n(t)H_1(E_{\tilde\rho}(L);\mathbb{C})}\right]_\zeta.$$

On the other hand, by the proof of Proposition 6.2,

$$\dim\big[H_1\big(E_{\rho\pi}(L);\mathbb{C}\big)\big]_\zeta \geq \#\{i \mid \mathcal{P}_i \subset \operatorname{Ker}(\zeta\pi)\}.$$

The latter number is equal to $m(B;\pi) = \#\{i \mid \pi(\mathcal{P}_i) \subset B\}$. Thus we have

$$\dim\left(\frac{H_1\big(E_{\tilde\rho}(L);\mathbb{C}\big)}{\phi_n(t)H_1\big(E_{\tilde\rho}(L);\mathbb{C}\big)}\right) \geq \varphi(n)m(B;\pi),$$

where φ denotes the Möbius function, *i.e.*, $\varphi(n)$ is equal to the number of positive integers less than n that is relatively prime to n. In particular, if $m(B;\pi) \neq 0$, then $H_1\big(E_{\tilde\rho}(L)\big)/\phi_n(t)H_1\big(E_{\tilde\rho}(L)\big)$ has a nontrivial rank, and hence the corresponding resultant vanishes; so, in Theorem 8.1(1), we ignore the order of $|H_1\big(E_{\rho\pi}(L)\big)|_B^{(p)}$. [The author does not know an estimate of the above order in terms of the Alexander invariants in case $m(B;\pi) \neq 0$.]

Keeping the above observation in mind, we prove the assertion (i) in Remark 8.2(1). Suppose the condition (a) is satisfied. Then, for each $\zeta \in \mathcal{Z}_A$ with $\operatorname{Ker}(\zeta) = B$, the following holds:

$$0 = \big[H_1\big(M_{\rho\pi}(L);\mathbb{C}\big)\big]_\zeta \quad \text{since } \beta_1\big(M_\pi(L)\big) = 0$$
$$= \left[\frac{H_1\big(E_{\tilde\rho}(L);\mathbb{C}\big)}{\phi_n(t)H_1\big(E_{\tilde\rho}(L);\mathbb{C}\big)}\right]_\zeta \quad \text{since } L = L_{\zeta\pi} \text{ by the indivisibility of } \pi.$$

So, the corresponding resultant does not vanish. Hence, we obtain the desired result.

Next, suppose the condition (b) is satisfied. Then for each sublink $L_{\rho\pi}$ of L, $\pi_1\big(E(L_{\rho\pi})\big)$ is generated by two elements. Hence the $\mathbb{Z}\langle t\rangle$-module $H_1\big(E_{\tilde\rho}(L_{\rho\pi})\big)$ is generated by one element. So, if the corresponding resultant is zero, then $H_1\big(E_{\tilde\rho}(L)\big)/\phi_n(t)H_1\big(E_{\tilde\rho}(L)\big)$ is isomorphic to $\mathbb{Z}\langle t\rangle/\langle\phi_n(t)\rangle$ (since $\phi_n(t)$ is irreducible over $\mathbb{Z}$), and therefore, it is $\mathbb{Z}$-torsion free. Hence, we obtain the desired result.

The assertion (ii) in Remark 8.2(1) is proved similarly.

In the rest of this section we prove Lemma 13.1.

LEMMA 13.2. *Let $\mathcal{R}$ be the $\mathbb{Z}\langle t\rangle$-module determined by the following short exact sequence;*

$$0 \longrightarrow \frac{\mathbb{Z}\langle t\rangle}{\langle t^n-1\rangle} \longrightarrow \bigoplus_{d|n} \frac{\mathbb{Z}\langle t\rangle}{\langle \phi_d(t)\rangle} \longrightarrow \mathcal{R} \longrightarrow 0.$$

Here the map from the first term to the middle term is the natural one. Then $\mathcal{R}$ is a finite $\mathbb{Z}\langle t\rangle$-module such that $|\mathcal{R}|$ contains only prime factors of n.

PROOF. Let $\{d_1, \ldots, d_k\}$ be the set of the divisors of n, and let $\mathcal{R}_i$ $(1 \le i \le k-1)$ be the $\mathbb{Z}\langle t\rangle$-module determined by the following exact sequence;

$$0 \longrightarrow \frac{\mathbb{Z}\langle t\rangle}{\langle \prod_{j=i}^k \phi_{d_j}(t)\rangle} \longrightarrow \frac{\mathbb{Z}\langle t\rangle}{\langle \phi_{d_i}(t)\rangle} \oplus \frac{\mathbb{Z}\langle t\rangle}{\langle \prod_{j=i+1}^k \phi_{d_j}(t)\rangle} \longrightarrow \mathcal{R}_i \longrightarrow 0.$$

Here the first homomorphism is the natural one. Then we have $|\mathcal{R}| = \prod_{i=1}^{k-1} |\mathcal{R}_i|$, since $t^n - 1 = \prod_{j=1}^k \phi_{d_j}(t)$. On the other hand, we see

$$\mathcal{R}_i \cong \frac{\mathbb{Z}\langle t\rangle}{\langle \phi_{d_i}(t), \prod_{j=i+1}^k \phi_{d_j}(t)\rangle},$$

and hence,

$$\begin{aligned} |\mathcal{R}_i| &= \Big|R\Big(\phi_{d_i}(t), \prod_{j=i+1}^{k} \phi_{d_j}(t)\Big)\Big| \\ &= \prod_{j=i+1}^{k} \Big|R\Big(\phi_{d_i}(t), \phi_{d_j}(t)\Big)\Big|. \end{aligned}$$

Note that

$$\Big|R\Big(\phi_{d_i}(t), \phi_{d_j}(t)\Big)\Big| = \Big|\prod(\omega_1 - \omega_2)\Big| = \Big|\prod(1 - \bar{\omega}_1\omega_2)\Big|,$$

where ω_1 [resp. ω_2] runs over all primitive d_i-th [resp. d_j-th] roots of 1; so $\bar{\omega}_1\omega_2$ is a d-th root of 1 for some nontrivial divisor d of $d_1 d_2$. Thus the above number is a product of $|\phi_d(1)|$'s and therefore it contains only prime factors of $d_i d_j$. Hence $|\mathcal{R}_i|$ contains only prime factors of $\prod_{j=i}^n d_j$. So $|\mathcal{R}|$ contains only prime factors of n.

PROOF OF LEMMA 13.1. Let M be a presentation matrix of the $\mathbb{Z}\langle t\rangle$-module H, *i.e.*,

$$\mathbb{Z}\langle t\rangle^p \xrightarrow{M} \mathbb{Z}\langle t\rangle^q \longrightarrow H \longrightarrow 0.$$

Then we have the following exact sequence

$$\Lambda_{(n)}^p \xrightarrow{M} \Lambda_{(n)}^q \longrightarrow \frac{H}{(t^n-1)H} \longrightarrow 0,$$

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Hence, we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
\Lambda_{(n)}^p & \xrightarrow{M} & \Lambda_{(n)}^q & \longrightarrow & \frac{H}{(t^n-1)H} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\left(\bigoplus_{d|n} \frac{\Lambda_{(n)}}{\langle\phi_d(t)\rangle}\right)^p & \xrightarrow{\bigoplus_{d|n} M} & \left(\bigoplus_{d|n} \frac{\Lambda_{(n)}}{\langle\phi_d(t)\rangle}\right)^q & \longrightarrow & \bigoplus_{d|n} H_d & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\mathcal{R}^p & \xrightarrow{\bar{M}} & \mathcal{R}^q & & & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

Here $\mathcal{R}$ is as in Lemma 13.2, $\bigoplus_{d|n} M$ and $\bar{M}$ are the natural homomorphisms induced by M, and

$$\begin{aligned} H_d &= \operatorname{Coker}\left[M\colon \left(\frac{\Lambda}{\langle\phi_d(t)\rangle}\right)^p \to \left(\frac{\Lambda}{\langle\phi_d(t)\rangle}\right)^q\right] \\ &\cong \frac{H}{\phi_d(t)H}. \end{aligned}$$

By the snake lemma, we have the exact sequence;

$$\operatorname{Ker}(\bar{M}) \longrightarrow \frac{H}{(t^n-1)H} \longrightarrow \bigoplus_{d|n} \frac{H}{\phi_d(t)H} \longrightarrow \operatorname{Coker}(\bar{M}).$$

By Lemma 13.2, $\operatorname{Ker}(\bar{M})$ and $\operatorname{Coker}(\bar{M})$ are finite abelian groups whose orders are not divisible by p. Hence we have

$$\left[\frac{H}{(t^n-1)H}\right]^{(p)} \cong \left[\bigoplus_{d|n} \frac{H}{\phi_d(t)H}\right]^{(p)},$$

and therefore we obtain the first part of Lemma 13.1. The second part follows from [Sk1, Lemma 2] or [W].

3. **Homology of abelian coverings of spatial graphs.**

14. Statement of results. Throughout this chapter, homology groups are considered to be with coefficients $\mathbb{Z}$. Let Γ be a finite graph embedded in a homology 3-sphere M, such that the valency of any vertex of Γ is 2 or 3. In this chapter, we use the term "edge" to denote the closure of a component of $\Gamma - \{\text{vertices with valency } 3\}$. For each edge e of Γ, let t_e be the homology class in $H_1(M-\Gamma)$ represented by a small simple loop around e. We call it the meridian of e. Let A be a finite abelian group isomorphic to $\mathbb{Z}_2^d$ for some positive integer d, and let $\pi\colon H_1(M-\Gamma) \longrightarrow A$ be an epimorphism such that $\pi(t_e) \neq 1$ for any edge e of Γ. Then Γ can be considered as a "colored" graph with color set $A - \{1\}$ satisfying the following condition: Let e_1, e_2, e_3 be edges of Γ with a common vertex, and let a_1, a_2, a_3 be their colors; then $a_1a_2a_3 = 1$. Let $M_\pi(\Gamma)$ be the covering of M branched over Γ determined by π. Then $M_\pi(\Gamma)$ is a 3-manifold. In fact, an abelian covering of a 3-manifold branched over a graph is again a 3-manifold, if and only if each of the vertices of the graph has valency 2 or 3, and the monodromy group is $\mathbb{Z}_2^d$ for some $d > 0$. This is the reason why we restrict our attention to the situation described in the above. As in Section 9, let $\mathcal{C}_A$ be the set of all subgroups B of A such that A/B is cyclic. Put $\mathcal{C}_A^* = \mathcal{C}_A - \{A\}$. For an element B of $\mathcal{C}_A^*$, let Γ_B be the subgraph of Γ consisting of those edges e such that $\pi(t_e) \neq 1$ in $A/B \cong \mathbb{Z}_2$. Then Γ_B is a link in M, and the double cover $M_2(\Gamma_B)$ of M branched over Γ_B is homeomorphic to $M_\pi(\Gamma)/B$.

THEOREM 14.1. *(1)* $H_1\big(M_\pi(\Gamma);\mathbb{C}\big) \cong \oplus_{B\in\mathcal{C}_A^*} H_1\big(M_2(\Gamma_B);\mathbb{C}\big)$.
(2) For any odd prime p,

$$H_1\big(M_\pi(\Gamma)\big)^{(p)} \cong \bigoplus_{B\in\mathcal{C}_A^*} H_1\big(M_2(\Gamma_B)\big)^{(p)}.$$

For special cases, we can also determine the 2-torsion parts, and obtain the following:

THEOREM 14.2. *(1) Suppose $A \cong \mathbb{Z}_2^2$, and Γ_B is connected for any $B \in \mathcal{C}_A^*$. Then*

$$H_1\big(M_\pi(\Gamma)\big) \cong \Big(\bigoplus_{B\in\mathcal{C}_A^*} H_1\big(M_2(\Gamma_B)\big)\Big) \oplus \mathbb{Z}_2^{\nu/3-1},$$

where ν is the number of edges of Γ.
(2) Suppose $A \cong \mathbb{Z}_2^3$ and Γ is the complete graph with 4 vertices. Then

$$H_1\big(M_\pi(\Gamma)\big) \cong \bigoplus_{B\in\mathcal{C}_A^*} H_1\big(M_2(\Gamma_B)\big).$$

EXAMPLE 14.3. Let Γ_d = (a circle) $\cup$ (d arcs) be the colored graph as illustrated in Figure 14.1. Then it satisfies the assumption of Theorem 14.2(1). Hence,

$$H_1\big(M_\pi(\Gamma_d)\big) \cong \Big(\bigoplus_{B\in\mathcal{C}_A^*} H_1\Big(M_2\big((\Gamma_d)_B\big)\Big)\Big) \oplus \mathbb{Z}_2^{d-1}.$$

The cases where $d = 1$ and 2 correspond to the results of [Nk1] and [Nk2] respectively.

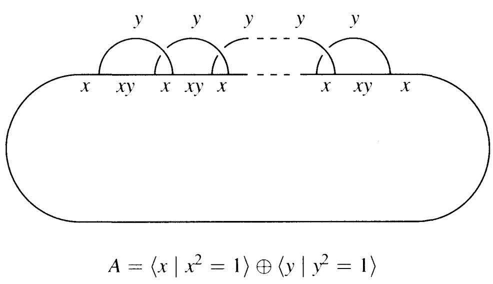

FIGURE 14.1

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15. Proof of Theorems 14.1 and 14.2. Note that the generator of the covering transformation group of each of the double coverings $M_2(\Gamma_B) \longrightarrow M$ acts on $H_1\big(M_2(\Gamma_B)\big)$ as multiplication by -1 (see [Br, pp. 118–120]). Theorem 14.1(1) and (2) can be proved by using this fact and the arguments in Steps 0 and 1 in the proofs of Theorems 1.1 and 8.1. Theorem 14.2(2) follows from Theorem 14.1(2) and the fact that $M_\pi(\Gamma)$ is a $\mathbb{Z}_2$-homology 3-sphere when Γ and π satisfy the assumption of Theorem 14.2(2); this fact follows from Proposition 15.5, which is proved later.

In the following we prove Theorem 14.2(1). To do this, we need to refine the arguments in Section 9. This refinement is based on the following observation: Let $A \cong \mathbb{Z}_2^d$, B an element of $\mathcal{C}_A^*$, and a an element of A which is not contained in B; then $\mathcal{E}_B = (1-a)\operatorname{Tr} B$. The following can be proved by direct calculation.

LEMMA 15.1. *(1) For $B, B' \in \mathcal{C}_A^*$, we have*

$$\operatorname{Tr} B \cdot \operatorname{Tr} B' = \begin{cases} 2^{d-1}\operatorname{Tr} B & \text{if } B = B', \\ 2^{d-2}\operatorname{Tr} A & \text{if } B \neq B'. \end{cases}$$

(2) $\sum_{B\in \mathcal{C}_A} \operatorname{Tr} B = 2^{d-1} + 2^{d-1}\operatorname{Tr} A$.

Since $H_1\big(M_\pi(L)/A\big) \cong H_1(M) \cong 0$, we see $\operatorname{Tr} A = 0$ as an endmorphism of $H_1\big(M_\pi(L)\big)$ (see [Br, pp. 118–120]). Thus from the above lemma, we have

LEMMA 15.2. *In* $\operatorname{End}\Big(H_1\big(M_\pi(L)\big)\Big)$, *we have the following identities;*
(1) For $B, B' \in \mathcal{C}_A^*$,

$$\operatorname{Tr} B \cdot \operatorname{Tr} B' = \begin{cases} 2^{d-1}\operatorname{Tr} & \text{if } B = B', \\ 0 & \text{if } B \neq B'. \end{cases}$$

(2) $\sum_{B\in \mathcal{C}_A^*} \operatorname{Tr} B = 2^{d-1}$.

Suppose Γ_B is connected for any $B \in \mathcal{C}_A^*$. Then $M_2(\Gamma_B)$ is a $\mathbb{Z}_2$-homology 3-sphere (*cf.* Sublemma 15.4), and therefore we have the following by the argument of [Br, pp. 118–120];

$$H_1\big(M_2(\Gamma_B)\big) \cong (\operatorname{Tr} B) H_1\big(M_\pi(\Gamma)\big).$$

Hence, by using Lemma 15.2, and noting the fact that the above groups are of odd order, we can see

$$\Big(\sum_{B\in \mathcal{C}_A^*} \operatorname{Tr} B\Big) H_1\big(M_\pi(\Gamma)\big) \cong \bigoplus_{B\in \mathcal{C}_A^*} (\operatorname{Tr} B) H_1\big(M_\pi(\Gamma)\big) \cong \bigoplus_{B\in \mathcal{C}_A^*} H_1\big(M_2(\Gamma_B)\big).$$

On the other hand, by Lemma 15.2(2),

$$\frac{H_1\big(M_\pi(\Gamma)\big)}{(\sum_{B\in \mathcal{C}_A^*} \operatorname{Tr} B) H_1\big(M_\pi(\Gamma)\big)} \cong \frac{H_1\big(M_\pi(\Gamma)\big)}{2^{d-1} H_1\big(M_\pi(\Gamma)\big)}.$$

Hence, we obtain the following short exact sequence:

$$0 \longrightarrow \bigoplus_{B\in \mathcal{C}_A^*} H_1\big(M_2(\Gamma_B)\big) \longrightarrow H_1\big(M_\pi(\Gamma)\big) \longrightarrow \frac{H_1\big(M_\pi(\Gamma)\big)}{2^{d-1} H_1\big(M_\pi(\Gamma)\big)} \longrightarrow 0.$$

In this short exact sequence, the first term has an odd order and the last term has an even order. Thus $H_1\big(M_\pi(\Gamma)\big)$ is the direct sum of these two terms. Suppose, further, $d = 2$. Then the last term is isomorphic to $H_1\big(M_\pi(\Gamma); \mathbb{Z}_2\big)$. Thus, Theorem 14.2(1) follows from the following lemma.

LEMMA 15.3. *Suppose* Γ, π, *and* A *satisfy the condition of Theorem 14.2(1). Then*

$$H_1\big(M_\pi(\Gamma); \mathbb{Z}_2\big) \cong \mathbb{Z}_2^{\nu/3-1},$$

where ν *is as in Theorem 14.2(1).*

To prove this we need the following fact, which is probably well-known.

SUBLEMMA 15.4. *Let* M *be a* $\mathbb{Z}_2$*-homology* 3*-sphere, and* L *a* μ*-component link in* M. *Let* $M_2(L)$ *be the double cover of* M *branched over* L. *Then*

$$H_1\big(M_2(L); \mathbb{Z}_2\big) \cong \mathbb{Z}_2^{\mu-1}.$$

PROOF. Since M is a $\mathbb{Z}_2$-homology sphere, there is a compact surface V in M with $\partial V = L$. As in [GL, Section 2], we define a bilinear form $\mathcal{G}_V \colon H_1(V; \mathbb{Z}_2) \times H_1(V; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ as follows: Suppose $\alpha, \beta \in H_1(V; \mathbb{Z}_2)$ are represented by 1-cycles a, b. Then $2b$ can be pushed off V into $M - V$, obtaining $\tilde{b}$, say. Define $\mathcal{G}_V(\alpha, \beta)$ to be the $\mathbb{Z}_2$-linking number of a and $\tilde{b}$. Let $\{a_1, \ldots, a_n\}$ be a basis of $H_1(V; \mathbb{Z}_2)$, and let G be the matrix $\big(\mathcal{G}_V(a_i, a_j)\big)$. Then as in [Ro, p. 212] (*cf.* [GL, Section 3]), we see G is a relation matrix of $H_1\big(M_2(L); \mathbb{Z}_2\big)$. On the other hand, since $\mathcal{G}_V$ is equal to the $\mathbb{Z}_2$-intersection pairing on $H_1(V; \mathbb{Z}_2)$ (*cf.* [Ro, p. 202]), we see the nullity of G is equal to $\mu - 1$. Hence, we obtain the desired result.

PROOF OF LEMMA 15.3. Choose an element B of $\mathcal{C}_A^*$. Then we have the following tower of double branched coverings;

$$M_\pi(\Gamma) \xrightarrow{p_2} M_2(\Gamma_B) \xrightarrow{p_1} M.$$

Here the branch set for p_2 is $p_1^{-1}(\Gamma_B^c)$, where $\Gamma_B^c = cl(\Gamma - \Gamma_B)$. Since Γ_B is connected by the assumption, $M_2(\Gamma_B)$ is a $\mathbb{Z}_2$-homology 3-sphere by Sublemma 15.4. Each component of Γ_B^c is an arc with endpoints in Γ_B. [Proof: Since Γ_B^c has the single color b where $\{b\} = B - \{1\}$, each component of Γ_B^c is an arc or a circle. If there is a circle component, then some constituent link $\Gamma_{B'}$ ($B' \in \mathcal{C}_A^*$) is disconnected, contradicting the assumption.] Let $\nu(B)$ be the number of the components of Γ_B^c. Then $p_1^{-1}(\Gamma_B^c)$ is a link of $\nu(B)$-components, and therefore $H_1\big(M_\pi(\Gamma); \mathbb{Z}_2\big) \cong \mathbb{Z}_2^{\nu(B)-1}$ by Sublemma 14.4. Noting that $\nu(B)$ is equal to the number of edges of Γ whose colors are equal to b, we have $\nu(B_1)+\nu(B_2)+\nu(B_3) = \nu$, where $\mathcal{C}_A^* = \{B_1, B_2, B_3\}$. Since all $\nu(B_i)$'s are equal, $\nu(B) = \nu/3$.

Finally, we present a proposition, which completes the proof of Theorem 14.2(2), and justifies its condition.

PROPOSITION 15.5. *Let M be a $\mathbb{Z}_2$-homology 3-sphere, and let $\Gamma, \pi, A \cong \mathbb{Z}_2^d$, and $M_\pi(\Gamma)$ be as described in the beginning of Section 14. Then $M_\pi(\Gamma)$ is again a $\mathbb{Z}_2$-homology 3-sphere, if and only if one of the following conditions is satisfied;*

(1) $d = 1$ and Γ is a knot,

(2) $d = 2$ and Γ is a 2-component link with linking number 1 $\pmod 2$,

(3) $d = 2$ and Γ is a θ-curve,

(4) $d = 3$ and Γ is the complete graph with 4 vertices.

PROOF. By repeatedly using Sublemma 15.4, we can prove the "if" part. So, we prove the "only if" part. In case $d = 1$, the proposition is a special case of Sublemma 15.4. So, we assume $d \geq 2$. Choose an element B of $\mathcal{C}_A^*$, and identify A with $B \oplus \mathbb{Z}_2$. Let $\rho: A \cong B \oplus \mathbb{Z}_2 \longrightarrow B$ be the projection, and let Γ_0 be the subgraph of Γ consisting of those edges e such that $\rho\pi(t_e) \neq 1$. Then we obtain the following tower of coverings;

$$M_\pi(\Gamma) \xrightarrow{p_2} M_{\rho\pi}(\Gamma_0) \xrightarrow{p_1} M.$$

Here p_2 is a double covering whose branch set is the link $p_1^{-1}(\Gamma_0^c)$, where $\Gamma_0^c = \mathrm{cl}(\Gamma - \Gamma_0)$. Since p_2 induces an epimorphism between the fundamental groups, we see $M_\pi(\Gamma)$ is a $\mathbb{Z}_2$-homology sphere, if and only if $M_{\rho\pi}(\Gamma_0)$ is a $\mathbb{Z}_2$-homology sphere and $p_1^{-1}(\Gamma_0^c)$ is a knot. Suppose these conditions hold. Then Γ_0^c is connected, and therefore it is either a circle or an arc.

CASE 1. $d = 2$: Then Γ_0 is a knot by Sublemma 15.4. If Γ_0^c is a circle, then $\mathrm{lk}(\Gamma_0, \Gamma_0^c) \equiv 1 \pmod 2$, since $p_1^{-1}(\Gamma_0^c)$ is a knot. Thus (2) is satisfied. If Γ_0^c is an an arc, then (3) is satisfied.

CASE 2. $d = 3$: Then, by the above, Γ_0 satisfies (2) or (3).

SUBCASE (I). Γ_0 satisfies (2); *i.e.* Γ_0 is a 2-component link $K_1 \cup K_2$ with $\mathrm{lk}(K_1, K_2) \equiv 1 \pmod 2$: Note that the covering projection p_1 is the composite of the following covering projections

$$M_{\rho\pi}(\Gamma_0) \xrightarrow{p_1''} M_2(K_1) \xrightarrow{p_1'} M,$$

where the branch set for p_1'' is the inverse image $\tilde{K_2}$ of K_2 in $M_2(K_1)$. If Γ_0^c is a circle, then the inverse image $\tilde{\Gamma_0^c}$ of Γ_0^c in $M_2(K_1)$ satisfies $\mathrm{lk}(\tilde{\Gamma_0^c}, \tilde{K}_2) \equiv 2\,\mathrm{lk}(\Gamma_0, K_2) \equiv 0 \pmod 2$, and therefore $p_1^{-1}(\Gamma_0^c) = (p_1'')^{-1}(\tilde{\Gamma}_0^c)$ is a 2-component link. Thus Γ_0^c must be an arc. If Γ_0^c joins K_1 and K_2, then the meridian t of Γ_0^c is null-homologous, and therefore $\pi(t) = 1$. If $\partial\Gamma_0^c$ is contained in either K_1 or K_2, then Γ is the disjoint union of a θ-curve and a circle. This situation is treated in the next subcase, where it is concluded that this does not occure. Hence this subcase does not occur.

SUBCASE (II). Γ_0 satisfies (3); *i.e.*, Γ_0 is a θ-curve: Suppose Γ_0^c is a circle. Then we can see $\sum_{i=1}^3 \mathrm{lk}(\Gamma_0^c, K_i) \equiv 0 \pmod 2$, where K_1, K_2, and K_3 are the constituent knots of Γ_0. [This following from the fact that $\sum_{i=1}^3 [K_i] \equiv 0$ in $H_1(\Gamma_0; \mathbb{Z}_2)$.] Thus $\mathrm{lk}(\Gamma_0^c, K_i) \equiv 0 \pmod 2$ for some K_i, and therefore we see $p_1^{-1}(\Gamma_0^c)$ is not connected. Hence Γ_0^c must be an arc. Suppose $\partial\Gamma_0^c$ lies in an edge of Γ_0. Then there is a constituent knot K_0 of Γ_0 which is disjoint from Γ_0^c. The covering p_1 is the composite of the following covering projections;

$$M_{\rho\pi}(\Gamma_0) \xrightarrow{p_1''} M_2(K_0) \xrightarrow{p_1'} M.$$

We see $(p_1')^{-1}(\Gamma_0^c)$ is a union of two arcs; so $p_1^{-1}(\Gamma_0^c)$ is a union of two loops, a contradiction. Hence Γ_0^c joins different edges of Γ_0, and therefore Γ is the complete graph with 4 vertices.

CASE 3. $d = 4$: Then Γ_0 is the complete graph with 4 vertices. If Γ_0^c is a circle, then as in the argument in the first half of Subcase (ii) of Case 2, we can find a constituent knot K_i of Γ_0^c such that $\mathrm{lk}(\Gamma_0^c, K_i) \equiv 0 \pmod 2$, and hence $p^{-1}(\Gamma_0^c)$ is not connected. Thus Γ_0^c is an arc. Then, by an argument similar to that in the latter half of Subcase (ii) of Case 2, we see this case does not occur.

CASE 4. $d \geq 5$: Then we have the following tower of coverings

$$M_\pi(\Gamma) \xrightarrow{p_2'} M_{\pi'}(\Gamma') \xrightarrow{p_1'} M,$$

where p_1' is a $\mathbb{Z}_2^4$-branched covering, and p_2' is a iteration of double branched coverings. By Case 3, $M_{\pi'}(\Gamma')$ is not a $\mathbb{Z}_2$-homology sphere. Since p_2' induces an epimorphism between the fundamental groups, $M_\pi(\Gamma)$ is not a $\mathbb{Z}_2$-homology sphere; so this case does not occur. Now the proof of Proposition 15.5 is complete.

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Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560
Japan
e-mail: sakuma@math.wani.osaka-u.ac.jp