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Ruled Exceptional Surfaces and the Poles of Motivic Zeta Functions

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Abstract. In this paper we study ruled surfaces which appear as exceptional surface in a succession of blowing-ups. In particular we prove that the e-invariant of such a ruled exceptional surface E is strictly positive whenever its intersection with the other exceptional surfaces does not contain a fiber (of E). This fact immediately enables us to resolve an open problem concerning an intersection configuration on such a ruled exceptional surface consisting of three nonintersecting sections. In the second part of the paper we apply the non-vanishing of e to the study of the poles of the well-known topological, Hodge and motivic zeta functions.

1 Introduction and Preliminaries

1.1 Let $f \in \mathbb{C}[x_1, \ldots, x_n] \setminus \mathbb{C}$ with $f(0) = 0$. One can associate with f the *Hodge zeta function* $Z_{\mathrm{Hod}}(f, s)$, which is a geometric invariant depending in particular on the singularities on the surface $f^{-1}\{0\}$ in $\mathbb{A}^n_{\mathbb{C}}$. It can be defined in terms of an embedded resolution $h\colon X \to \mathbb{A}^n_{\mathbb{C}}$ of $f^{-1}\{0\}$ in $\mathbb{A}^n_{\mathbb{C}}$ as follows. Denote by E_i, $i \in T$, the irreducible components of $h^{-1}(f^{-1}\{0\})$. For each $i \in T$ we denote by N_i and $\nu_i - 1$ the multiplicities of E_i in the divisor on X of $f \circ h$ and $h^*(dx_1 \wedge \cdots \wedge dx_n)$, respectively. For $i \in T$ and $I \subseteq T$ we put $E_i^\circ := E_i \setminus (\bigcup_{j \neq i} E_j)$, $E_I := \bigcap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus (\bigcup_{j \notin I} E_j)$. In particular, when $I = \varnothing$ we have $E_\varnothing = X$. Remark that X is the disjoint union of the E_I°. Then the formula of $Z_{\mathrm{Hod}}(f, s)$ in terms of the embedded resolution h is

$$Z_{\mathrm{Hod}}(f, s) := \sum_{I \subseteq T} H(E_I^\circ \cap h^{-1}\{0\}) \prod_{i \in I} \frac{uv - 1}{(uv)^{\nu_i + sN_i} - 1} \in \mathbb{Q}(u, v)(S), \tag{1.1}$$

where $H(\,\cdot\,)$ denotes the Hodge polynomial and where we consider $(uv)^{-s}$ as a variable S. By taking Euler characteristics instead of Hodge polynomials, this function specializes to the topological zeta function $Z_{\mathrm{top}}(f, s)$, and, on the other hand, the function itself is a specialization of the well-known motivic zeta function $Z_{\mathrm{mot}}(f, s)$. In terms of the embedded resolution h we have

$$Z_{\mathrm{top}}(f, s) = \sum_{I \subseteq T} \chi(E_I^\circ \cap h^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i} \in \mathbb{Q}(s),$$

where $\chi(\,\cdot\,)$ denotes the topological Euler characteristic, and

$$sZ_{\mathrm{mot}}(f, s) = \sum_{I \subseteq T} [E_I^\circ \cap h^{-1}\{0\}] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i + sN_i} - 1} \in \mathcal{M}_{\mathbb{C}}[[S]],$$

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where we consider $\mathbb{L}^{-s}$ as a variable S and where $\mathcal{M}_{\mathbb{C}}$ denotes the ring obtained from the Grothendieck ring of complex varieties by inverting the affine line. Here $[\,\cdot\,]$ denotes the class in $\mathcal{M}_{\mathbb{C}}$ and $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$.

It is interesting to note that the motivic zeta function was originally introduced in by Denef and Loeser [3] as a power series over $\mathcal{M}_{\mathbb{C}}$. They showed that it is in fact a rational function by proving the above formula in terms of an embedded resolution [3, Theorem 2.2.1]. Of course, this immediately implies that the above formulas are independent of the chosen resolution h.

1.2 We are especially interested in the poles of these functions. Of course, for $Z_{\mathrm{top}}(f,s)$ the notion of a pole is clear. For $Z_{\mathrm{Hod}}(f,s)$, we say that a rational number q is a pole if $(uv)^{-q}$ is, considering $Z_{\mathrm{Hod}}(f,s)$ as a rational function in the variable $S = (uv)^{-s}$. For the definition of a pole of $Z_{\mathrm{mot}}(f,s)$, we refer the reader to [7, Definition 4.7]. Obviously the possible rational poles are all of the form $-\nu_i/N_i$ for some $i \in T$. In this context, we say that $-\nu_i/N_i$ is the candidate pole induced by E_i. One can, for example, easily see that the absolute value $\min_{i\in T}\{\nu_i/N_i\}$ of the largest candidate pole, which is in fact just the *log canonical threshold* of f at 0, really is a pole of $Z_{\mathrm{Hod}}(f,s)$. The whole set $\{-\nu_i/N_i \mid i \in T\}$ is, of course, not an invariant of f, but its subset consisting of the poles of $Z_{\mathrm{Hod}}(f,s)$ is. Philosophically, the poles of these zeta functions are induced by "important" components E_i, which occur in every resolution. So we search for geometric conditions on the components E_i telling us whether or not the corresponding candidate pole $-\nu_i/N_i$ is a pole.

1.3 When the number n of variables is 2, Veys has already given a complete geometric determination of the poles [10, Theorem 4.3]. For $n \geq 3$, one may consider for the moment the contribution $\mathcal{R}$ to the residue of $Z_{\mathrm{Hod}}(f,s)$ at $-\nu_i/N_i$ of only one component E_i, which then is supposed not to intersect any other component E_j with $-\nu_i/N_i = -\nu_j/N_j$. Of course, by the contribution of E_i to the residue of $Z_{\mathrm{Hod}}(f,s)$ at $-\nu_i/N_i$ we mean the residue at $-\nu_i/N_i$ of the function obtained from $Z_{\mathrm{Hod}}(f,s)$ by restricting its defining expression, see (1.1), to the terms which correspond to the subsets I of T containing i. Note that by the above condition on the intersecting components, the order of the candidate pole $-\nu_i/N_i$ of this restricted sum is at most 1. One easily obtains the following explicit formula for $\mathcal{R}$:

$$\mathcal{R} = \frac{1-uv}{N_i(uv)^{-\nu_i/N_i}}\Big(\sum_{i\in I\subseteq T} H(E_I^\circ \cap h^{-1}\{0\}) \prod_{j\in I\setminus\{i\}} \frac{uv-1}{(uv)^{\nu_j-(\nu_i/N_i)N_j}-1}\Big).$$

We will denote the occurring expressions $\nu_j - (\nu_i/N_i)N_j$ by $\alpha_j^{(i)}$.

1.4 Henceforth, let us suppose that $n = 3$, that $h(E_i) = \{0\}$ and that $\alpha_j^{(i)} \notin \mathbb{Z}$ for all $j \in T$ with $i \neq j$ and $E_i \cap E_j \neq \varnothing$. Veys proved the following result, which he actually stated for Igusa's local zeta function, and hence for the p-adic analogue of $\mathcal{R}$, but which is easily seen to remain true for the zeta function $Z_{\mathrm{Hod}}(f,s)$.

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Theorem 1.1 ([9, Theorem 5.9]) *Suppose that the surface E_i is created in the resolution process as the exceptional surface E of a blowing-up at a nonsingular projective curve D, and suppose that the strict transform of $f^{-1}\{0\}$ before this blowing-up is nonsingular. Denote by D_j, $j \in A$, the irreducible components of the intersection of E with the other components of the inverse image of $f^{-1}\{0\}$ (at the stage where E is just created). Then we have $\mathcal{R} = 0$, except for the situation when the genus of D is 1 and $\bigcup_{j\in A} D_j$ consists of three nonintersecting sections of the ruled surface E.*

Moreover, Veys proved [9, Proposition 5.12] that when the origin of $\mathbb{A}^3_{\mathbb{C}}$ is an absolutely isolated singularity of $f^{-1}\{0\}$, *i.e.,* an isolated singularity which is resolvable by only performing point-centered blowing-ups, the exceptional situation of Theorem 1.1 cannot occur. But, of course, we would like the exceptional situation also to be impossible for more general singularities.

Secondly, consider the following special case of a result of a previous paper.

Theorem 1.2 ([6, Theorem2.3]) *If $\chi(E_i^\circ) \neq 0$, then $\mathcal{R} \neq 0$.*

What about the reverse implication? Recall the monodromy conjecture, stating that *if s_0 is a pole of $Z_{\mathrm{Hod}}(f,s)$, then $e^{2\pi i s_\circ}$ is an eigenvalue of the local monodromy of f at some point of $f^{-1}\{0\}$*, and combine it with the formula of A'Campo [1, Theorem 3]. This formula says that the alternating product of all characteristic polynomials of the local monodromy of f at 0 is

$$\prod_{j\in T}(1-t^{N_j})^{-\chi(E_j^\circ\cap h^{-1}\{0\})},$$

making the reverse implication at least plausible. Denef and Jacobs even conjecture this implication, but in the p-adic setting [2, Conjecture 1.2.2]. Veys already proved some partial results about it [9]. It is interesting to note that when E_i is again created as a ruled surface E over a nonsingular projective curve D of genus 1 and when the intersection configuration $\bigcup_{j\in A} D_j$ on E consists of $k \geq 3$ nonintersecting sections of E, the reverse implication seems to fail at first sight. Indeed, $\chi(E_i^\circ) = 0$ in this case, but there is no obvious reason for the residue to be zero.

Furthermore, it is intriguing that one can easily see that such an intersection configuration consisting of $k \geq 3$ nonintersecting sections is in fact impossible when E is a ruled surface over a curve D of any genus g different from 1. Indeed, since $k \geq 3$, we know[8, Example 7.2] that in E all the self-intersection numbers of the D_j are zero. Hence we obtain by [8, Example 6.6(c)] that $0 = 2g - 2$, providing a contradiction when $g \neq 1$.

1.5 In this paper we consider ruled surfaces which are created as exceptional surfaces of a blowing-up in a resolution process. In particular we study their e-invariant (see [4, §V.2] for the definition of this integer.) An immediate consequence of our first main result is the fact that *the e-invariant of a ruled exceptional surface E in a resolution process is (strictly) positive whenever the intersection configuration $\bigcup_{j\in A} D_j$ on E does not contain a fiber.* This result will immediately imply that the intersection configuration on such a ruled surface in a resolution process, no matter what the genus is of the curve over which it is a ruled surface, can never consist of three disjoint curves, clarifying the two problem situations sketched out above.

In the second part of the paper, we apply this result to the study of the poles of the zeta functions $Z_{\text{top}}(f,s), Z_{\text{Hod}}(f,s)$ and $Z_{\text{mot}}(f,s)$. As a contribution to a geometric determination of their poles, we prove the reverse implication of Theorem 1.2 in the following case.

Theorem 1.3 *Suppose that the intersection configuration $\bigcup_{j\in A} D_j$ on E is not connected. Then, except for the situation described below, we have*

$$\chi\big(E \setminus \bigcup_{j\in A} D_j\big) = 0 \;\Rightarrow\; \mathcal{R} = 0,$$

where $\mathcal{R}$ stands for the contribution of E_i to the residue at $-\nu_i/N_i$ of any of the three zeta functions mentioned above.

We are in the exceptional situation when E is rational, $e \geq 3$ and $\bigcup_{j\in A} D_j$ consists of precisely two connected components, one of them consisting only of a section and the other one containing at least one singular curve.

Veys [13] studied the case of a connected intersection configuration.

As a part of the proof of our result we find, under the conditions of the theorem, all the possible intersection configurations $\bigcup_{j\in A} D_j$ on a rational ruled exceptional surface with all the curves D_j nonsingular and with $\chi(E \setminus \bigcup_{j\in A} D_j) = 0$. It turns out that there are only two possible configurations, both of them consisting only of sections. Except for a section which is disjoint from all the other curves D_j, say D_0, the two possibilities for the intersection configuration have the following form. Either there exist two points on E, say P and Q, such that $D_j \cap D_k = \{P, Q\}$ for $j,k \in A\setminus\{0\}$ with $j \neq k$ (Figure 1) or there exist $|A|-1$ points, say P_j for $j \in A\setminus\{0\}$, on one of the sections D_j, say D_{j_1}, such that $D_{j_1} \cap D_j = \{P_{j_1}, P_j\}$ for $j \in A \setminus \{0, j_1\}$ and $D_j \cap D_k = \{P_{j_1}\}$ for $j,k \in A \setminus \{0, j_1\}$ with $j \neq k$ (Figure 2).

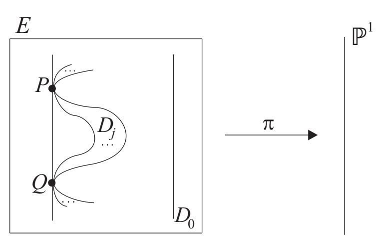

Figure 1

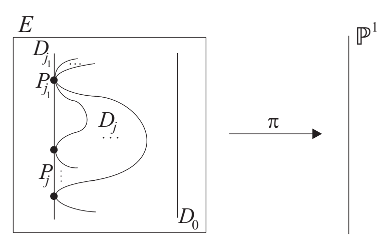

Figure 2

1.6 Embedded Resolutions

Let $f \in \mathbb{C}[x_1,\dots,x_n] \setminus \mathbb{C}$. By Hironaka's Main Theorem II [5, p. 142] we know that we can construct an embedded resolution $h\colon X \to \mathbb{A}^n_{\mathbb{C}}$ of $f^{-1}\{0\}$ in $\mathbb{A}^n_{\mathbb{C}}$ by means of blowing-ups. We explain what we mean by this.

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Set $X_0 = \mathbb{A}^n_{\mathbb{C}}$ and $Y^{[0]} = f^{-1}\{0\}$. Then Hironaka's result says that we can find a finite succession of blowing-ups $\pi_i \colon X_{i+1} \to X_i$, for $i = 0, \dots, r-1$, with irreducible nonsingular center $D_i \subset X_i$ and exceptional variety $E^{[i+1]}_{i+1} \subset X_{i+1}$ subject to the following conditions. Let $E^{[i+1]}_j$ and $Y^{[i+1]}$ denote the strict transform of $E^{[i]}_j$ and $Y^{[i]}$, respectively, in X_{i+1} by π_i for $j = 1, \dots, i$. Then

(i) $X = X_r$ and $h = \pi_0 \circ \cdots \circ \pi_{r-1}$;

(ii) for $i = 0, \dots, r-1$ we have $D_i \subset Y^{[i]}$ and $\operatorname{codim}(D_i, X_i) \geq 2$;

(iii) $\bigcup_{1 \leq j \leq i} E^{[i]}_j$ is a normal crossings divisor and it has only normal crossings with D_i for $i = 1, \dots, r-1$;

(iv) $(\bigcup_{1 \leq j \leq r} E^{[r]}_j) \cup Y^{[r]} = h^{-1}(f^{-1}\{0\})$ is a normal crossings divisor.

From now on by *embedded resolution* we will mean an embedded resolution which is constructed as a succession of blowing-ups satisfying the above conditions.

Let $n = 3$ and fix an exceptional surface $E^{[m]}_m$, for some $m \in \{1, \dots, r\}$. Of course, the center D_{m-1} can only be a point or a nonsingular curve. We are especially interested in the latter case, in which we know by [4, Theorem II.8.24] that $E^{[m]}_m$, together with the induced projection map $\pi = \pi_{m-1}|_{E^{[m]}_m} \colon E^{[m]}_m \to D_{m-1}$, is a ruled surface.

1.7 Ruled Surfaces

For a projective ruled surface E we denote by e and C_0 the invariant and the section, respectively, introduced in [4, Section V.2]. In particular, $C_0^2 = -e$. When more than one ruled surface comes into the picture at the same time, we use the notations $C_0(E)$ and $e(E)$ to avoid confusion. Moreover, by f, or sometimes $f(E)$, we denote any fiber of the ruled surface E. When we use the symbols e, f or C_0 throughout this paper we will implicitly assume that the ruled surface E with which we are dealing is projective.

The following result will be used frequently.

Proposition 1.4 ([4, Propositions V.2.20 andV.2.21]) *Let E be a ruled surface and let D be an irreducible curve on E, different from C_0 and f. Write $D \equiv aC_0 + bf$ with $a, b \in \mathbb{Z}$.*

(i) *If $e \geq 0$, then $a > 0$, $b \geq ae$.*

(ii) *If $e < 0$, then either $a = 1, b \geq 0$ or $a \geq 2, b \geq \frac{1}{2}ae$.*

2 The Positivity of the e-Invariant

Let X and Y be two 3-folds and let $h \colon Y \to X$ be a composition of blowing-ups. Let E be the exceptional surface of the last blowing-up in this succession. Suppose that E is a ruled surface and that $h(E)$ is a point. We will show in this section that the extra condition saying that the intersection of E and the exceptional locus minus E contains no fiber, gives us a lot of information about the surface E and the possible ways it can be created. More concretely, we will first find a restriction on (the position of) the centers of the successive blowing-ups, caused by the extra condition above. This is Theorem 2.1. Secondly, from this restricting fact we will be able to deduce that the e-invariant of E is (strictly) positive in this case.

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2.1 Let us fix the data for the theorem providing a restriction on the centers. Let X and Y be 3-folds and put $X_0 = X$. Let $h\colon Y \to X$ be a finite succession of blowing-ups $\pi_i\colon X_{i+1} \to X_i$ with irreducible nonsingular center $D_i \subset X_i$ and exceptional variety $E_{i+1}^{[i+1]} \subset X_{i+1}$. Suppose that $h = \pi_0 \circ \cdots \circ \pi_{m-1}$, such that $Y = X_m$, and put $E := E_m^{[m]}$. We denote by $E_j^{[i+1]}$ the strict transform of $E_j^{[i]}$ in X_{i+1} under π_i for $j = 1, \ldots, i$. Now we can state the following theorem.

Theorem 2.1 *Suppose that D_{m-1} is a curve (hence E is a ruled surface over D_{m-1}) and that $h(E)$ is a point. If the intersection $E \cap (\bigcup_{1\leq j\leq m-1} E_j^{[m]})$ does not contain any fiber of the ruled surface E, then there exists a number $k \in \{1, \ldots, m-2\}$ such that, starting from X_k, the part of the succession of blowing-ups being relevant for the creation of E,* i.e., *the composition of those π_i, $i = k, \ldots, m-1$, for which $D_i \cap (\pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_{m-1})(E) \neq \varnothing$, satisfies the following. We may as well assume that all π_i, $i = k, \ldots, m-1$ are relevant. All the centers D_i, $i = k, \ldots, m-1$ are (nonsingular) projective curves. The first center,* i.e., *D_k, is a fiber of the ruled surface $E_k^{[k]}$ when D_{k-1} is a curve, and it is an arbitrary nonsingular projective curve in the projective plane $E_k^{[k]} \cong \mathbb{P}^2_{\mathbb{C}}$ when D_{k-1} is a point. Each of the following centers D_i, $i = k+1, \ldots, m-1$ is contained in $E_i^{[i]}$, is not a fiber of $E_i^{[i]}$ and is either disjoint from or contained in $\bigcup_{k\leq j\leq i-1} E_j^{[i]}$.*

Remark It is easily seen that the intersection $E_i^{[i]} \cap (\bigcup_{k\leq j\leq i-1} E_j^{[i]})$ then precisely consists of one or two disjoint sections. So, for $i = k+1, \ldots, m-1$ the center D_i is either one of these sections or a curve disjoint from them.

Proof We take for k the largest integer in $\{1, \ldots, m-2\}$ such that $(\pi_{k-1} \circ \pi_k \circ \cdots \circ \pi_{m-1})(E)$ is a point. This number certainly exists by our assumption that $h(E)$ is a point. By the definition of k we know that $C := (\pi_k \circ \cdots \circ \pi_{m-1})(E)$ is a curve. So, either D_{k-1} is a curve and C is a fiber of the ruled surface $E_k^{[k]}$, or D_{k-1} is a point and C is contained in $E_k^{[k]}$. Let us suppose from now on that all the blowing-ups π_i, $i = k, \ldots, m-1$, are relevant for the creation of E.

We will show that $C = D_k$. We already have that $C \cap D_k \neq \varnothing$, since π_k is supposed to be relevant for the creation of E. If $C \neq D_k$, then $E_{k+1}^{[k+1]} \cap \widetilde{C} \neq \varnothing$ consists of a finite number of points, where we denote by $\widetilde{C}$ the strict transform of C under π_k. Note moreover that $\widetilde{C} = (\pi_{k+1} \circ \cdots \circ \pi_{m-1})(E)$. By Lemma 2.2 we then obtain that there exists an exceptional surface $E_i^{[m-1]}$ such that $E_i^{[m-1]} \cap D_{m-1} \neq \varnothing$ consists of a finite number of points. Blowing up at D_{m-1} results in fibers on the ruled surface E, which is a contradiction. Hence $C = D_k$.

Since $\pi_k((\pi_{k+1} \circ \cdots \circ \pi_{m-1})(E)) = D_k$, it is clear that the curve $(\pi_{k+1} \circ \cdots \circ \pi_{m-1})(E)$ is contained in $E_{k+1}^{[k+1]}$ and that it is not a fiber of $E_{k+1}^{[k+1]}$. As above we find that

$$(\pi_{k+1} \circ \cdots \circ \pi_{m-1})(E) = D_{k+1}.$$

The fact that D_{k+1} is either disjoint from or contained in $\bigcup_{k\leq j\leq (k+1)-1} E_j^{[k+1]}$ is again an easy corollary of Lemma 2.2. Obviously the reader can finish this argument by induction. Note that this implies, for example, that $D_i = (\pi_i \circ \cdots \circ \pi_{m-1})(E)$ for each $i \in \{k, \ldots, m-1\}$. In particular, $(\pi_0 \circ \cdots \circ \pi_{i-1})(D_i) = h(E)$ is a point, yielding that the curve D_i is complete. ■

Lemma 2.2 *Using the notation of Section* 2.1*, suppose that D_{m-1} is a curve. Let $i \leq j$ be elements of $\{1, \ldots, m-1\}$ such that $C := (\pi_j \circ \cdots \circ \pi_{m-1})(E)$ is a curve and $E_i^{[j]} \cap C \neq \varnothing$ consists of a finite number of points. Then there exists an exceptional surface $E_l^{[m-1]}$ in X_{m-1} such that $E_l^{[m-1]} \cap D_{m-1} \neq \varnothing$ consists of a finite number of points.*

Proof Put $D := (\pi_{j+1} \circ \cdots \circ \pi_{m-1})(E)$. Obviously it is sufficient to show that there exists an exceptional surface $E_l^{[j+1]}$ in X_{j+1} such that $E_l^{[j+1]} \cap D \neq \varnothing$ consists of a finite number of points.

Let us first treat the case that $D_j \neq C$. If $D_j \cap C = \varnothing$, we take $l = i$. If $D_j \cap C \neq \varnothing$, we take $l = j+1$.

When $D_j = C$, the (irreducible) curve D is contained in $E_{j+1}^{[j+1]}$ and is not a fiber of it. Because $E_i^{[j]} \cap C \neq \varnothing$ consists of a finite number of points, we have that $E_i^{[j+1]} \cap E_{j+1}^{[j+1]} \neq \varnothing$ consists of a finite number of fibers of $E_{j+1}^{[j+1]}$. Hence we can take $l = i$. ■

Remark We will often use Theorem 2.1 in the case that h is an embedded resolution. Then the centers of the blowing-ups are chosen in such a way that for each i the exceptional locus $\bigcup_{1 \leq j \leq i} E_j^{[i]}$ of $\pi_0 \circ \cdots \circ \pi_{i-1}$ is a normal crossings divisor. In particular the surfaces $E_j^{[i]}$ are all nonsingular. In this case the center D_{k-1} in Theorem 2.1 is either a point or a curve which is not entirely mapped onto one point by $\pi_0 \circ \cdots \circ \pi_{k-2}$. Indeed, suppose that D_{k-1} is a curve and that $(\pi_0 \circ \cdots \circ \pi_{k-2})(D_{k-1})$ is a point. Then D_{k-1} belongs to the exceptional locus of $\pi_0 \circ \cdots \circ \pi_{k-2}$. Thus, there exists an exceptional surface $E_i^{[k-1]}$ for some $i \in \{1, \ldots, k-1\}$, such that $D_{k-1} \subseteq E_i^{[k-1]}$. By the smoothness of $E_i^{[k-1]}$ this implies that $E_k^{[k]} \cap E_i^{[k]}$ is exactly a section of the ruled surface $E_k^{[k]}$. Since we already know that D_k is a fiber of $E_k^{[k]}$, we see that $D_k \cap E_i^{[k]} \neq \varnothing$ consists of a finite number of points. As in the proof of the theorem, this leads to a contradiction.

Now let $f \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$ such that the origin 0 of $\mathbb{A}^3_{\mathbb{C}}$ is an isolated singularity of the surface $f^{-1}\{0\}$. Then it is reasonable to consider an embedded resolution $h \colon X \to \mathbb{A}^3_{\mathbb{C}}$ of $f^{-1}\{0\}$ in $\mathbb{A}^3_{\mathbb{C}}$ which satisfies the condition that $h(E_i) = \{0\}$ for any exceptional surface E_i of h with $h^{-1}\{0\} \cap E_i \neq \varnothing$. For the surface E of Theorem 2.1, we take an exceptional surface of this resolution map h and suppose that it is mapped onto the origin of $\mathbb{A}^3_{\mathbb{C}}$. Then obviously the center D_{k-1} of Theorem 2.1 also has to be mapped entirely onto 0, implying that in this situation of an isolated surface singularity (and an appropriate embedded resolution) the only possibility for D_{k-1} is a point.

Now we are ready to prove the positivity of the e-invariant of E. We will use the following lemma. Its proof is trivial using Proposition 1.4.

Lemma 2.3 *Let E be a ruled surface and let D be an irreducible curve on E with $D^2 < 0$. Then $e > 0$ and $D = C_0$.*

Theorem 2.4 *We use the notation of Section 2.1 and we suppose that D_{m-1} is a curve and that $h(E)$ is a point. Denote by $g(D_{m-1})$ the geometric genus of D_{m-1}. If the intersection $E \cap (\bigcup_{1\le j\le m-1} E_j^{[m]})$ does not contain any fiber of the ruled surface E, then*

(i) $2g(D_{m-1}) < 2 + e$;
(ii) $e > 0$;
(iii) $C_0 \subseteq E \cap \big(\bigcup_{k\le j\le m-1} E_j^{[m]}\big)$.

Proof Fix $k \in \{1, \dots, m-2\}$ satisfying Theorem 2.1. Let us assume again that all the π_i, $i = k, \dots, m-1$, are relevant for the creation of E. We will show by induction that for all $i \in \{k+1, \dots, m\}$ we have $2g(D_{i-1}) < 2 + e(E_i^{[i]})$, $e(E_i^{[i]}) > 0$ and $C_0(E_i^{[i]}) \subseteq E_i^{[i]} \cap (\bigcup_{k\le j\le i-1} E_j^{[i]})$.

Let us first treat the case $i = k+1$. We distinguish two possibilities: the center D_{k-1} is either a point or a curve. Suppose that D_{k-1} is a point. Then $E_k^{[k]} \cong \mathbb{P}^2_{\mathbb{C}}$ and D_k is a nonsingular projective curve in $\mathbb{P}^2_{\mathbb{C}}$, say of degree d. We must consider the exceptional surface $E_{k+1}^{[k+1]}$, which is a ruled surface over D_k and which intersects $E_k^{[k+1]} \cong \mathbb{P}^2_{\mathbb{C}}$ in a curve isomorphic to D_k, say F. Now we introduce some notation. When C is a projective curve on a nonsingular projective surface S, possibly also belonging to some other surfaces, we write $C^2|_S$ to indicate that we consider C as a curve on S while calculating its self-intersection number. So, for example, we have $F^2|_{E_k^{[k+1]}} = d^2$. We will show that $F^2|_{E_{k+1}^{[k+1]}} < 0$. Restricting everything to some Zariski open neighbourhood of the point D_{k-1}, we clearly have

$$K(X_{k+1}) = 2E_k^{[k+1]} + 3E_{k+1}^{[k+1]}$$

in $\operatorname{Pic} X_{k+1}$. By the adjunction formula (see [4, Proposition II.8.20]), we obtain in $\operatorname{Pic} E_k^{[k+1]}$ that

$$3E_k^{[k+1]} \cdot E_k^{[k+1]} = -3L - 3F,$$

where L stands for an arbitrary line in $\mathbb{P}^2_{\mathbb{C}}$. Using the commutative diagram

$$\begin{tikzcd}
E_{k+1}^{[k+1]} \arrow[r] & X_{k+1} \\
F \arrow[u] \arrow[r, "p"'] & E_k^{[k+1]} \arrow[u, "q"']
\end{tikzcd}$$

where all maps are the canonical inclusions, we see by calculating $(q \circ p)^*(E_k^{[k+1]})$ in two different ways (and taking degrees) that

$$F^2|_{E_{k+1}^{[k+1]}} = \deg p^*(E_k^{[k+1]} \cdot E_k^{[k+1]}) = (-L - F) \cdot F = -d - d^2 \quad < \ 0.$$

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By Lemma 2.3, we then know that $e(E_{k+1}^{[k+1]}) > 0$ and that

$$C_0(E_{k+1}^{[k+1]}) = F \subseteq E_{k+1}^{[k+1]} \cap E_k^{[k+1]}.$$

Moreover, we have

$$2g(D_k) - 2 - e(E_{k+1}^{[k+1]}) = 2\frac{(d-1)(d-2)}{2} - 2 - d(d+1) = -4d \quad < \ 0,$$

ending the argument for the first case of the induction basis. When D_{k-1} is a curve, we can use a very similar argument. In this case one obtains that $e(E_{k+1}^{[k+1]}) = 1$. The details are left to the reader.

Fix $i \in \{k+2,\dots,m\}$ and suppose that for each $j \in \{k+1,\dots,i-1\}$ we have $2g(D_{j-1}) < 2 + e(E_j^{[j]})$, $e(E_j^{[j]}) > 0$ and $C_0(E_j^{[j]}) \subseteq E_j^{[j]} \cap (\bigcup_{k\le l\le j-1} E_l^{[j]})$. Furthermore we suppose that for each projective curve C which belongs to two exceptional surfaces $E_{l_1}^{[j]}$ and $E_{l_2}^{[j]}$, with $l_1, l_2 \in \{k,\dots,j\}$, we have that the product $C^2|_{E_{l_1}^{[j]}} C^2|_{E_{l_2}^{[j]}}$ is (strictly) negative. An exception is when l_1 or l_2 equals k, say for example $l_1 = k$, and when moreover D_{k-1} is a curve, since in this case the intersection curve $C := E_k^{[j]} \cap E_{l_2}^{[j]}$ is a fiber of the ruled surface $E_k^{[j]}$, implying that $C^2|_{E_k^{[j]}} = 0$. In this case we will still suppose that $C^2|_{E_{l_2}^{[j]}} < 0$. Note that we can make this extra assumption on the self-intersection numbers since they are fulfilled in the induction basis. Again we distinguish two possibilities, this time according to the position of the center D_{i-1}.

We start with the situation that the center $D_{i-1} \subseteq E_{i-1}^{[i-1]}$ is disjoint from $\bigcup_{k\le l\le i-2} E_l^{[i-1]}$. Blowing up at D_{i-1} results in a ruled surface $E_i^{[i]}$, which intersects the ruled surface $E_{i-1}^{[i]}$ (over D_{i-2}) in a curve F isomorphic to D_{i-1}. Write $D_{i-1} \equiv aC_0 + bf$ (on $E_{i-1}^{[i-1]}$) with $a, b \in \mathbb{Z}$. Because D_{i-1} is disjoint from $\bigcup_{k\le l\le i-2} E_l^{[i-1]}$, we have $D_{i-1} \cap C_0 = \varnothing$ implying that $b = ae$. Hence $F^2|_{E_{i-1}^{[i]}} = D_{i-1}^2 = a^2(-e+2e) = ea^2$, which is (strictly) positive by the induction hypothesis and Proposition 1.4. We will now prove that $F^2|_{E_i^{[i]}} < 0$, following the argument for the induction basis. Here we have, eventually after restricting everything to some Zariski open in X_{k-1}, that

$$K(X_i) = \sum_{k\le l\le i} (\nu_l - 1) E_l^{[i]}$$

with $\nu_l \in \mathbb{Z}_{>1}$. Denote the inclusion $F \to E_{i-1}^{[i]}$ by p. By the adjunction formula and [4, Corollary V.2.11] we obtain that

$$F^2|_{E_i^{[i]}} = \deg p^*(E_{i-1}^{[i]} \cdot E_{i-1}^{[i]}) = -e(E_{i-1}^{[i]})a^2 + \frac{a}{\nu_{i-1}}\big(2g(D_{i-2}) - 2 - e(E_{i-1}^{[i]})\big),$$

which is (strictly) negative by the induction hypothesis. Finally we have

$$\begin{aligned} 2g(D_{i-1}) - 2 - e(E_i^{[i]}) &= \deg K(F) - e(E_i^{[i]}) \\ &= \deg\big((K(E_{i-1}^{[i]}) + F)\cdot F\big) - e(E_i^{[i]}) \\ &= a\big(2g(D_{i-2}) - 2 - e(E_{i-1}^{[i]})\big)\left(1 + \frac{1}{\nu_{i-1}}\right), \end{aligned}$$

which is also (strictly) negative by the induction hypothesis, ending the proof of the first case.

Now we consider the situation where the center $D_{i-1} \subseteq E_{i-1}^{[i-1]}$ is contained in $\bigcup_{k\le l\le i-2} E_l^{[i-1]}$. Then D_{i-1} is contained in precisely two exceptional surfaces $E_l^{[i-1]}$ with $l \in \{k,\dots,i-1\}$, say, for example, $E_{l_1}^{[i-1]}$ and $E_{l_2}^{[i-1]}$. We may suppose that $D_{i-1}^2|_{E_{l_1}^{[i-1]}} > 0$ and $D_{i-1}^2|_{E_{l_2}^{[i-1]}} < 0$ (or $D_{i-1}^2|_{E_{l_2}^{[i-1]}} < 0$, $D_{i-1}^2|_{E_{l_1}^{[i-1]}} = 0$, $l_1 = k$ and D_{k-1} is a curve). Blowing up at D_{i-1} results in a ruled surface $E_i^{[i]}$ which intersects $E_{l_1}^{[i]}$ and $E_{l_2}^{[i]}$ in the disjoint curves F and G, respectively. Obviously $F^2|_{E_{l_1}^{[i]}} > 0$ (or $F^2|_{E_{l_1}^{[i]}} = 0$, $l_1 = k$ and D_{k-1} is a curve) and $G^2|_{E_{l_2}^{[i]}} < 0$. We will now prove that $F^2|_{E_i^{[i]}} < 0$. In exactly the same way as in the previous case, one finds that

$$F^2|_{E_i^{[i]}} = \frac{1}{\nu_{l_1}}\big(-(\nu_i-1)F^2|_{E_{l_1}^{[i]}} + K(E_{l_1}^{[i]})\cdot F\big) \le \frac{1}{\nu_{l_1}}K(E_{l_1}^{[i]})\cdot F.$$

Suppose from now on that $l_1 \neq k$. We leave the easy case $l_1 = k$ to the reader. Then $E_{l_1}^{[i]}$ is a ruled surface over D_{l_1-1} and $F \neq C_0(E_{l_1}^{[i]})$. This last fact is true since $e(E_{l_1}^{[i]}) > 0$ (induction hypothesis) and $F^2|_{E_{l_1}^{[i]}} > 0$. We then have

$$F^2|_{E_i^{[i]}} \le \frac{1}{\nu_{l_1}}\big(2g(D_{l_1-1}) - 2 - e(E_{l_1}^{[i]})\big)\, f(E_{l_1}^{[i]})\cdot F,$$

which is (strictly) negative by the induction hypothesis. Let us calculate $2g(D_{i-1}) - 2 - e(E_i^{[i]})$ by the same method as in the previous case:

$$\begin{aligned}2g(D_{i-1}) - 2 - e(E_i^{[i]}) &= \deg\big((K(E_{l_1}^{[i]}) + F)\cdot F\big) - e(E_i^{[i]})\\ &= \Big(1 + \frac{1}{\nu_{l_1}}\Big)\big(2g(D_{l_1-1}) - 2 - e(E_{l_1}^{[i]})\big)\, f(E_{l_1}^{[i]})\cdot F\\ &\quad + F^2|_{E_{l_1}^{[i]}}\Big(1 - \frac{\nu_i - 1}{\nu_{l_1}}\Big),\end{aligned}$$

which is (strictly) negative by the induction hypothesis and the fact that $\nu_i = \nu_{l_1} + \nu_{l_2} \ge \nu_{l_1} + 1$. Finally we prove that $G^2|_{E_i^{[i]}} > 0$. Since $F = C_0(E_i^{[i]})$, G is a section of $E_i^{[i]}$ and $F \cap G = \varnothing$, we have $G \equiv C_0 + ef$ in Pic $E_i^{[i]}$. Hence $G^2|_{E_i^{[i]}} = e$, which was just proved to be positive. This ends the proof of Theorem 2.4. ■

Now we come to the generalization of [9, Proposition 5.12], concerning the problem case of an intersection configuration on E consisting of three nonintersecting sections. We will need the following proposition.

Proposition 2.5 *Let E be a ruled surface and let D_1 and D_2 be two irreducible curves on E such that D_1, D_2 and C_0 are disjoint in twos. Then $e = 0$.*

Proof Trivial, using Proposition 1.4. ■

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Corollary 2.6 *Let $f \in \mathbb{C}[x,y,z] \setminus \mathbb{C}$. Fix an embedded resolution $h\colon X \to \mathbb{A}^3_{\mathbb{C}}$ of $f^{-1}\{0\}$ in $\mathbb{A}^3_{\mathbb{C}}$ and use the notation of Section* 1.6. *Let $\widetilde{E}$ be an exceptional surface with $h(\widetilde{E}) = \{0\}$ and which is created as a ruled surface E, say as an exceptional surface of the blowing-up π_{m-1} in the resolution process. Denote by E_i the (reduced) irreducible components, different from E, of $(f \circ \pi_0 \circ \cdots \circ \pi_{m-1})^{-1}\{0\}$. Then the intersection $E \cap (\bigcup_i E_i)$ cannot consist of three nonintersecting sections of E.*

Proof This is an immediate consequence of Theorem 2.4 and Proposition 2.5. ■

3 Ruled Surfaces E with $\chi(E^\circ) = 0$

3.1 Let $f \in \mathbb{C}[x,y,z] \setminus \mathbb{C}$. Fix an embedded resolution $h\colon X \to \mathbb{A}^3_{\mathbb{C}}$ of $f^{-1}\{0\}$ in $\mathbb{A}^3_{\mathbb{C}}$ and use the notation of Section 1. Let $E_m^{[r]}$ be an exceptional surface with $h(E_m^{[r]}) = \{0\}$ and suppose that it is created as a ruled surface $\pi\colon E_m^{[m]} \to D_{m-1}$. Note that in fact we only suppose here that the center D_{m-1} is a nonsingular projective curve. Put $E := E_m^{[m]}$, $D := D_{m-1}$, $\nu := \nu_m$ and $N := N_m$. Let $E_i, i \in T'$, be the irreducible components (different from $E_m^{[r]}$) of $h^{-1}(f^{-1}\{0\})$ intersecting $E_m^{[r]}$ and put $\alpha_i := \nu_i - (\nu/N)N_i$. Then we suppose that $\alpha_i \notin \mathbb{Z}$ for $i \in T'$. We denote by E° the part of E that does not belong to any other irreducible component of $(f \circ \pi_0 \circ \cdots \circ \pi_{m-1})^{-1}\{0\}$. Note that $E^\circ \cong E_m^{[r]} \setminus (\bigcup_{i \in T'} E_i)$. Then in this (and the next) section we study the implication

$$\chi(E^\circ) = 0 \ \Rightarrow\ \mathcal{R} = 0,$$

where $\mathcal{R}$ stands for the contribution of $E_m^{[r]}$ to the residue at $-\nu/N$ of any of the zeta functions defined in Section 1.

3.2 Clearly $E \setminus E^\circ$ is a union of irreducible curves. The case that this union is a connected subset of E is studied by Veys in [13]. From now on we suppose that $E \setminus E^\circ$ is not connected. Hence none of the curves is a fiber of E, implying by Theorem 2.4 that C_0 is one of them. We denote the other curves by D_i, for $i = 1, \ldots, s$.

Moreover, by Theorem 2.4 we know that $e > 0$. So Proposition 2.5 implies that $E \setminus E^\circ$ consists of at most two connected components. Hence we put ourselves in the situation of exactly two connected components. Proposition 1.4 easily implies that the component containing C_0 only consists of the curve C_0.

3.3 Let D_j be a curve on $E \setminus E^\circ$. We consider the restriction

$$\varphi := (\pi_m \circ \cdots \circ \pi_{r-1})|_{E_m^{[r]}} \colon E_m^{[r]} \to E_m^{[m]}$$

to $E_m^{[r]}$ of the morphism $\pi_m \circ \cdots \circ \pi_{r-1} \colon X_r \to X_m$. Then we know that the strict transform of D_j in $E_m^{[r]}$ under φ is equal to some irreducible component of the intersection of $E_m^{[r]}$ with another component of $h^{-1}(f^{-1}\{0\})$, say with $E_{i(j)}$. For $j = 1, \ldots, s$ we define

$$\beta_j := \alpha_{i(j)} = \nu_{i(j)} - \frac{\nu}{N} N_{i(j)}.$$

Analogously we associate a rational number β to the curve C_0. Note that

$$\beta, \beta_j \notin \mathbb{Z} \quad \text{for } j = 1, \dots, s.$$

3.4 Let κ_j denote the self-intersection number of D_j. Then by [8, Example 6.6(b),(c)] we have

$$\sum_{j=1}^{s} d_j(\beta_j - 1) + (\beta - 1) = -2,$$

$$\sum_{j=1}^{s} \frac{\kappa_j}{2d_j}(\beta_j - 1) - \frac{e}{2}(\beta - 1) = 2g - 2,$$

where g stands for the genus of D and d_j, $j = 1, \dots, s$, is the number of intersections of the curve D_j with a "general" fiber of $\pi\colon E \to D$.

Write $D_j \equiv a_j(C_0 + ef)$ in $\operatorname{Pic} E$, with $a_j \in \mathbb{Z}_{>0}$. Then $\kappa_j = D_j^2 = ea_j^2$ and $d_j = D_j \cdot f = a_j$. From the two relations above we now obtain

$$\beta = \frac{-2}{e}(g - 1).$$

Hence we may suppose that $e \geq 3$ and $g \neq 1$, because otherwise $\beta \in \mathbb{Z}$.

3.5 Suppose that $g \geq 2$. By [9, Proposition 5.13] we know that $\chi(E^\circ) = 0$ can only occur when $s = 1$ and D_1 itself is a section. Obviously $\mathcal{R} = 0$ in this case; see also [9, Proposition 5.1(i)]. Thus the only case which still must be investigated is the case $g = 0$.

3.6 From now on we will study the case that $g = 0$ and that $E \setminus E^\circ$ consists of two connected components, one of them consisting only of the curve C_0. Moreover, we restrict ourselves to the situation when all the curves D_i, $i = 1, \dots, s$, are nonsingular. In this case, we will be able to prove the implication we are after, *i.e.*, the vanishing of $\chi(E^\circ)$ implies the vanishing of $\mathcal{R}$. For this, we first try to find out what the possible intersection configurations on E (only consisting of nonsingular curves) with $\chi(E^\circ) = 0$ look like. We need some lemmata.

Notation 3.1 For two curves C_1 and C_2 on E and a point $P \in C_1 \cap C_2$ we denote by $(C_1, C_2)_P$ the intersection multiplicity of C_1 and C_2 at P.

Lemma 3.2

(i) *The intersection $D_i \cap D_j$ is nonempty for each pair $\{i, j\} \subseteq \{1, \dots, s\}$.*
(ii) *If the nonsingular curve D_i, $i \in \{1, \dots, s\}$, on E is rational, then it is a section.*

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Proof (i) Trivial. (ii) By the adjunction formula, we have

$$(K(E)+D_i)\cdot D_i = 2g(D_i)-2=-2.$$

Because $D_i \cap C_0 = \varnothing$, we can write $D_i \equiv a(C_0+ef)$ with $a \in \mathbb{Z}_{>0}$. Since

$$K(E) \equiv -2C_0-(2+e)f,$$

see [4, Corollary V.2.11], we obtain that $(a-1)(ae-2)=0$, implying that $a=1$. Hence D_i is a section. ■

Lemma 3.3 *Let C_i, C_j, C_k be three nonsingular curves on E and let $P \in C_i \cap C_j \cap C_k$. If $(C_i, C_j)_P < (C_i, C_k)_P$, then $(C_j, C_k)_P = (C_i, C_j)_P$.*

Proof Easy. ■

Proposition 3.4 *We use the notation of Sections* 3.1 *and* 3.2*. Suppose that $g(D)=0$ and that $E \setminus E^\circ$ consists of two connected components, one of them consisting only of the curve C_0. If all the curves D_i, $i=1,\dots,s$, are nonsingular and $\chi(E^\circ)=0$, then all the curves D_i are sections.*

Proof By Lemma 3.2(ii), it is sufficient to prove that all curves D_i are rational. Let us suppose that (at least) one of the curves D_i is not rational, say D_1. The condition $\chi(E^\circ)=0$ is equivalent to $\chi(\bigcup_{i=1}^s D_i)=2$. Recall that $\chi(D_i)=2-2g(D_i)$. Hence $\chi(D_1)\le 0$. By Lemma 3.2(i) and the additivity of $\chi(\cdot)$, we see that the condition $\chi(\bigcup_{i=1}^s D_i)=2$ implies that there exist (at least) two rational curves, say D_2 and D_3, which moreover satisfy $D_1 \cap D_2 = D_1 \cap D_3 = \{P\}$ for some point P on E.

Since D_2 and D_3 are sections by Lemma 3.2(ii), we see that

$$(D_2, D_3)_P \le D_2 \cdot D_3 = e.$$

Write $D_1 \equiv a(C_0+ef)$ with $a \in \mathbb{Z}_{>1}$. Then for $i \in \{2,3\}$ we have

$$(D_1, D_i)_P = D_1 \cdot D_i = a(C_0+ef)^2 = ae > e,$$

contradicting Lemma 3.3. ■

From now on we suppose that $\bigcup_{i=1}^s D_i$ is a union of sections. We will investigate the possible connected configurations $\bigcup_{i=1}^s D_i$ of sections on the rational ruled surface E, with all sections disjoint from C_0 and such that $\chi(\bigcup_{i=1}^s D_i)=2$.

Lemma 3.5 *Let $\pi\colon E \to \mathbb{P}^1$ be a projective ruled surface, let f be a fiber on E and let C be a section. Then there exists an isomorphism $\varphi\colon E \setminus (f \cup C) \to \mathbb{A}^2$ such that $p_1 \circ \varphi = \pi$, where p_1 stands for the first projection $\mathbb{A}^2 \to \mathbb{A}^1$.*

Proof By [4, Proposition V.2.2] and [4, Corollary V.2.14] it is clear that there exists an isomorphism between $E \setminus f$ and $\mathbb{A}^1 \times \mathbb{P}^1$ under which π corresponds to the first projection $\mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{A}^1$.

Since C is a section, its equation in $\mathbb{A}^1 \times \mathbb{P}^1$ is of the form $C\colon f(x)y_0 + g(x)y_1 = 0$, with x the affine coordinate on $\mathbb{A}^1$, $(y_0 : y_1)$ homogeneous coordinates on $\mathbb{P}^1$ and $f(x), g(x) \in \mathbb{C}[x]$. Clearly $f(x)$ and $g(x)$ do not have a common zero.

By the Nullstellensatz there exist polynomials $\tilde{f}, \tilde{g} \in \mathbb{C}[x]$ such that $f\tilde{g} + g\tilde{f} = 1$. We consider the family of curves which are given by the following equations:

$$c(fy_0 + gy_1) + (\tilde{f}y_0 + \tilde{g}y_1) = 0,$$

with $c \in \mathbb{C}$. One can check that all these curves are disjoint from C, and moreover that they are disjoint in twos. We obtain an appropriate isomorphism $(\mathbb{A}^1 \times \mathbb{P}^1) \setminus C \cong \mathbb{A}^2$ by transforming the curves above into the "canonical" sections of the first projection $\mathbb{A}^2 \to \mathbb{A}^1$. Concretely, we consider the isomorphism

$$\begin{array}{rccc} \varphi\colon & (\mathbb{A}^1 \times \mathbb{P}^1) \setminus C & \longrightarrow & \mathbb{A}^2 \\ & (a, t_0 : t_1) & \longmapsto & \left(a, \dfrac{-(\tilde{f}(a)t_0 + \tilde{g}(a)t_1)}{f(a)t_0 + g(a)t_1}\right). \end{array}$$

This ends the proof of Lemma 3.5. ■

Proposition 3.6 *Let $\pi\colon E \to \mathbb{P}^1$ be a ruled surface with invariant $e \in \mathbb{Z}_{>0}$ and let C_1, C_2, C_3 be sections of E. Suppose that C_1, C_2 and C_3 are disjoint from C_0. Let P and Q be different points on C_1 such that $C_1 \cap C_2 = \{P\}$ and $C_1 \cap C_3 = \{Q\}$; see Figure 3. Then C_2 and C_3 intersect each other in e different points.*

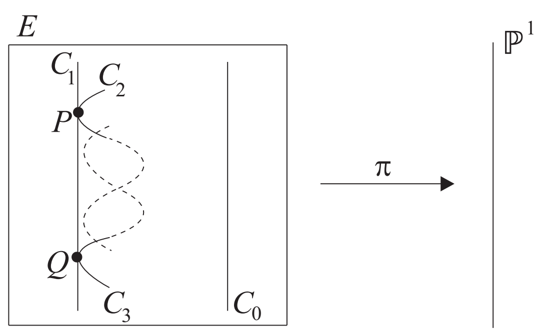

Figure 3

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Proof We first note that $(C_1, C_3)_Q = C_1 \cdot C_3 = (C_0 + ef)^2 = e$. By removing from E the section C_0 and the fiber through P and by applying Lemma 3.5 to this situation, we reduce the proof to the following problem in $\mathbb{A}^2$. Let C_1 and C_2 be two disjoint sections of $p_1 \colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, and let C_3 be a section that intersects C_1 in precisely one point, say Q, with, moreover, $(C_1, C_3)_Q = e$. Then we have to prove that C_2 and C_3 intersect each other in e different points.

Obviously the equations of the sections C_i of $p_1 \colon \mathbb{A}^2 \to \mathbb{A}^1$ are of the form

$$C_i : y = f_i(x),$$

where x and y are the affine coordinates on $\mathbb{A}^2$ and $f_i(x) \in \mathbb{C}[x]$. Since $C_1 \cap C_2 = \varnothing$, there exists a nonzero complex number c such that $f_2(x) = f_1(x) + c$. The condition on C_3 implies that there exist complex numbers a and c', with $c' \neq 0$, such that $f_3(x) - f_1(x) = c'(x-a)^e$. Then C_2 and C_3 clearly intersect each other in e different intersection points. ■

Now we are ready to prove the following theorem.

Theorem 3.7 *Let $\pi \colon E \to \mathbb{P}^1$ be a ruled surface with invariant $e \in \mathbb{Z}_{>2}$ and let $\bigcup_{i=1}^{s} D_i$ be a connected configuration of sections on E. Suppose that all sections D_i are disjoint from C_0 and that $\chi\big(E \setminus \big((\bigcup_{i=1}^{s} D_i) \cup C_0\big)\big) = 0$. Then $\bigcup_{i=1}^{s} D_i$ has one of the following two forms:*

(i) *there exist two points on E, say P and Q, such that $D_i \bigcap D_j = \{P, Q\}$ for $i, j = 1, \dots, s$ with $i \neq j$ (Figure 4),*

(ii) *there exist s points, say $P_1, \dots, P_s$, on one of the sections D_i, say D_1, such that $D_1 \cap D_i = \{P_1, P_i\}$ for $i = 2, \dots, s$ and $D_i \cap D_j = \{P_1\}$ for $i, j = 2, \dots, s$ with $i \neq j$ (Figure 5).*

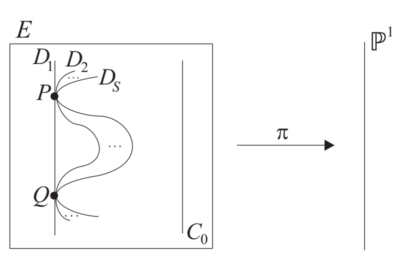

Figure 4

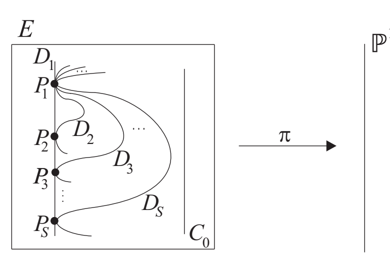

Figure 5

Remark In Figures 4 and 5 the curves D_i and D_j do not have to be tangent in their intersection points.

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Proof We search for the configurations $\bigcup_{i=1}^s D_i$ with $\chi(\bigcup_{i=1}^s D_i) = 2$. First recall that for any two sections D_i and D_j we have $D_i \cdot D_j = (C_0 + ef)^2 = e$. We distinguish two possibilities: either there exists a section D_i, say D, for which there exists another section D_j intersecting D in precisely one point, or there does not exist such a section D.

We start with the first possibility. Denote by r the number of points R on D such that there exists a section D_j with $D \cap D_j = \{R\}$. For each of these points R_i, $i = 1, \dots, r$, let $D_j^{(i)}$, $j = 1, \dots, s_i$, be the sections satisfying $D \cap D_j^{(i)} = \{R_i\}$. Then, by Lemma 3.2(i) and the additivity of $\chi(\,\cdot\,)$, we have

$$\chi\big(\bigcup_{i=1}^{s} D_i\big) \le \chi\big(D \cup \big(\bigcup_{i=1}^{r}\bigcup_{j=1}^{s_i} D_j^{(i)}\big)\big).$$

Note that $D_{j_1}^{(i)} \cap D_{j_2}^{(i)} = \{R_i\}$ for $j_1 \neq j_2$. Indeed, since $(D, D_{j_1}^{(i)})_{R_i} = D \cdot D_{j_1}^{(i)} = e$ and $(D, D_{j_2}^{(i)})_{R_i} = e$, we have by Lemma 3.3 that

$$(D_{j_1}^{(i)}, D_{j_2}^{(i)})_{R_i} = e = D_{j_1}^{(i)} \cdot D_{j_2}^{(i)}.$$

Furthermore, by Proposition 3.6 the intersection $D_{j_1}^{(i)} \cap D_{j_2}^{(1)}$, with $i \neq 1$, consists of e different points. Hence we see that

$$\chi\big(\bigcup_{i=1}^{s} D_i\big) \le 2 + s_1 + \sum_{i=2}^{r} s_i(1 - s_1 e).$$

If $r \ge 2$, then we have

$$\chi\big(\bigcup_{i=1}^{s} D_i\big) \le 2 + s_1 + (1 - s_1 e) \le 1.$$

So we suppose that $r = 1$. Then one easily finds that

$$\chi\big(D \cup \big(\bigcup_{j=1}^{s_1} D_j^{(1)}\big)\big) = 2 + s_1 \ge 3.$$

Thus, there still has to be another section D_i, say D'. Of course,

$$\chi\big(\bigcup_{i=1}^{s} D_i\big) \le \chi\big(D \cup \big(\bigcup_{j=1}^{s_1} D_j^{(1)}\big) \cup D'\big).$$

Suppose that $R_1 \notin D'$. Since D' has to intersect D in at least two different points and all the curves $D_j^{(1)}$ in at least one point, we obtain that

$$\chi\big(D \cup \big(\bigcup_{j=1}^{s_1} D_j^{(1)}\big) \cup D'\big) \le 2 + s_1 + (2 - 2 - s_1) = 2,$$

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the equality occurring when D' intersects D precisely twice and each $D_j^{(1)}$ precisely once. But this configuration realizing the equality is impossible on a rational ruled surface E with invariant $e \geq 3$. This follows immediately from Proposition 3.6 by taking $D_1^{(1)}, D$ and D' for C_1, C_2 and C_3, respectively.

We now treat the case that $R_1 \in D'$. By Lemma 3.3 the curve D' still has to intersect the curves $D_j^{(1)}$ in a point different from R_1. Hence we obtain again that

$$\chi\big(D \cup (\bigcup_{j=1}^{s_1} D_j^{(1)}) \cup D'\big) \leq 2 + s_1 + (2 - 2 - s_1) = 2,$$

the equality occurring when D' intersects all the curves D and $D_j^{(1)}$ in precisely one point different from R_1. The reader can easily check that this configuration $D \cup (\bigcup_{j=1}^{s_1} D_j^{(1)}) \cup D'$ satisfies condition (ii) of the statement of the theorem. There cannot be another section D_i anymore. Indeed, suppose that D_3 still is another section. Then D_3 has to intersect $D \cup (\bigcup_{j=1}^{s_1} D_j^{(1)}) \cup D'$ in precisely two points. Since D_3 also has to intersect D in at least two points, we obtain that $R_1 \in R_3$. But then also D_3 has to intersect the curves $D_j^{(1)}$ in a point different from R_1, which yields a contradiction.

We come to the second possibility, *i.e.,* we suppose that there does not exist a section D_i intersecting some other section D_j in precisely one point. An important remark is that this assumption implies that for any $j \in \{1, \dots, s\}$ we have

$$\chi\big(\bigcup_{i=1}^{s} D_i\big) \leq \chi\Big(\bigcup_{\substack{i=1\\ i\neq j}}^{s} D_i\Big),$$

the equality occurring when D_j intersects $\bigcup_{i\neq j} D_i$ in precisely two points. In fact, we see that $\chi(\bigcup_{i=2}^{s} D_i) = 2$ can only occur when each curve D_j intersects $\bigcup_{i\neq j} D_i$ in precisely two points. The case of one section obviously satisfies both conditions (i) and (ii). So we assume that $s \geq 2$. Denote by P and Q the two points in which D_1 and D_2 intersect. Since any two sections have to intersect each other in precisely two points, we obtain that each section D_j, with $j \in \{1, \dots, s\}$, has to intersect $\bigcup_{i\neq j} D_i$ precisely in P and Q. Of course, this Theorem 3.7(i). ■

4 Calculation of the Residue

4.1 We continue using the notation of Sections 3.1, 3.2 and 3.3. In this section we verify that for the two possible configurations on $E = E_m^{[m]}$ of Theorem 3.7 we have $\mathcal{R} = 0$, where $\mathcal{R}$ stands for the contribution of $E_m^{[r]}$ to the residue at $-\nu_m/N_m$ of any of the zeta functions defined in Section 1. We will only give the details for the Hodge zeta function, the argument for the motivic and topological zeta function being similar and easier, respectively.

4.2 To calculate the residue $\mathcal{R}$, we will use the technique of Veys in [9, Section 2], which is explained there for Igusa's local zeta function, but which can easily be seen also to be true for the zeta functions of the present paper. Hence, as soon as E is created we

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will "forget" the remaining part of the resolution process h, and we will construct the *minimal* embedded resolution of $C_0 \cup (\bigcup_{i=1}^{s} D_i)$ in E. For this we have to perform a finite number of point-centered blowing-ups, say f_j, $j = 1, \dots, t$, each of them creating a new exceptional curve F_j. We will also denote by C_0, D_i and F_l the strict transforms of the corresponding curves under the blowing-ups f_j further on in the embedded resolution process. For each $j \in \{1, \dots, t\}$ we associate to F_j the rational number

$$\gamma_j := \sum_{i=1}^{s} \mu_i(\beta_i - 1) + \sum_{l=1}^{j-1} \delta_l(\gamma_l - 1) + 2,$$

where μ_i denotes the multiplicity on D_i of the center of the blowing-up f_j and where $\delta_l = 1$ if this center belongs to F_l and $\delta_l = 0$ otherwise.

Let C_i, $i \in S$, be a finite number of irreducible curves on some nonsingular surface V, such that $\bigcup_{i \in S} C_i$ is a normal crossings divisor on V. Suppose that to each C_i we have associated a rational number $a_i \in \mathbb{Q} \setminus \{0\}$. Then we define

$$\mathcal{R}\big(\bigcup_{i \in S} C_i\big) := H\big(V \setminus \bigcup_{i \in S} C_i\big) + \sum_{i \in S} H\big(C_i \setminus \bigcup_{l \neq i} C_l\big)\, \frac{uv - 1}{(uv)^{a_i} - 1} + \sum_{\substack{\{i,j\} \subseteq S \\ i \neq j}} \operatorname{card}(C_i \cap C_j) \frac{(uv-1)^2}{((uv)^{a_i} - 1)((uv)^{a_j} - 1)}.$$

With this notation the technique of Veys yields that

$$\mathcal{R}_{\mathrm{Hod}} = \frac{(1 - uv)}{N_m (uv)^{-\nu_m / N_m}}\, \mathcal{R}\big(C_0 \cup \big(\bigcup_{i=1}^{s} D_i\big) \cup \big(\bigcup_{j=1}^{t} F_j\big)\big).$$

4.3 Let P be a point on a nonsingular surface V and let $C_1, \dots, C_r$ be nonsingular irreducible curves on V such that $P \in C_i$ for each $i \in \{1, \dots, r\}$. To each of the curves C_i we associate a rational number $a_i \in \mathbb{Q} \setminus \{0\}$. Suppose that there exists a positive integer $n \in \mathbb{Z}_{>1}$ such that $(C_i, C_j)_P = n$ for all pairs $\{i, j\} \subseteq \{1, \dots, r\}$; recall Notation 3.1. To obtain an embedded resolution of the germ of $\bigcup_{i=1}^{r} C_r$ at P we have to blow up precisely n times. Denote by F_j, $j = 1, \dots, n$, the corresponding exceptional curves. Then the resolution graph looks as in Figure 6.

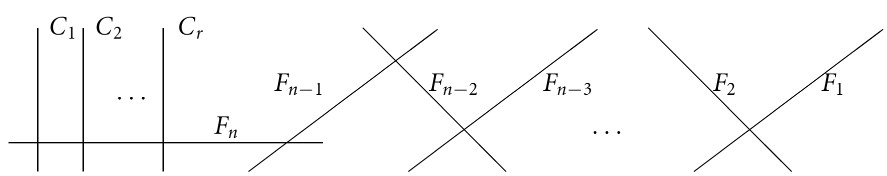

Figure 6

To the curves F_j we associate rational numbers γ_j as in Section 4.2. We will need the following lemmata.

Lemma 4.1

(i) *For* $i \in \{1,\dots,n\}$ *we have* $\gamma_i = i\gamma_1 - (i-1)$.
(ii) *For* $i, j \in \{1,\dots,n\}$ *we have* $\gamma_i + \gamma_{j-1} = \gamma_{i-1} + \gamma_j$, *where we put* $\gamma_0 := 1$.

Proof (i) By induction. (ii) Trivial, using (i). ∎

The following lemma is a special case of [11, Theorem 6.1, Example 6.2]. Again Veys worked in his paper with Igusa's local zeta function, but it is easily seen that the argument remains true in our situation. Using Lemma 4.1(ii), this fact can also easily be checked by induction on n.

Lemma 4.2 *Suppose that* $n \geq 2$. *Then we have*

$$\frac{uv(uv-1)}{(uv)^{\gamma_1}-1} + \sum_{j=2}^{n-1} \frac{(uv-1)^2}{(uv)^{\gamma_j}-1} + \sum_{j=1}^{n-1} \frac{(uv-1)^2}{((uv)^{\gamma_j}-1)((uv)^{\gamma_{j+1}}-1)} = \frac{(uv-1)(1+\sum_{j=1}^{n-1}(uv)^{\gamma_j})}{(uv)^{\gamma_n}-1}.$$

Remark Of course, when we take for V and the C_i the surface E and the curves D_i, respectively, the expression in the left-hand side of the equality above is just (except for some nonzero factor) the contribution of $\bigcup_{j=1}^{n-1} F_j$ to the residue $\mathcal{R}$.

Lemma 4.3 *Let* C_1, C_2 *and* C_3 *be three sections on a ruled surface E, each of them disjoint from* C_0. *Suppose that there exists points P and Q on E such that* $C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3 = \{P, Q\}$. *Then there exists an integer* $n \in \{1,\dots,e-1\}$ *such that for all pairs* $\{i,j\} \subseteq \{1,2,3\}$ *we have* $(C_i, C_j)_P = n$ *and* $(C_i, C_j)_Q = e - n$.

Proof First recall that $C_i \cdot C_j = (C_0 + ef)^2 = e$ for all $i, j \in \{1,2,3\}$ with $i \neq j$. Hence $e = (C_i, C_j)_P + (C_i, C_j)_Q$. By Lemma 3.3 we know that either the three self-intersection numbers at P are equal or one of them is strictly larger than the other two. In the second case we would obtain that one of the self-intersection numbers at Q is strictly smaller than the other two, which of course contradicts Lemma 3.3. ∎

We now can prove the first vanishing result.

Proposition 4.4 *We use the notation of Sections* 4.1 *and* 4.2. *Then for configuration* (i) *of Theorem* 3.7 *we have* $\mathcal{R} = 0$.

Proof If $s = 1$, one easily computes that $\mathcal{R} = 0$ using the fact that $\beta_1 + \beta = 0$, see Section 3.4. Now suppose that $s \geq 2$. By Lemma 4.3 we know that all the self-intersection numbers $(D_i, D_j)_P$ are equal, say equal to $n \in \{1,\dots,e-1\}$. Then $(D_i, D_j)_Q = e - n$ for all pairs $\{i,j\} \subseteq \{1,\dots,s\}$. For the situations with $n = 1$

or $e - n = 1$, the reader is invited to make the trivial modifications to the following argument. Now we proceed for both the points P and Q as in Subsection 4.3. We use the notation F_j^P, F_j^Q, γ_j^P and γ_j^Q to indicate the point to which these curves and numbers are related. Then we must show that

$$R := \mathcal{R}\Big(C_0 \cup \big(\bigcup_{i=1}^{s} D_i\big) \cup \big(\bigcup_{j=1}^{n} F_j^P\big) \cup \big(\bigcup_{j=1}^{e-n} F_j^Q\big)\Big) = 0.$$

Recall that $\beta = 2/e$, see Section 3.4. By Lemma 4.2 we obtain that

$$R = (uv+1)\frac{uv-1}{(uv)^{2/e}-1} + \frac{uv-1}{(uv)^{\gamma_n^P}-1}\Big(\sum_{j=1}^{n}(uv)^{\gamma_j^P}\Big) + \frac{uv-1}{(uv)^{\gamma_{e-n}^Q}-1}\Big(\sum_{j=1}^{e-n}(uv)^{\gamma_j^Q}\Big)$$
$$+ \Big(1 + \frac{1}{(uv)^{\gamma_n^P}-1} + \frac{1}{(uv)^{\gamma_{e-n}^Q}-1}\Big)\Big(\sum_{i=1}^{s}\frac{(uv-1)^2}{(uv)^{\beta_i}-1} + (uv-1)(uv-s)\Big).$$

By Section 3.4 we have $\gamma_1 := \gamma_1^P = \gamma_1^Q = \sum_{i=1}^{s}(\beta_i - 1) + 2 = 1 - \frac{2}{e}$, such that Lemma 4.1(i) implies that $\gamma_n^P + \gamma_{e-n}^Q = n\gamma_1 - (n-1) + (e-n)\gamma_1 - (e-n-1) = 0$. This easily yields that

$$1 + \frac{1}{(uv)^{\gamma_n^P}-1} + \frac{1}{(uv)^{\gamma_{e-n}^Q}-1} = 0,$$

implying by Lemma 4.1(i) that

$$\begin{aligned} R &= (uv-1)\Bigg(\frac{uv+1}{(uv)^{2/e}-1} \\ &\qquad + \frac{uv}{(uv)^{\gamma_n^P}-1}\Big(\sum_{j=1}^{n}\Big(\frac{(uv)^{\gamma_1}}{uv}\Big)^j - (uv)^{\gamma_n^P}\sum_{j=1}^{e-n}\Big(\frac{(uv)^{\gamma_1}}{uv}\Big)^j\Big)\Bigg) \\ &= (uv-1)\Big(\frac{uv+1}{(uv)^{2/e}-1} + \frac{(uv)^{\gamma_1}((uv-(uv)^{\gamma_n^P}) - (uv)^{\gamma_n^P}(uv-(uv)^{-\gamma_n^P}))}{((uv)^{\gamma_n^P}-1)(uv-(uv)^{\gamma_1})}\Big) \\ &= 0. \end{aligned}$$ ■

Before we prove the second vanishing fact, we reduce the possibilities in configuration (ii) of Theorem 3.12.

Lemma 4.5 *If $s \geq 3$ in configuration* (ii) *of Theorem* 3.7*, we have that for each $i \in \{2, \dots, s\}$ the curves D_1 and D_i intersect transversally at P_i.*

Proof We have to prove that $(D_1, D_i)_{P_i} = 1$ for all $i \in \{2, \dots, s\}$. Fix two different indices in $\{2, \dots, s\}$, say i_1 and i_2, and suppose that $(D_1, D_{i_1})_{P_{i_1}} > 1$. By removing from E the section C_0 and the fiber through P_1 and by applying Lemma 3.5 to this situation, we find ourselves in precisely the same affine settings as in the proof of Proposition 3.6. Now rereading the argument of that proof shows that D_1 and D_{i_2} have to intersect each other in $(D_1, D_{i_1})_{P_{i_1}}$ different points, all of them of course different from P_1 by construction. This is a contradiction. ■

Proposition 4.6 *We use the notation of Sections* 4.1 *and* 4.2. *Then for configuration* (ii) *of Theorem* 3.7 *we have* $\mathcal{R} = 0$.

Proof We may assume that $s \geq 3$, since the configurations with $s = 1$ or $s = 2$ were already part of Proposition 4.4. By Lemma 4.5, we then know that $(D_1, D_i)_{P_i} = 1$ for all $i \in \{2, \dots, s\}$. Hence $(D_1, D_i)_{P_1} = e - 1 > 1$ for all $i \in \{2, \dots, s\}$. Furthermore we have $(D_i, D_j)_{P_1} = D_i \cdot D_j = e$ for all pairs $\{i, j\} \subseteq \{2, \dots, s\}$. To obtain the minimal embedded resolution of $C_0 \cup (\bigcup_{i=1}^{s} D_i)$ in E we have to blow up precisely e times, creating the exceptional curves F_i, $i = 1, \dots, e$, as in Figure 7 below. Note that Figure 7 only shows the resolution graph of the germ $C_0 \cup (\bigcup_{i=1}^{s} D_i)$ at P_1.

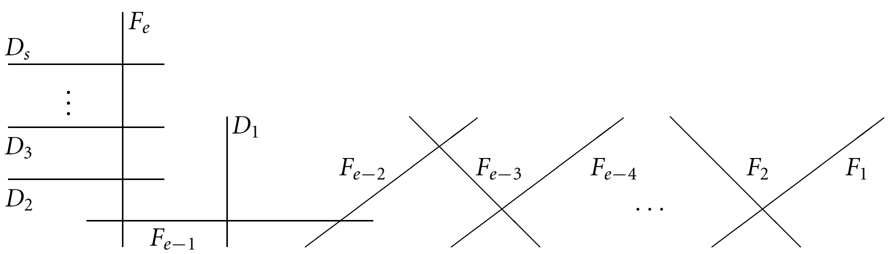

Figure 7

We have to prove that

$$R := \mathcal{R}\Big(C_0 \cup \big(\bigcup_{i=1}^{s} D_i \big) \cup \big(\bigcup_{j=1}^{e} F_j \big) \Big) = 0.$$

One easily checks that Lemma 4.1 and Lemma 4.2 are also true in this situation when we put $n := e - 1$. Then the contribution R_1 of $\bigcup_{j=1}^{e-1} F_j$ to R is equal to

$$R_1 = \frac{(uv-1)(1 + \sum_{j=1}^{e-2} (uv)^{\gamma_j})}{(uv)^{\gamma_{e-1}} - 1} + (uv - 2)\frac{uv - 1}{(uv)^{\gamma_{e-1}} - 1} + \frac{(uv-1)^2}{(uv)^{\gamma_{e-1}} - 1}\Big(\frac{1}{(uv)^{\beta_1} - 1} + \frac{1}{(uv)^{\gamma_e} - 1} \Big).$$

Since

$$\begin{aligned} \gamma_e &= \sum_{i=2}^{s} (\beta_i - 1) + (\gamma_{e-1} - 1) + 2 \\ &= -\beta_1 + \gamma_{e-1} + \sum_{i=1}^{s} (\beta_i - 1) + 2 = -\beta_1, \end{aligned}$$

we have

$$R_1 = \frac{(uv-1)(uv)^{\gamma_1}(uv - (uv)^{\gamma_{e-2}})}{((uv)^{\gamma_{e-1}} - 1)(uv - (uv)^{\gamma_1})},$$

following from Lemma 4.1(i) as in the proof of Proposition 4.4. The relations $\gamma_1 + \gamma_{e-2} = \gamma_{e-1} + 1$ and $\gamma_1 = -\gamma_{e-1} = 1 - 2/e$, which follow from Lemma 4.1, imply that

$$\begin{aligned} R &= uv - 1 + (uv+1)\frac{uv-1}{(uv)^{2/e}-1} + R_1 \\ &\quad + \Big(1 + \frac{1}{(uv)^{\beta_1}-1} + \frac{1}{(uv)^{\gamma_e}-1}\Big)\Big((uv-(s-1))(uv-1) + \sum_{i=2}^{s}\frac{(uv-1)^2}{(uv)^{\beta_i}-1}\Big) \\ &= (uv-1)(uv+1)\Big(\frac{1}{(uv)^{2/e}-1} + \frac{(uv)^{\gamma_1}-1}{((uv)^{\gamma_{e-1}}-1)(uv-(uv)^{\gamma_1})}\Big) = 0. \qquad \blacksquare \end{aligned}$$

As some kind of conclusion of Sections 3 and 4 we can state the following theorem.

Theorem 4.7 *Let $f \in \mathbb{C}[x,y,z] \setminus \mathbb{C}$. Fix an embedded resolution $h\colon X \to \mathbb{A}^3_{\mathbb{C}}$ of $f^{-1}\{0\}$ in $\mathbb{A}^3_{\mathbb{C}}$ and use the notation of Section 1.6. Let $\tilde{E}$ be an exceptional surface with $h(\tilde{E}) = \{0\}$ and which is created as a ruled surface E, say as exceptional surface $E_m^{[m]}$ of the blowing-up π_{m-1} in the resolution process. Suppose that $\nu_i - (\nu_m/N_m)N_i \notin \mathbb{Z}$ for each irreducible component E_i of $h^{-1}(f^{-1}\{0\})$ which intersects $\tilde{E}$ but is different from it. Denote by D_i, $i \in A$, the irreducible components of the intersection of E with the union of the other irreducible components of $(f \circ \pi_0 \circ \cdots \circ \pi_{m-1})^{-1}\{0\}$. Suppose that $\bigcup_{i\in A} D_i$ is not connected. Then, except for the situation described below, we have*

$$\chi\big(E \setminus \bigcup_{i\in A} D_i\big) = 0 \quad \Rightarrow \quad \mathcal{R} = 0,$$

where $\mathcal{R}$ stands for the contribution of $\tilde{E} = E_m^{[r]}$ to the residue at $-\nu_m/N_m$ of any of the zeta functions defined in Section 1.

We are in the exceptional situation when E is rational, $e \geq 3$ and $\bigcup_{i\in A} D_i$ consists of precisely two connected components, one of them consisting only of the section C_0 and the other one containing at least one singular curve.

Recall that Veys [13] studied the case that $\bigcup_{i\in A} D_i$ is connected. Furthermore we would like to recall that the other implication, *i.e.*,

$$\mathcal{R} = 0 \quad \Rightarrow \quad \chi\big(E \setminus \bigcup_{i\in A} D_i\big) = 0,$$

is also true. This follows from [6, Theorem 2.3] and is actually true for any exceptional surface $\tilde{E} = E_m^{[r]}$ with $h(\tilde{E}) = 0$ and satisfying $\nu_i - (\nu_m/N_m)N_i \notin \mathbb{Z}$ for all other irreducible components E_i of $h^{-1}(f^{-1}\{0\})$ which intersect $\tilde{E}$.

We end with the following theorem, which is a nice consequence of some older results and the results of the present paper.

Theorem 4.8 *We use the notation and conditions of Theorem 4.7, except for the fact that here the ruled surface E is supposed to be non-rational and the union $\bigcup_{i\in A} D_i$ is also allowed to be connected. Then the following conditions are equivalent.*

(i) $\mathcal{R} = 0$*;*
(ii) $\chi(E \setminus \bigcup_{i\in A} D_i) = 0$*;*
(iii) $\bigcup_{i\in A} D_i$ *consists of two sections, a fiber through each of their intersection points, and any number of other fibers.*

Proof The implication (i) $\Rightarrow$ (ii) is explained in the remark above.

Since $h(\tilde{E}) = \{0\}$, at least one of the curves D_i is a section of the ruled surface E. Hence the implication (ii) $\Rightarrow$ (iii) follows from [12, Theorem 5.1], except when E is a ruled surface over an elliptic curve. In this problem case we know by [12, Theorem 5.5] that $\bigcup_{i\in A} D_i$ has either the form of condition (iii) or consists of $k \geq 1$ irreducible nonsingular curves of genus 1 which are all disjoint. But the latter possibility can be excluded, since by Theorem 2.4(iii) one of the curves D_i equals C_0 and by Section 3.4 the rational number β associated to C_0 is an integer.

Finally, the implication (iii) $\Rightarrow$ (i) is proven in [9, Proposition 5.3] for Igusa's local zeta function. It is an easy exercise to verify that the arguments remain true in our situation. ∎

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