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Simultaneously preperiodic points for a family of polynomials in positive characteristic

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Abstract. In the goundbreaking paper [BD11] (which opened a wide avenue of research regarding unlikely intersections in arithmetic dynamics), Baker and DeMarco prove that for the family of polynomials $f_\lambda(x) := x^d + \lambda$ (parameterized by $\lambda \in \mathbb{C}$), given two starting points a and b in $\mathbb{C}$, if there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic under the action of f_λ, then $a^d = b^d$. In this paper, we study the same question, this time working in a field of characteristic $p > 0$. The answer in positive characteristic is more nuanced, as there are three distinct cases: (i) both starting points a and b live in $\overline{\mathbb{F}}_p$; (ii) d is a power of p; and (iii) not both a and b live in $\overline{\mathbb{F}}_p$, while d is not a power of p. Only in case (iii), one derives the same conclusion as in characteristic 0 (i.e., that $a^d = b^d$). In case (i), one has that for each $\lambda \in \overline{\mathbb{F}}_p$, both a and b are preperiodic under the action of f_λ, while in case (ii), one obtains that *also* whenever $a - b \in \overline{\mathbb{F}}_p$, then for each parameter λ, we have that a is preperiodic under the action of f_λ if and only if b is preperiodic under the action of f_λ.

1 Introduction

We start by setting up some basic notation for our paper in Subsection 1.1.

1.1 Notation

Throughout this paper, given a self-map f on some quasiprojective variety X, we denote by f^n its n-th compositional power; by convention, f^0 represents the identity map id_X on X. A preperiodic point $x \in X$ for f has the property that $f^m(x) = f^n(x)$ for some $0 \le m < n$; if $m = 0$ (i.e., $f^n(x) = x$), then the point x is called periodic (under the action of f).

1.2 Our results

We prove the following main result.

Theorem 1.1 *Let $d \ge 2$ be an integer, let L be a field of characteristic $p > 0$, and let $\alpha, \beta \in L$. We let $\overline{L}$ be a fixed algebraic closure of L, and we let $\overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$ inside $\overline{L}$. We consider the family of polynomials*

$$f_\lambda(x) := x^d + \lambda \text{ parameterized by } \lambda \in \overline{L}.$$

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Then there exist infinitely many $\lambda \in \overline{L}$ such that both α and β are preperiodic under the action of f_λ if and only if at least one of the following statements holds:

(1) $\alpha, \beta \in \overline{\mathbb{F}}_p \cap L$.
(2) $d = p^\ell$ *for some positive integer* ℓ *and* $\beta - \alpha \in \overline{\mathbb{F}}_p \cap L$.
(3) $\alpha^d = \beta^d$.

Moreover, if either one of the conditions (1)–(3) holds, then for each $\lambda \in \overline{L}$, we have that α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

1.3 The principle of unlikely intersections

Our Theorem 1.1 fits into the general principle of unlikely intersections; for a wonderful introduction to this area stemming from classical arithmetic geometry, we refer the reader to the book [Zan12].

Inspired by the results of Masser and Zannier from [MZ10], Baker and DeMarco [BD11] proved a first outstanding result for unlikely intersections in a purely dynamical context. So, given an integer $d \geq 2$ and given complex numbers a and b, Baker-DeMarco [BD11] prove that if there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic under the action of $f_\lambda(x) = x^d + \lambda$, then $a^d = b^d$. In other words, the infinite occurrence of the *unlikely* event that both a and b are preperiodic points for the same polynomial f_λ can only happen if both a and b have the same iterates under the entire family of maps $\{f_\lambda\}_{\lambda\in\mathbb{C}}$; so, a *very rigid* global condition is derived from the existence of infinitely many discrete unlikely events. The result of [BD11] was extended for more general families of polynomials and starting points (see [GHT13, BD13], for example), including families of polynomials parameterized by points in a higher dimensional space (see [GHT15, GHT16]). Also, extensions of [BD11] were obtained for certain families of rational maps (see [DWY15, GHT15]), and also for arbitrary families of Lattés maps (see [DM20]). Each time, the proof of any of the above results had two distinct parts. First, one proves that a certain equidistribution theorem for points of small height holds for the given dynamical system, which leads to knowing that certain canonical heights (suitably normalized) computed for the two starting points with respect to our family of maps are equal. Second, using the equality of the above canonical heights, one derives the precise relation between the two starting points.

Now, the key ingredient for establishing the first part of the above strategy comes from any of the equidistribution theorems of Baker-Rumely [BR06], Chambert-Loir [CL06], Favre-Rivera-Letelier [FRL06], or Yuan [Yua08]. Verifying the hypotheses of the aforementioned equidistribution theorems is the difficult part and requires a detailed analysis of the arithmetical properties of the given dynamical system. Usually, completing the second step of the above strategy is easier, and it generally relies on two ingredients: a complex dynamics argument (which in turn uses crucially some key features of complex analytic functions, such as the Open Mapping Theorem), along with the refined characterization provided by Medvedev and Scanlon [MS14] of the subvarieties of $\mathbb{A}^N$, which are invariant under the coordinatewise action of N one-variable polynomials. All of the above results hold over fields of characteristic 0, essentially because in positive characteristic, one lacks *completely* the tools for dealing with the aforementioned second step. In the present paper, we obtain a first complete answer to an unlikely intersection problem for a *dynamical system* in characteristic p.

1.4 The picture in positive characteristic

Overall, there are only a handful of results for the unlikely intersection principle in characteristic p. These known results are valid for Drinfeld modules (see [BM17, BM22, GH13, Ghi24]) since the Drinfeld modules are the natural vehicle in positive characteristic for many of the classical questions in arithmetic geometry, such as the André-Oort conjecture (see [Bre05]), the Bogomolov conjecture (see [Bos02]), the Mordell-Lang conjecture (see [Ghi05, GT08]), the Manin-Mumford conjecture (see [Sca02]), and the Siegel's theorem (see [GT07]). Generally, if one tries to prove results in characteristic p beyond the world of Drinfeld modules, then one encounters significant difficulties, especially in a purely dynamical setting.

In Theorem 1.1, we establish the counterpart of the main result of [BD11] in positive characteristic. The three different possibilities (1)–(3) from Theorem 1.1 show the distinct *three* scenarios one has to deal with when working arithmetic questions in characteristic p. First, we have the case when the starting points α and β live in $\overline{\mathbb{F}}_p$ – that is, the so-called *isotrivial* case, which is always *very special*. Second, we have the case when $d = p^\ell$ is a power of the characteristic; this is special since then each polynomial $f_\lambda(x) = x^d + \lambda$ from our family is an *affine* map on $\mathbb{G}_a$ (i.e., it is a composition of an additive polynomial $x \mapsto x^{p^\ell}$ with a translate $x \mapsto x + \lambda$). Third, we have the *generic* case; that is, in the absence of the above two cases, then indeed the only possibility for α and β to admit infinitely many parameters λ such that both starting points are preperiodic under the action of f_λ is when $\alpha^d = \beta^d$ (same as in characteristic 0).

Remark 1.2 The second case above (i.e., case (2) in Theorem 1.1) appears due to the fact that when $d = p^\ell$, our family of polynomials $f_\lambda = x^{p^\ell} + \lambda$ commutes with additional polynomials (besides the identity map). In fact, given any translate $T_\xi(x) = x + \xi$ for some $\xi \in \overline{\mathbb{F}}_p$, then T_ξ commutes with f_λ^m, where m is a positive integer so that $\xi \in \mathbb{F}_{p^{\ell m}}$.

1.5 The strategy for our proof

We also prove (see Theorem 6.1) a generalization of Theorem 1.1 by replacing the hypothesis that there exist infinitely many parameters λ for which both α and β are preperiodic under the action of f_λ with the weaker hypothesis that (in a suitable product formula field L) there exist infinitely many parameters λ_n such that

$$\lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\alpha) = \widehat{h}_{f_{\lambda_n}}(\beta) = 0; \tag{1.5.1}$$

for more details regarding the global canonical heights $\widehat{h}_{f_\lambda}$, see Subsection 3.5. The fact that we can reduce in our Theorem 1.1 to the case L is a product formula field is explained in Subsection 6.1 (especially, see Proposition 6.2). Also, as noted in Remark 3.9, once α (or β) is preperiodic under the action of f_λ, then its global

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canonical height (with respect to f_λ) equals 0; hence, the condition (1.5.1) is weaker than the hypothesis from Theorem 1.1.

Similar to the proof of Baker-DeMarco [BD11], the first move is proving that the equidistribution theorem from [BR10] holds, which allows us to conclude that certain local canonical heights constructed with respect to the two starting points α and β are equal (for more details, see Section 3 and, also, see Theorem 4.1). In order to state the equidistribution theorem that we will employ in our proof (see Theorem 2.2), we need a technical setup both from the theory of Berkovich spaces and also from arithmetic dynamics; this is done in Section 2. We continue by introducing canonical heights (both local and global) associated to our family of polynomials; this is done in Section 3. Our results from Section 3 provide the technical background for obtaining the crucial Theorem 4.1 in Section 4. Theorem 4.1 says that the existence of an infinite sequence of parameters λ_n satisfying equation (1.5.1) yields that for *each* parameter λ and for each nonarchimedean place v of L, we have

$$\widehat{h}_{v,\lambda}(\alpha) = \widehat{h}_{v,\lambda}(\beta); \tag{1.5.2}$$

for the precise definition of the local canonical heights $\widehat{h}_{v,\lambda}$, we refer the reader to Section 3.

In Section 5, we prove Proposition 5.1, which says that assuming equation (1.5.2) holds (for each place v and each parameter λ), and also assuming that d is a not a power of p and that not both α and β live in $\overline{\mathbb{F}}_p$, then condition (3) from Theorem 1.1 must hold. Its proof requires a *refined* analysis of the valuations for $\alpha^d - \beta^d$, obtained by employing equation (1.5.2) for suitably chosen parameters λ. Theorem 4.1 coupled with Proposition 5.1 proves the direct implication (which is the much harder part) from the conclusion of Theorem 1.1. Finally, in Section 6, we conclude our proof of Theorem 1.1. We actually state and prove the more general Theorem 6.1 and show first how to deduce Theorem 1.1 as a consequence of Theorem 6.1. The main part of Section 6 is devoted to proving Theorem 6.1; once again, the key ingredient is our Proposition 5.1.

2 Equidistribution for points of small height

As mentioned in Section 1, we will need to apply the arithmetic equidistribution discovered independently by Baker-Rumely [BR06], Chambert-Loir [CL06], and Favre-Rivera-Letelier [FRL06]; when the base field is a nonarchimedean field, the equidistribution theorem is best stated over the Berkovich space associated to the underlying variety in question. We will introduce briefly the desired equidistribution theorem for points of small height (see Theorem 2.2); for a comprehensive introduction to Berkovich spaces, we refer the reader to [BR10]. In our presentation, we use the approach of Baker-Rumely, which connects the equidistribution theorem to the theory of arithmetic capacities. Hence, the material presented in this Section 2 is mainly from the book [BR10] by Baker and Rumely.

So, following [BR10, Definition 7.51], we let L be a field of characteristic p endowed with a product formula (i.e., there exists a set Ω_L of (pairwise inequivalent) absolute values satisfying the following conditions):

(i) for each nonzero $x \in L$, we have $|x|_v = 1$ for all but finitely many $v \in \Omega_L$; and
(ii) for each nonzero $x \in L$, we have

$$\prod_{v \in \Omega_L} |x|_v = 1. \tag{2.0.1}$$

We note that usually, one asks that the product formula (2.0.1) holds in a slighly more general form: $\prod_{v\in\Omega_L} |x|_v^{N_v} = 1$ for some given positive integers N_v; however, since all the absolute values from Ω_L are nonarchimedean, we can absorb the exponents N_v in the definition of the respective absolute values $|\cdot|_v$ (see also [GH13, Equation 2.2]). Furthermore, as mentioned in [BR10, Chapter 7], one does not require L to be a global (function) field, but rather one needs that L is a general product formula field (see equations (i)–(ii) above). In particular, we can let L_0 be the perfect closure of the rational function field (in one variable) over $\overline{\mathbb{F}}_p$, that is,

$$L_0 := \overline{\mathbb{F}}_p\left(t, t^{1/p}, t^{1/p^2}, \dots, t^{1/p^n}, \dots\right), \tag{2.0.2}$$

and then take L to be any finite extension of L_0; then L is a product formula field. Indeed, each place of $\overline{\mathbb{F}}_p(t)$ (which geometrically, corresponds to a point of $\mathbb{P}^1(\overline{\mathbb{F}}_p)$) extends uniquely to a place w of L_0, thus making L_0 a product formula field. Above each given place w of L_0, there exist finitely many places v of L; we denote by $\Omega := \Omega_L$ this set of places of L. Then L is a product formula field with respect to Ω. Furthermore, the separable closure L^{sep} of L coincides with its algebraic closure $\overline{L}$ (see also [GS22, Remark 1.1]). Finally, we have the following fact: only the points in $\overline{\mathbb{F}}_p$ are the points $x \in L$ which are integral at each place in Ω; that is,

$$\text{if } |x|_v \le 1 \text{ for each } v \in \Omega, \text{ then } x \in \overline{\mathbb{F}}_p. \tag{2.0.3}$$

In the rest of this section, we work with an arbitrary product formula field L; however, the relevant case for our results is a finite extension of the field from (2.0.2). Now, for each $v \in \Omega_L$, we let $\mathbb{C}_v$ be an algebraically closed field containing L, which is also complete with respect to a fixed extension of $|\cdot|_v$ to $\mathbb{C}_v$. Let $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ denote the Berkovich affine line over $\mathbb{C}_v$ (see [BR10] or [BD11, Section 2] for more details). In order to apply the main equidistribution result from [BR10, Theorem 7.52], we recall briefly the potential theory on the affine line over $\mathbb{C}_v$. The right setting for nonarchimedean potential theory is the potential theory on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ developed in [BR10]. We quote here part of a nice summary of the theory from [BD11, Section 2] without going into details (we refer the reader to [BR10, BD11] for all the details and proofs).

So, let E be a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. Then analogous to the complex case, the logarithmic capacity $\mathbf{c}(E) = e^{-V(E)}$, and the Green's function G_E of E relative to ∞ can be defined where $V(E)$ is the infimum of the *energy integral* with respect to all possible probability measures supported on E. If $\mathbf{c}(E) > 0$, then the exists a unique probability measure μ_E, also called the *equilibrum measure on E*, attaining the infimum of the energy integral. Furthermore, the support of μ_E is contained in the boundary of the unbounded component of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \setminus E$. The Green's function $G_E(z)$ of E relative to infinity is a well-defined nonnegative real-valued subharmonic function on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$

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which is harmonic on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}\backslash E$ (in the sense of [BR10, Chapter 8]). The following result (see [BD11, Lemma 2.5]) summarizes the key features of the Green's function.

Lemma 2.1 *Let E be a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ and let U be the unbounded component of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}\backslash E$.*

(1) *If $\mathbf{c}(E) > 0$ (i.e., $V(E) < \infty$), then $G_E(z) = V(E) + \log|z|_v$ for all $z \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ such that $|z|_v$ is sufficiently large.*
(2) *If $G_E(z) = 0$ for all $z \in E$, then G_E is continuous on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, $\mathrm{Supp}(\mu_E) = \partial U$ and $G_E(z) > 0$ if and only if $z \in U$.*
(3) *If $G : \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \to \mathbb{R}$ is a continuous subharmonic function which is harmonic on U, identically zero on E, and such that $G(z) - \log^+|z|_v$ is bounded, then $G = G_E$. Furthermore, if $G(z) = \log|z|_v + V + o(1)$ (as $|z|_v \to \infty$) for some $V < \infty$, then $V(E) = V$, and so, $\mathbf{c}(E) = e^{-V}$.*

To state the equidistribution result from [BR10], we consider the compact *Berkovich adèlic sets*, which are of the following form

$$\mathbb{E} := \prod_{v \in \Omega_L} E_v, \tag{2.0.4}$$

where E_v is a nonempty compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ for each $v \in \Omega_L$, and furthermore, E_v is the closed unit disk $\mathcal{D}(0,1)$ in $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ for all but finitely many $v \in \Omega_L$. The *logarithmic capacity* $\mathbf{c}(\mathbb{E})$ of $\mathbb{E}$ is defined as follows:

$$\mathbf{c}(\mathbb{E}) = \prod_{v \in \Omega_L} \mathbf{c}(E_v). \tag{2.0.5}$$

Note that in (2.0.5), there is a finite product as for all but finitely many $v \in \Omega_L$, we have $\mathbf{c}(E_v) = \mathbf{c}(\mathcal{D}(0,1)) = 1$. Let $G_v := G_{E_v}$ be the Green's function of E_v relative to ∞ for each $v \in \Omega_L$. For every $v \in \Omega_L$, we fix an embedding of the separable closure L^{sep} of L into $\mathbb{C}_v$. Let $S \subset L^{\mathrm{sep}}$ be any finite subset that is invariant under the action of the Galois group $\mathrm{Gal}(L^{\mathrm{sep}}/L)$. We define the height $h_{\mathbb{E}}(S)$ of S relative to $\mathbb{E}$ by

$$h_{\mathbb{E}}(S) = \sum_{v \in \Omega_L} \left(\frac{1}{|S|} \sum_{z \in S} G_v(z) \right). \tag{2.0.6}$$

Note that this definition is independent of the particular embedding L^{sep} into $\mathbb{C}_v$ that we choose at each place $v \in \Omega_L$. Finally, for each $v \in \Omega_L$, we let μ_v be the equilibrum measure on E_v. The following is a special case of the equidistribution result [BR10, Theorem 7.52] that we need for our application.

Theorem 2.2 *With the above notation, let $\mathbb{E} = \prod_{v \in \Omega} E_v$ be a compact Berkovich adèlic set with $\mathbf{c}(\mathbb{E}) = 1$. Suppose that S_n is a sequence of $\mathrm{Gal}(L^{\mathrm{sep}}/L)$-invariant finite subsets of L^{sep} with $|S_n| \to \infty$ and $h_{\mathbb{E}}(S_n) \to 0$ as $n \to \infty$. For each $v \in \Omega_L$ and for each n, let δ_n be the discrete probability measure supported equally on the elements of S_n. Then the sequence of measures $\{\delta_n\}$ converges weakly to μ_v the equilibrium measure on E_v.*

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3 Dynamics and heights associated to our family of polynomials

Throughout this section, we let L_0 be the perfect closure of $\overline{\mathbb{F}}_p(t)$ (see its definition from (2.0.2)), and then we let L be a given finite extension of L_0. Then each finite extension of L is separable (i.e., $L^{\text{sep}} = \overline{L}$); so, from now on, we fix an algebraic closure $\overline{L}$ of L. Also, for the sake of simplifying our notation, we let $\Omega := \Omega_L$ be the set of inequivalent places of L witnessing the fact that L is a product formula field.

3.1 Preperiodic parameters for a given starting point

We let $d \geq 2$ be an integer. We work with a family of polynomials as given in Theorem 1.1 (i.e., $f_\lambda(x) = x^d + \lambda$ parameterized by $\lambda \in \overline{L}$). Given $\gamma \in L$, we define

$$P_{n,\gamma}(\lambda) := f_\lambda^n(\gamma) \text{ for each } n \in \mathbb{N}; \tag{3.1.1}$$

then $P_{n,\gamma}(\lambda)$ is a polynomial in λ. A simple induction on n yields the following result.

Lemma 3.1 *With the above hypothesis, for each $n \in \mathbb{N}$, the polynomial $P_{n,\gamma}(\lambda)$ is monic and has degree d^{n-1} in λ.*

Remark 3.2 We immediately obtain as a corollary of Lemma 3.1 the fact that γ is not preperiodic for the entire family of polynomials f_λ. Furthermore, we obtain that if γ is preperiodic for f_λ, then $\lambda \in \overline{L}$.

3.2 Generalized Mandelbrot sets

From now on, in Section 3, we fix a place $v \in \Omega$.

Following the same approach as in [BD11], one defines the *generalized Mandelbrot set* $M_{\gamma,v} \subset \mathbb{A}^1_{\text{Berk},\mathbb{C}_v}$ associated to γ; roughly speaking, $M_{\gamma,v}$ is the subset of $\mathbb{C}_v$ consisting of all $\lambda \in \mathbb{C}_v$ such that $P_{n,\gamma}(\lambda)$ is v-adic bounded, as we let $n \to \infty$.

Let $\lambda \in \mathbb{C}_v$ and define the local canonical height $\widehat{h}_{v,\lambda}(x)$ of $x \in \mathbb{C}_v$ with respect to the polynomial f_λ; more precisely, we have the formula

$$\widehat{h}_{v,\lambda}(x) := \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(x)|_v}{d^n}, \tag{3.2.1}$$

where $\log^+(z) = \log\max\{z, 1\}$ for each real number z. Clearly, $\widehat{h}_{v,\lambda}(x)$ is a continuous function of both λ and x on $\mathbb{C}_v$. Also, we will be using the following easy fact:

$$\widehat{h}_{v,\lambda}(x) = \frac{\widehat{h}_{v,\lambda}(f_\lambda^m(x))}{d^m} \text{ for each } m \in \mathbb{N} \text{ and for each } x \in \mathbb{C}_v. \tag{3.2.2}$$

As $\mathbb{C}_v$ is a dense subspace of $\mathbb{A}^1_{\text{Berk},\mathbb{C}_v}$, continuity in λ implies that the canonical local height function $\widehat{h}_{v,\lambda}(\gamma)$ has a natural extension on $\mathbb{A}^1_{\text{Berk},\mathbb{C}_v}$ (note that the topology on $\mathbb{C}_v$ is the restriction of the weak topology on $\mathbb{A}^1_{\text{Berk},\mathbb{C}_v}$, so any continuous function on $\mathbb{C}_v$ will automatically have a unique extension to $\mathbb{A}^1_{\text{Berk},\mathbb{C}_v}$). Then $\lambda \in M_{\gamma,v}$ if and only if $\widehat{h}_{v,\lambda}(\gamma) = 0$. Thus, $M_{\gamma,v}$ is a closed subset of $\mathbb{A}^1_{\text{Berk},\mathbb{C}_v}$; in fact, the following is true (as previously proved in [BD11]).

Proposition 3.3 *$M_{\gamma,v}$ is a compact subset of $\mathbb{A}^1_{\text{Berk},\mathbb{C}_v}$.*

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Proof Since we already know that $M_{\gamma,v}$ is a closed subset of the locally compact space $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, then in order to prove Proposition 3.3, it suffices to show that $M_{\gamma,v}$ is a bounded subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. This last fact follows immediately from Lemma 3.4, part (iii). ∎

The following Lemma 3.4 is not only used in the proof of Proposition 3.3, but it is also repeatedly used throughout Section 5; its proof is easy but its findings are important.

Lemma 3.4 *Let $\gamma, \lambda \in \mathbb{C}_v$.*

(i) *If* $\max\{|\lambda|_v, |\gamma|_v\} \le 1$, *then*

$$\widehat{h}_{v,\lambda}(\gamma) = 0.$$

(ii) *If* $|\gamma|_v^d > \max\{1, |\lambda|_v\}$, *then*

$$\widehat{h}_{v,\lambda}(\gamma) = \log|\gamma|_v > 0. \tag{3.2.3}$$

(iii) *If* $|\lambda|_v > \max\left\{1, |\gamma|_v^d\right\}$, *then*

$$\widehat{h}_{v,\lambda}(\gamma) = \frac{\log|\lambda|_v}{d} > 0.$$

Proof of Lemma 3.4. We first note that conclusion (i) is immediate since knowing that both λ and γ are integral at the place v yields that each $f_\lambda^n(\gamma)$ is integral at v, thus showing that $\widehat{h}_{v,\lambda}(\gamma) = 0$.

Next, we work under the hypotheses from part (ii). The fact that $|\gamma|_v^d > \max\{1, |\lambda|_v\}$ yields that

$$|f_\lambda(\gamma)|_v = |\gamma^d + \lambda|_v = |\gamma|_v^d > |\gamma|_v.$$

An easy induction on n shows that for each $n \ge 1$, we have that

$$|f_\lambda^n(\gamma)|_v = |\gamma|_v^{d^n};$$

then the desired conclusion in part (ii) follows.

Finally, part (iii) is a consequence of part (ii) because the inequality $|\lambda| > \max\left\{1, |\gamma|_v^d\right\}$ yields

$$|f_\lambda(\gamma)|_v = \left|\gamma^d + \lambda\right|_v = |\lambda|_v > |\lambda|_v^{\frac{1}{d}}. \tag{3.2.4}$$

Equation (3.2.4) allows us to apply the conclusion from part (ii) to the point $f_\lambda(\gamma)$ and the parameter λ, and thus, we get

$$\widehat{h}_{f_\lambda}(f_\lambda(\gamma)) = |f_\lambda(\gamma)|_v = |\lambda|_v.$$

Then equation (3.2.2) yields the desired conclusion in Lemma 3.4, part (iii). ∎

3.3 The logarithmic capacities of the generalized Mandelbrot sets

Next, our goal is to compute the logarithmic capacities of the v-adic generalized Mandelbrot sets $M_{\gamma,v}$ associated to γ for our given family f polynomials f_λ.

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Theorem 3.5 *The logarithmic capacity of $M_{\gamma,\nu}$ is $\mathbf{c}(M_{\gamma,\nu}) = 1$.*

The strategy for the proof of Theorem 3.5 is to construct a continuous subharmonic function $G_{\lambda,\nu} : \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_\nu} \to \mathbb{R}$ satisfying Lemma 2.1 (3); the technical steps follow identically as in the proof of the similar result from [BD11]. So, we let

$$G_{\gamma,\nu}(\lambda) := \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\gamma)|_\nu}{d^{n-1}} = d \cdot \widehat{h}_{\nu,\lambda}(\gamma). \tag{3.3.1}$$

Note that $G_{\gamma,\nu}(\lambda) \geq 0$ for all $\lambda \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_\nu}$; also, $\lambda \in M_{\gamma,\nu}$ if and only if $G_{\gamma,\nu}(\lambda) = 0$.

The proof of the next result is essentially the same as the proof of [BD11, Proposition 3.7].

Lemma 3.6 *$G_{\gamma,\nu}$ is the Green's function for $M_{\gamma,\nu}$ relative to ∞.*

Now we are ready to prove Theorem 3.5.

Proof of Theorem 3.5. Lemma 3.4 (iii) yields that

$$G_{\gamma,\nu}(\lambda) = \log|\lambda|_\nu, \text{ for } |\lambda|_\nu \text{ sufficiently large.} \tag{3.3.2}$$

Combining Lemma 2.1 (3) and Lemma 3.6, along with equation (3.3.2), we find that $V(M_{\gamma,\nu}) = 0$. Hence, the logarithmic capacity of $M_{\gamma,\nu}$ is 1, as desired. ∎

3.4 The generalized adèlic Mandelbrot set

Let us call $\mathbb{M}_\gamma = \prod_{\nu\in\Omega} M_{\gamma,\nu}$ the *generalized adèlic Mandelbrot set* associated to γ. As a corollary to Theorem 3.5 and Lemma 3.4, we get that $\mathbb{M}_\gamma$ satisfies the hypothesis of Theorem 2.2; the proof of the next result is identical as its counterpart from [BD11].

Corollary 3.7 *For all but finitely many nonarchimedean places ν, we have that $M_{\gamma,\nu}$ is the closed unit disk $\mathcal{D}(0;1)$ in $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_\nu}$; furthermore, $\mathbf{c}(\mathbb{M}_\gamma) = 1$.*

3.5 Global canonical heights

For each $\lambda \in \overline{L}$ (again note that $\overline{L} = L^{\mathrm{sep}}$), we will use the notation

$$h_{\mathbb{M}_\gamma}(\lambda) := h_{\mathbb{M}_\gamma}(S) \text{ where } S \text{ is the } \mathrm{Gal}(L^{\mathrm{sep}}/L)\text{-orbit of } \lambda. \tag{3.5.1}$$

The notation from (3.5.1) is connected to the global canonical height associated to the polynomials f_λ.

Definition 3.8 For each $x \in \overline{L}$, we define its Weil height as

$$h(x) := \frac{1}{[L(x):L]} \cdot \sum_{\nu\in\Omega} \sum_{y\in\mathrm{Gal}(L^{\mathrm{sep}}/L)\cdot x} \log^+ |y|_\nu. \tag{3.5.2}$$

For each $\lambda \in \overline{L}$, we define the global canonical height of $x \in \overline{L}$ with respect to the polynomial f_λ as

$$\widehat{h}_{f_\lambda}(x) = \lim_{n\to\infty} \frac{h\left(f_\lambda^n(x)\right)}{d^n}. \tag{3.5.3}$$

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Remark 3.9 If γ is preperiodic under the action of f_λ, then it is immediate to see (based on equation (3.5.3)) that $\widehat{h}_{f_\lambda}(\gamma) = 0$ (since there are finitely many distinct points $f_\lambda^n(\gamma)$).

However, using [Ben05, Theorem B], one can also establish the converse statement as well (i.e., once $\widehat{h}_{f_\lambda}(\gamma) = 0$, then γ must be preperiodic under the action of f_λ). Indeed, as long as $\lambda \notin \overline{\mathbb{F}}_p$, then f_λ is not isotrivial, and therefore, [Ben05, Theorem B] shows that a point is preperiodic if and only if its canonical height equals 0. Finally, if $\lambda \in \overline{\mathbb{F}}_p$, then it is immediate to see that γ is preperiodic if and only if also $\gamma \in \overline{\mathbb{F}}_p$. Similarly, if $\widehat{h}_{f_\lambda}(\gamma) = 0$ (and $\lambda \in \overline{\mathbb{F}}_p$), then we must have that $|\gamma|_v \le 1$ for each place $v \in \Omega$ (see Lemma 3.4 (ii)), and therefore, we must also have that $\gamma \in \overline{\mathbb{F}}_p$ (see (2.0.3)).

The following fact follows easily from the decomposition of the global canonical height as a sum of local canonical heights; a similar result was obtained in the proof of [GH13, Theorem 2.6] in the context of Drinfeld modules.

Lemma 3.10 *Let $\gamma \in L$. Then for each $\lambda \in \overline{L}$, we have $h_{\mathbb{M}_\gamma}(\lambda) = d \cdot \widehat{h}_{f_\lambda}(\gamma)$.*

4 Equality of the respective local canonical heights

We continue with the notation as in Section 3. In particular, L is a finite extension of the perfect closure of the rational function field in one variable over $\overline{\mathbb{F}}_p$ (see (2.0.2)). Also, for any point $\gamma \in L$, we construct the generalized adèlic Mandelbrot set $\mathbb{M}_\gamma$ and then define the associated height $h_{\mathbb{M}_\gamma}$.

The following result is the key technical ingredient which we extract from Theorem 2.2. Its proof is essentially the same as its counterpart from [BD11] because we are both dealing with the same family of polynomials; the technical ingredients used in the proof of Theorem 4.1 are contained in Theorem 2.2 and Lemma 3.10. We also note that a similar result was proven in [GH13, Theorem 2.6] for dynamical systems coming from Drinfeld modules.

Theorem 4.1 *Let L, f_λ, $\widehat{h}_{f_\lambda}$, $\widehat{h}_{v,\lambda}$ be defined as in Section 3; also, let $\alpha, \beta \in L$. Assume there exists an infinite sequence $\{\lambda_n\}$ in $\overline{L}$ with the property that*

$$\lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\alpha) = \lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\beta) = 0. \tag{4.0.1}$$

Then for each $\lambda \in \overline{L}$ and for each $v \in \Omega$, we have that $\widehat{h}_{v,\lambda}(\alpha) = \widehat{h}_{v,\lambda}(\beta)$.

5 Proof of the precise relation between the starting points.

In this section, we prove the following result.

Proposition 5.1 *Let $L_0 := \overline{\mathbb{F}}_p\left(t, t^{1/p}, t^{1/p^2}, \cdots, t^{1/p^n}, \cdots\right)$ and let L be a finite extension of L_0. We denote by $\Omega := \Omega_L$ the set of inequivalent places of L. We let $\alpha, \beta \in L$, not both of them contained in $\overline{\mathbb{F}}_p$. Let $d \ge 2$ be an integer, which is not a power of the prime p. We let*

$$f_\lambda(x) := x^d + \lambda$$

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be a family of polynomials parameterized by $\lambda \in \overline{L}$. As in Section 3, for each $\lambda \in \overline{L}$ and for each place $v \in \Omega$, we let

$$\widehat{h}_{v,\lambda}(\alpha) = \lim_{n\to\infty} \frac{\log^+ \left|f_\lambda^n(\alpha)\right|_v}{d^n} \text{ and } \widehat{h}_{v,\lambda}(\beta) = \lim_{n\to\infty} \frac{\log^+ \left|f_\lambda^n(\beta)\right|_v}{d^n}.$$

If for each $\lambda \in \overline{L}$ and for each place $v \in \Omega$, we have that

$$\widehat{h}_{v,\lambda}(\alpha) = \widehat{h}_{v,\lambda}(\beta), \tag{5.0.1}$$

then we must have that $\alpha^d = \beta^d$.

Proposition 5.1 constitutes the bridge in our arguments between Theorem 4.1 and Theorem 1.1 (see also Theorem 6.1).

5.1 The strategy for proving Proposition 5.1

From now on, we work under the hypotheses from Proposition 5.1. We split its proof into Subsections 5.2, 5.3, 5.4, and 5.5.

So, we let S be the (finite) set of places $v \in \Omega$ with the property that

$$\max\{|\alpha|_v, |\beta|_v\} > 1. \tag{5.1.1}$$

Note that our hypothesis from Proposition 5.1 that not both α and β live in $\overline{\mathbb{F}}_p$ yields that S is a *nonempty* set. Our strategy will be to prove that

$$\left|\alpha^d - \beta^d\right|_v \le 1 \text{ for each } v \in S. \tag{5.1.2}$$

Indeed, since S consists of all the places v where α or β is not v-adic integral (see inequality (5.1.1)), then the only places of Ω for which $\alpha^d - \beta^d$ may not be a v-adic integer are exactly the ones from the set S. So, inequality (5.1.2) would prove that $\alpha^d - \beta^d$ is integral at each place $v \in \Omega$. Due to the product formula (2.0.1) on L (see also (2.0.3)), this means that $\alpha^d - \beta^d \in \overline{\mathbb{F}}_p$, which is sufficient to deduce that $\alpha^d = \beta^d$ if $d = 2$ (see Lemma 5.12). Now, in the case $d > 2$, we can actually prove that the inequality in (5.1.2) is strict; this is sufficient to deduce that $\alpha^d = \beta^d$ (see Lemma 5.11). We also note (see Remark 5.10) that it is exactly in the last part of our argument (the proof of Lemma 5.9) where we employ the hypothesis from Proposition 5.1 that d is not a power of p.

In order to deduce the inequality (5.1.2), we employ the hypothesis (5.0.1) from Proposition 5.1 for various well-chosen λ's in $\overline{L}$. Also, we first prove that

$$\left|\alpha^d - \beta^d\right|_v \le |\alpha|_v = |\beta|_v \text{ for each } v \in S; \tag{5.1.3}$$

this is done in Subsection 5.2.

5.2 First step in the proof of Proposition 5.1

In this subsection, we will establish (5.1.3). We first prove the following easy fact which will be used repeatedly in our proof of Proposition 5.1.

Lemma 5.2 *For each place $v \in S$, we have $|\alpha|_v = |\beta|_v > 1$.*

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Proof of Lemma 5.2. The desired conclusion is an easy corollary of Lemma 3.4, parts (i)–(ii) using $\lambda = 0$ and $v \in S$ (see (5.1.1)), along with the hypothesis (5.0.1) of Proposition 5.1. ∎

Corollary 5.3 *With the hypothesis as in Proposition 5.1, we have that neither α nor β live in $\overline{\mathbb{F}}_p$.*

Proof Indeed, Lemma 3.4 shows that *both* α and β are not integral at the places from S; hence, neither one can live in $\overline{\mathbb{F}}_p$. ∎

In particular, Corollary 5.3 yields that α and β are nonzero. The following Lemma will finish the proof for the assertion from (5.1.3).

Lemma 5.4 *For each place v in S, we have that $|\beta^d - \alpha^d|_v \le |\alpha|_v$.*

Proof of Lemma 5.4. We argue by contradiction, and so we assume that $|\beta^d - \alpha^d|_v > |\alpha|_v$, and we will derive a contradiction.

Indeed, we consider $\lambda_0 := -\alpha^d$ and then apply Lemma 3.4 for λ_0 and $\gamma := f_{\lambda_0}(\beta) = \beta^d - \alpha^d$; since $|\beta^d - \alpha^d|_v^d > |\lambda_0|_v = |\alpha|_v^d$ (according to our assumption), Lemma 3.4 (ii) yields that

$$\widehat{h}_{v,\lambda_0}(\gamma) = \log\left|\beta^d - \alpha^d\right|_v.$$

But then using the fact that $\widehat{h}_{v,\lambda_0}(\beta) = \frac{\widehat{h}_{v,\lambda_0}(\gamma)}{d}$ (see equation (3.2.2)), we obtain that

$$\widehat{h}_{v,\lambda_0}(\beta) = \frac{\log|\beta^d - \alpha^d|_v}{d}. \tag{5.2.1}$$

However, we compute

$$f_{\lambda_0}(\alpha) = 0 \text{ and } f_{\lambda_0}^2(\alpha) = -\alpha^d.$$

Then using again Lemma 3.4 (ii), this time for λ_0 and $-\alpha^d$ (note that $|\alpha^d|_v^d > |\alpha^d|_v$), we conclude that

$$\widehat{h}_{v,\lambda_0}(-\alpha^d) = \log|\alpha^d|_v.$$

But then again, using equation (3.2.2), we get that

$$\widehat{h}_{v,\lambda_0}(\alpha) = \frac{\widehat{h}_{v,\lambda_0}(f_{\lambda_0}^2(\alpha))}{d^2} = \frac{\log|\alpha|_v}{d}. \tag{5.2.2}$$

However, our assumption that $|\alpha^d - \beta^d|_v > |\alpha|_v$ coupled with equations (5.2.1) and (5.2.2) contradict the main hypothesis from our Proposition 5.1 that $\widehat{h}_{v,\lambda_0}(\alpha) = \widehat{h}_{v,\lambda_0}(\beta)$. In conclusion, indeed, we must have that

$$|\alpha^d - \beta^d|_v \le |\alpha|_v = |\beta|_v, \tag{5.2.3}$$

for each place v in S. ∎

5.3 Second step in the proof of Proposition 5.1

The inequality from equation (5.1.3) says that $|\alpha^d - \beta^d|_v$ is *much smaller* than one would expect it to be; that is, since $|\alpha|_v = |\beta|_v > 1$ (for $v \in S$), then *normally* one would

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expect $|\alpha^d - \beta|_v$ to be *larger* than $|\alpha|_v = |\beta|_v$. The next Lemma refines further the inequality from (5.1.3) showing that we actually have a *strict* inequality in (5.1.3).

Lemma 5.5 *For each place v in S, we have that $|\beta^d - \alpha^d|_v < |\alpha|_v$.*

Proof of Lemma 5.5. We argue by contradiction and therefore assume that $|\alpha^d - \beta^d|_v \geq |\alpha|_v$. Then Lemma 5.4 yields that actually $|\alpha^d - \beta^d|_v = |\alpha|_v$.

Claim 5.6 *There exists some nonzero $\gamma_0 \in \overline{\mathbb{F}}_p$ such that*

$$\left|\beta^d - \alpha^d - \gamma_0 \cdot \alpha\right|_v < |\alpha|_v. \tag{5.3.1}$$

■

The existence of γ_0 as in the conclusion of Claim 5.6 is essential in the proof of Lemma 5.5. Furthermore, the argument used in the proof of Claim 5.6 will also be useful in our further arguments for proving Lemma 5.5.

Proof of Claim 5.6. Let $K = \overline{\mathbb{F}}_p(t, \alpha, \beta)$; then K is a subfield of L. Moreover, K is a global function field (of transcendence degree 1) over $\overline{\mathbb{F}}_p$ (it is a finite extension of $\overline{\mathbb{F}}_p(t)$). We let $\mathcal{O}_{K,v}$ be the ring of v-adic integers in K; then $\mathcal{O}_{K,v}$ is a discrete valuation ring. We let $\pi_v \in \mathcal{O}_{K,v} \subset L$ be a *uniformizer* for the restriction of $|\cdot|_v$ on K; that is,

$$|\pi_v|_v = \max\left\{|x|_v \colon |x|_v < 1 \text{ and } x \in K\right\}.$$

So, there exists some positive integer e with the property that both $(\beta^d - \alpha^d) \cdot \pi_v^e$ and $\alpha \cdot \pi_v^e$ are v-adic units in K (note that $e > 0$ since $|\beta^d - \alpha^d|_v = |\alpha|_v > 1$). We let $\mathrm{red}_v : \mathcal{O}_{K,v} \longrightarrow \overline{\mathbb{F}}_p$ be the reduction map at the place v; then we simply compute

$$\gamma_0 := \frac{\mathrm{red}_v\left(\left(\beta^d - \alpha^d\right) \cdot \pi_v^e\right)}{\mathrm{red}_v\left(\alpha \cdot \pi_v^e\right)} \in \overline{\mathbb{F}}_p^{\,*}, \tag{5.3.2}$$

which satisfies the desired conclusion from Claim 5.6. ■

So, we let $\gamma_0 \in \overline{\mathbb{F}}_p$ as in the conclusion of Claim 5.6.

Claim 5.7 *With the above notation, there exists some $\gamma_1 \in \overline{\mathbb{F}}_p$ such that*

$$(\gamma_1 + \gamma_0)^d = 1 \textit{ and } \gamma_1^d \neq 1. \tag{5.3.3}$$

Proof of Claim 5.7. We argue by contradiction and therefore we assume that for each $u \in \overline{\mathbb{F}}_p$ such that $u^d = 1$, we have that also $(u - \gamma_0)^d = 1$. This means that the d-th roots of unity in $\overline{\mathbb{F}}_p$ can be grouped in disjoint subsets of p elements (note that $\gamma_0 \neq 0$):

$$u, u - \gamma_0, u - 2\gamma_0, \cdots, u - (p-1)\gamma_0.$$

However, this would mean that there are $p \cdot k$ solutions (for some positive integer k) for the equation $x^d = 1$ in $\overline{\mathbb{F}}_p$. This is a contradiction because the equation $x^d = 1$ has s solutions in $\overline{\mathbb{F}}_p$, where d is written as $s \cdot p^\ell$ for some integer $\ell \geq 0$ and some positive integer s coprime with p. The fact that p does not divide s shows that indeed one can find some γ_1 satisfying conditions (5.3.3). ■

Then we consider (with γ_1 satisfying the conclusion of Claim 5.7)

$$\lambda_1 := \gamma_1 \cdot \alpha - \alpha^d;$$

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a simple computation shows that

$$f^2_{\lambda_1}(\alpha) = f_{\lambda_1}(\gamma_1\alpha) = (\gamma_1^d - 1)\alpha^d + \gamma_1\alpha.$$

Since $\gamma_1^d \neq 1$, we conclude that

$$|f^2_{\lambda_1}(\alpha)|_v = |\alpha|_v^d > |\alpha|_v = |\lambda_1|_v^{1/d},$$

and thus, an application of Lemma 3.4 (ii) (coupled with equation (3.2.2)) yields

$$\widehat{h}_{v,\lambda_1}(\alpha) = \frac{\widehat{h}_{v,\lambda_1}\left(f^2_{\lambda_1}(\alpha)\right)}{d^2} = \frac{\log|\alpha|_v^d}{d^2} = \frac{\log|\alpha|_v}{d}. \tag{5.3.4}$$

However, we compute

$$f^2_{\lambda_1}(\beta) = f_{\lambda_1}(\beta^d - \alpha^d + \gamma_1\alpha) = (\beta^d - \alpha^d + \gamma_1\alpha)^d + \gamma_1\alpha - \alpha^d. \tag{5.3.5}$$

Claim 5.8 *With our choice for γ_0 as in Claim 5.6 and for γ_1 as in Claim 5.7, we have that*

$$\left|(\beta^d - \alpha^d + \gamma_1\alpha)^d - \alpha^d\right|_v < |\alpha^d|_v. \tag{5.3.6}$$

Proof of Claim 5.8. In order to justify (5.3.6), we argue similarly as in our proof of Claim 5.6. So, letting as before, π_v be a uniformizer of $K = \overline{\mathbb{F}}_p(t,\alpha,\beta)$ at the place v, then for some positive integer e and some v-adic units u_1, u_2 in K (and therefore in L), we have

$$\beta^d - \alpha^d = \pi_v^{-e} \cdot u_1 \text{ and } \alpha = \pi_v^{-e} \cdot u_2; \tag{5.3.7}$$

in particular, (5.3.7) yields $|\alpha^d|_v = (|\pi_v|_v^{-e})^d = |\pi_v|_v^{-de}$. Then we obtain

$$\left|(\beta^d - \alpha^d + \gamma_1\alpha)^d - \alpha^d\right|_v = \left|\pi_v^{-de}\cdot(u_1+\gamma_1u_2)^d - \pi_v^{-de}\cdot u_2^d\right|_v = |\pi_v|_v^{-de}\cdot\left|(u_1+\gamma_1u_2)^d - u_2^d\right|_v.$$

So, in order to get inequality (5.3.6), it suffices to prove that

$$\left|(u_1+\gamma_1u_2)^d - u_2^d\right|_v < 1; \tag{5.3.8}$$

furthermore, using that u_1 and u_2 are v-adic units, inequality (5.3.8) is equivalent with asking that

$$\left|\left(\frac{u_1}{u_2}+\gamma_1\right)^d - 1\right|_v < 1. \tag{5.3.9}$$

We rewrite the left-hand side in (5.3.9) as

$$\left|\left(\frac{u_1-\gamma_0u_2}{u_2} + (\gamma_0+\gamma_1)\right)^d - 1\right|_v. \tag{5.3.10}$$

Equation (5.3.1) (see also (5.3.2)) yields that

$$\left|\frac{u_1-\gamma_0u_2}{u_2}\right|_v < 1; \tag{5.3.11}$$

so, coupling equations (5.3.11) and (5.3.10), along with the fact that $(\gamma_0+\gamma_1)^d = 1$, we obtain inequality (5.3.9) (which, in turn, delivers the desired inequality (5.3.6)).

This finishes our proof of Claim 5.8. ■

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Now, inequality (5.3.6) yields (see also equation (5.3.5)) that

$$\left|f_{\lambda_1}^2(\beta)\right|_v < |\alpha|_v^d. \tag{5.3.12}$$

We let $M := \max\left\{|\alpha|_v, \left|f_{\lambda_1}^2(\beta)\right|_v\right\}$; inequality (5.3.12) yields

$$M < |\alpha|_v^d. \tag{5.3.13}$$

Using inequality (5.3.12), we obtain

$$\left|f_{\lambda_1}^3(\beta)\right|_v = \left|\left(f_{\lambda_1}^2(\beta)\right)^d + \gamma_1\alpha - \alpha^d\right|_v \leq \max\left\{\left|f_{\lambda_1}^2(\beta)\right|_v^d, |\alpha|_v^d\right\} = M^d. \tag{5.3.14}$$

An easy induction (similar to deriving inequality (5.3.14)) shows then that for each $n \geq 2$,

$$\left|f_{\lambda_1}^n(\beta)\right|_v \leq M^{d^{n-2}}. \tag{5.3.15}$$

Inequality (5.3.15) yields that $\widehat{h}_{v,\lambda_1}(\beta) \leq \frac{\log(M)}{d^2}$; then coupling this last inequality with equations (5.3.4) and (5.3.13), we obtain that

$$\widehat{h}_{v,\lambda_1}(\beta) < \widehat{h}_{v,\lambda_1}(\alpha),$$

which contradicts our main hypothesis (5.0.1) from Proposition 5.1. This concludes our proof of Lemma 5.5.

5.4 Third step in the proof of Proposition 5.1

We continue our analysis for $|\beta^d - \alpha^d|_v$ (for $v \in S$) with the goal of proving the inequality from (5.1.2). This time, we need to split our proof depending whether $d = 2$ or $d > 2$.

Lemma 5.9 *For each $v \in S$, we must have that*

(i) *if $d > 2$, then $|\beta^d - \alpha^d|_v < 1$.*
(ii) *if $d = 2$, then $|\beta^d - \alpha^d|_v \leq 1$.*

Proof of Lemma 5.9. We let $\lambda_2 := \alpha - \alpha^d$. Then clearly, $f_{\lambda_2}(\alpha) = \alpha$, which means that for any place v (not just the ones from the set S), we have that

$$\widehat{h}_{v,\lambda_2}(\alpha) = 0. \tag{5.4.1}$$

From now on, we argue by contradiction and assume that for some $v \in S$, we have

- $|\beta^d - \alpha^d|_v \geq 1$ if $d > 2$.
- $|\beta^d - \alpha^d|_v > 1$ if $d = 2$.

However, we know from Lemma 5.5 that $|\beta^d - \alpha^d|_v < |\alpha|_v$.

Next, we write $d = p^\ell \cdot s$ for some integer $\ell \geq 0$ and some positive integer s coprime with p. Furthermore, due to our hypothesis that $d \neq p^\ell$, then we must have that $s \geq 2$. We compute

$$f_{\lambda_2}^2(\beta) = f_{\lambda_2}(\beta^d - \alpha^d + \alpha) = \left(\beta^d - \alpha^d + \alpha\right)^d + \alpha - \alpha^d.$$

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Then we let $\varepsilon := \beta^d - \alpha^d$ and proceed as follows:

$$f_{\lambda_2}^2(\beta) = (\varepsilon + \alpha)^d - \alpha^d + \alpha = \left(\varepsilon^{p^\ell} + \alpha^{p^\ell}\right)^s - \alpha^{sp^\ell} + \alpha. \tag{5.4.2}$$

Then we expand

$$\left(\varepsilon^{p^\ell} + \alpha^{p^\ell}\right)^s = \alpha^{sp^\ell} + s\alpha^{(s-1)p^\ell}\varepsilon^{p^\ell} + \sum_{i=2}^{s}\binom{s}{i}\cdot\alpha^{(s-i)p^\ell}\varepsilon^{ip^\ell}.$$

From our assumption that $|\beta^d - \alpha^d|_v \geq 1$ (for any $d \geq 2$) along with the conclusion of Lemma 5.5, we have that

$$1 \leq |\varepsilon|_v < |\alpha|_v \text{ (with the left inequality being strict if } d = 2). \tag{5.4.3}$$

Since p does not divide s, inequality (5.4.3) shows that

$$\left|s\alpha^{(s-1)p^\ell}\varepsilon^{p^\ell}\right|_v = \left|\alpha^{(s-1)p^\ell}\varepsilon^{p^\ell}\right|_v > \left|\binom{s}{i}\cdot\alpha^{(s-i)p^\ell}\varepsilon^{ip^\ell}\right|_v \tag{5.4.4}$$

for each $i = 2, \ldots, s$. Equation (5.4.4) allows us to conclude that

$$\left|\left(\varepsilon^{p^\ell} + \alpha^{p^\ell}\right)^s - \alpha^{sp^\ell}\right|_v = \left|\alpha^{(s-1)p^\ell}\varepsilon^{p^\ell}\right|_v. \tag{5.4.5}$$

Clearly, if $d > 2$, then $(s-1)p^\ell \geq \max\{s-1, p^\ell\} \geq 2$; hence, using again that $|\varepsilon|_v \geq 1$ if $d > 2$, we derive that

$$\left|\alpha^{(s-1)p^\ell}\varepsilon^{p^\ell}\right|_v \geq |\alpha|_v^2 > |\alpha|_v \text{ if } d > 2. \tag{5.4.6}$$

Furthermore, using our assumption that $|\varepsilon|_v > 1$ if $d = 2$, then we also derive that

$$\left|\alpha^{(s-1)p^\ell}\varepsilon^{p^\ell}\right|_v = |\alpha\cdot\varepsilon|_v > |\alpha|_v \text{ if } d = 2, \tag{5.4.7}$$

because then $p \neq 2$ and so, $\ell = 0$ and $s = 2$ if $d = 2$.

Remark 5.10 We note that it is precisely in deriving inequalities (5.4.6) and (5.4.7) that we used the hypothesis that d is not a power of the prime p, since this translates to the inequality $s \geq 2$, which is used in both of the above two inequalities (along with the fact that we cannot have $s = p = 2$). ■

Combining inequalities (5.4.6), (5.4.7), (5.4.5), and (5.4.2) yields that

$$\left|f_{\lambda_2}^2(\beta)\right|_v = \left|\alpha^{(s-1)p^\ell}\varepsilon^{p^\ell}\right|_v > |\alpha|_v = |\lambda_2|_v^{1/d}. \tag{5.4.8}$$

Inequality (5.4.8) along with Lemma 3.4 (ii) yields that

$$\widehat{h}_{v,\lambda_2}\left(f_{\lambda_2}^2(\beta)\right) = \log\left|\alpha^{(s-1)p^\ell}\varepsilon^{p^\ell}\right|_v > 0. \tag{5.4.9}$$

Finally, using equations (5.4.9) and (3.2.2), we conclude that $\widehat{h}_{v,\lambda_2}(\beta) > 0$. Coupled with equation (5.4.1), this contradicts the main hypothesis (5.0.1) that $\widehat{h}_{v,\lambda_2}(\alpha) = \widehat{h}_{v,\lambda_2}(\beta)$. This concludes our proof of Lemma 5.9.

5.5 Final step in the proof of Proposition 5.1

Now we can finish our proof of Proposition 5.1. Once again, we split our analysis into two cases: $d > 2$, respectively $d = 2$.

Lemma 5.11 *If $d > 2$, then the conclusion in Proposition 5.1 holds.*

Proof of Lemma 5.11. By definition of the set S (see (5.1.1)), we have that if $|\beta^d - \alpha^d|_v > 1$, then we must have that $v \in S$. However, Lemma 5.9 yields that $|\beta^d - \alpha^d|_v < 1$ if $v \in S$. Hence, $\beta^d - \alpha^d$ is integral at all places, and furthermore, for the places $v \in S$ (note that S is nonempty due to our assumption that not both α and β are in $\overline{\mathbb{F}}_p$), we have that $|\beta^d - \alpha^d|_v < 1$; this contradicts the product formula (2.0.1), unless $\alpha^d - \beta^d = 0$, which is precisely the desired conclusion from Proposition 5.1. ■

Lemma 5.12 *If $d = 2$, the conclusion in Proposition 5.1 must hold.*

Proof The same argument as in the proof of Lemma 5.11 yields that $|\beta^2 - \alpha^2|_v \le 1$ for each place $v \in S$. Since we already know that $|\beta^2 - \alpha^2|_v \le 1$ for $v \notin S$, then (2.0.3) yields that $\varepsilon = \beta^2 - \alpha^2 \in \overline{\mathbb{F}}_p$.

Furthermore, we note that $p \ne 2$ since we know that $d = 2$ is not a power of the prime p (according to the hypothesis in Proposition 5.1).

Again, we work with $\lambda_2 = \alpha - \alpha^2$ as in Lemma 5.9. We compute (with $\varepsilon = \beta^2 - \alpha^2$)

$$f_{\lambda_2}(\beta) = \beta^2 + \alpha - \alpha^2 = \alpha + \varepsilon \text{ and } f_{\lambda_2}^2(\beta) = (2\varepsilon + 1)\alpha + \varepsilon^2 \tag{5.5.1}$$

and then

$$f_{\lambda_2}^3(\beta) = (4\varepsilon^2 + 4\varepsilon)\alpha^2 + (4\varepsilon^3 + 2\varepsilon^2 + 1)\alpha + \varepsilon^4. \tag{5.5.2}$$

Now, if $4\varepsilon^2 + 4\varepsilon \ne 0$, then we get (note that $\varepsilon \in \overline{\mathbb{F}}_p$)

$$\left|f_{\lambda_2}^3(\beta)\right|_v = |\alpha|_v^2 \text{ for } v \in S. \tag{5.5.3}$$

Using inequality (5.5.3) along with Lemma 3.4 (ii) yields that for $v \in S$, we would have that $\widehat{h}_{v,\lambda_2}(f_{\lambda_2}^3(\beta)) > 0$, and therefore, we would also have that $\widehat{h}_{v,\lambda_2}(\beta) > 0$, which contradicts that $\widehat{h}_{v,\lambda_2}(\alpha) = 0$ because α is a fixed point under the action of f_{λ_2}. So, we must have instead that

$$4\varepsilon^2 + 4\varepsilon = 0. \tag{5.5.4}$$

Since $p \ne 2$, then equation (5.5.4) yields either $\varepsilon = 0$ (which provides the desired conclusion $\alpha^2 = \beta^2$), or we must have $\varepsilon = -1$. However, this last possibility would provide a contradiction since then we can run the same argument with α and β reversed, and we would have gotten that

$$\alpha^2 - \beta^2 = -1,$$

thus contradicting $\beta^2 - \alpha^2 = -1$. More precisely, we could consider next

$$\lambda_3 := \beta - \beta^2$$

and impose the condition that $\widehat{h}_{v,\lambda_3}(\alpha) = 0$ for each $v \in S$. Letting $\mu := -\varepsilon = \alpha^2 - \beta^2$ and then arguing along the same lines as in the derivation of equations (5.5.1), (5.5.2),

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(5.5.3), and (5.5.4), we would obtain that either $\mu = 0$ (which is the desired conclusion), or that $\mu = -1$. Since we cannot have that

$$\varepsilon = -1 \text{ and } \mu = -\varepsilon = -1$$

because the characteristic p of our field is not 2 (because $d = 2$ is not a power of the characteristic), we conclude that also when $d = 2$, we must have that $\alpha^2 = \beta^2$.

This concludes our proof of Lemma 5.12. ∎

Combining Lemmas 5.11 and 5.12, we obtain the desired conclusion in Proposition 5.1.

6 Proof of our main results.

In this section, we finish our proof for Theorem 1.1. We actually prove a more general result.

Theorem 6.1 *Let $L_0 = \overline{\mathbb{F}}_p\left(t, t^{1/p}, t^{1/p^2}, \dots, t^{1/p^n}, \dots\right)$ and let L be a finite extension of L_0. Let $d \geq 2$ be an integer and let $\alpha, \beta \in L$. We consider the family of polynomials*

$$f_\lambda(x) := x^d + \lambda \text{ parameterized by } \lambda \in \overline{L}.$$

Then there exists an infinite sequence $\{\lambda_n\}_{n\geq 1}$ in $\overline{L}$ with the property that

$$\lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\alpha) = \lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\beta) = 0, \tag{6.0.1}$$

if and only if at least one of the following statements holds:

(1) $\alpha, \beta \in \overline{\mathbb{F}}_p$.

(2) $d = p^\ell$ *for some positive integer* ℓ *and* $\beta - \alpha \in \overline{\mathbb{F}}_p$.

(3) $\alpha^d = \beta^d$.

Moreover, if either one of the conditions (1)–(3) holds, then for each $\lambda \in \overline{L}$, we have that α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

We start by proving Theorem 1.1 assuming that Theorem 6.1 holds; this is done in Subsection 6.1. Then we will prove Theorem 6.1 by splitting our argument into two steps in Subsections 6.3 and, respectively, 6.4.

6.1 Proof of Theorem 1.1 assuming Theorem 6.1 holds.

The next proposition shows that in Theorem 1.1, we may assume $\operatorname{trdeg}_{\mathbb{F}_p} L = 1$.

Proposition 6.2 *Let $d \geq 2$ be an integer, let L be a field of characteristic $p > 0$, and let $\alpha, \beta \in L$. If there exists $\lambda_1 \in \overline{L}$ such that both α and β are preperiodic for the polynomial $f_{\lambda_1}(x) = x^d + \lambda_1$, then $\operatorname{trdeg}_{\mathbb{F}_p} \mathbb{F}_p(\alpha, \beta) \leq 1$.*

Proof We recall the notation from Subsection 3.1 (see Lemma 3.1) that for each $\gamma \in L$, we have

$$P_{n,\gamma}(\lambda) := f_\lambda^n(\gamma), \tag{6.1.1}$$

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which is a monic polynomial of degree d^{n-1} (in λ). Furthermore, the constant term is $P_{n,\gamma}(0) = \gamma^{d^n}$. An easy induction yields that each coefficient of λ^i in $P_{n,\gamma}(\lambda)$ for $i = 1, \dots, d^{n-1} - 1$ is itself a polynomial *in* γ, that is,

$$P_{n,\gamma}(\lambda) = \lambda^{d^{n-1}} + \sum_{i=1}^{d^{n-1}-1} c_{n,i}(\gamma) \cdot \lambda^i + \gamma^{d^n}, \tag{6.1.2}$$

with each $c_{n,i} \in \mathbb{F}_p[x]$ being a polynomial of degree less than d^n. Therefore, imposing the condition that α is a preperiodic point under the action of some f_{λ_1} yields an equation of the form

$$P_{n,\alpha}(\lambda_1) = P_{m,\alpha}(\lambda_1) \text{ for some } 0 \le m < n.$$

Using equation (6.1.2) (along with the information about the degrees of each corresponding polynomials $c_{m,i}$ and $c_{n,j}$), we obtain that $\alpha \in \overline{\mathbb{F}_p(\lambda_1)}$. A similar reasoning, using this time that β is preperiodic under the action of f_{λ_1}, yields that also $\beta \in \overline{\mathbb{F}_p(\lambda_1)}$. Hence, we conclude that

$$\mathrm{trdeg}_{\mathbb{F}_p} \mathbb{F}_p(\alpha, \beta) \le \mathrm{trdeg}_{\mathbb{F}_p} \overline{\mathbb{F}_p(\lambda_1)} \le 1,$$

as desired for the conclusion of Proposition 6.2. ■

Proof of Theorem 1.1 as a consequence of Theorem 6.1. First, we note that Theorem 1.1 is left unchanged if we replace L by *any* field extension of $\overline{\mathbb{F}}_p(\alpha, \beta)$.

Second, Proposition 6.2 allows us to assume that $\mathrm{trdeg}_{\mathbb{F}_p} L = 1$ in Theorem 1.1; that is, α and β live in a fixed algebraic closure of $\mathbb{F}_p(t)$.

So, from now on, we take L_0 be the perfect closure of the rational function field in one variable over $\overline{\mathbb{F}}_p$, that is,

$$L_0 = \overline{\mathbb{F}}_p(t)^{\mathrm{per}} = \overline{\mathbb{F}}_p\left(t, t^{\frac{1}{p}}, t^{\frac{1}{p^2}}, \dots, t^{\frac{1}{p^n}}, \dots\right) \tag{6.1.3}$$

and we let L be a finite extension of L_0 containing both α and β.

Now, assume that we have infinitely many $\lambda_n \in \overline{L}$ with the property that both α and β are preperiodic under the action of f_{λ_n}. Then (according to Remark 3.9), we have that $\widehat{h}_{f_{\lambda_n}}(\alpha) = \widehat{h}_{f_{\lambda_n}}(\beta) = 0$ for each $n \ge 1$; hence, the direct implication from Theorem 6.1 yields that at least one of the conditions (1)–(3) are met.

Conversely, assuming that at least one of the conditions (1)–(3) are met, then the "moreover" statement from Theorem 6.1 yields that for each $\lambda \in \overline{L}$, α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ. So, in order to prove the converse statement in Theorem 1.1, it suffices to establish the following fact.

Proposition 6.3 *Let L be an arbitrary field of characteristic p and let $\gamma \in L$. Then there exist infinitely many $\lambda \in \overline{L}$ such that γ is preperiodic under the action of f_λ.* ■

If L were $\mathbb{C}$, then the conclusion from Proposition 6.3 follows from a more general result established in [DeM16]; however, since L is a field of characteristic p, once again we require a different proof.

Proof of Proposition 6.3. If $\gamma \in \overline{\mathbb{F}}_p$, then the statement is obvious because then γ is preperiodic under f_λ for each $\lambda \in \overline{\mathbb{F}}_p$. So, from now on, we assume $\gamma \in L \backslash \overline{\mathbb{F}}_p$. The desired conclusion in Proposition 6.3 follows from the next Lemma, which provides a more refined conclusion.

Lemma 6.4 *Assume $\gamma \notin \overline{\mathbb{F}}_p$. Then there exist infinitely many $\lambda \in \overline{L}$ with the property that there exists some prime number q such that $f_\lambda^q(\gamma) = \gamma$.* ∎

Proof of Lemma 6.4. We argue by contradiction and so assume the set

$$\mathcal{P} := \left\{\lambda \in \overline{L}\colon \text{ there exists a prime } q \text{ such that } f_\lambda^q(\gamma) = \gamma\right\}$$

is finite. In particular, this means that there exists a positive integer M with the property that for each prime $q > M$ and for each $\lambda \in \overline{L}$ such that

$$f_\lambda^q(\gamma) = \gamma, \tag{6.1.4}$$

there exists a prime $q_0 < M$ (with q_0 depending on λ, of course) such that

$$f_\lambda^{q_0}(\gamma) = \gamma. \tag{6.1.5}$$

However, since q and q_0 are distinct primes, then equations (6.1.4) and (6.1.5) yield that $f_\lambda(\gamma) = \gamma$ (i.e., $\lambda = \gamma - \gamma^d$). Hence, letting $P_{q,\gamma}(\lambda) := f_\lambda^q(\gamma)$ as before (see equation (6.1.1)), the only solution $\lambda \in \overline{L}$ to the equation $P_{q,\gamma}(\lambda) = \gamma$ is $\lambda_0 := \gamma - \gamma^d$. Now, using the shape of the polynomial $P_{q,\gamma}(\lambda)$ (see equation (6.1.2)), we conclude that

$$P_{q,\gamma}(\lambda) = (\lambda - \lambda_0)^{d^{q-1}}. \tag{6.1.6}$$

In particular, this means that the constant term in the polynomial $P_{q,\gamma}$ must be $(\gamma^d - \gamma)^{d^{q-1}}$. However, we know that the constant term in the polynomial $P_{q,\gamma}$ is γ^{d^q}; this leads to the equation

$$\left(\gamma^d - \gamma\right)^{d^{q-1}} = \gamma^{d^q}. \tag{6.1.7}$$

Any solution γ to equation (6.1.7) must live in $\overline{\mathbb{F}}_p$, thus contradicting the hypotheses of Lemma 6.4. Therefore, indeed, the set $\mathcal{P}$ must be infinite, as claimed in the conclusion of Lemma 6.4. ∎

Lemma 6.4 shows that also when $\gamma \notin \overline{\mathbb{F}}_p$, there exist infinitely many $\lambda \in \overline{L}$ such that γ is preperiodic under the action of f_λ. This concludes our proof for Proposition 6.3.

So, Proposition 6.3 shows that there exist infinitely many $\lambda \in \overline{L}$ such that α is preperiodic under the action of f_λ and therefore (according to the "moreover" claim in Theorem 6.1), also β is preperiodic under the action of f_λ. Hence, this establishes the converse statement in Theorem 1.1.

This concludes our proof of Theorem 1.1, assuming that Theorem 6.1 holds.

6.2 Strategy for proving Theorem 6.1

We split our proof of Theorem 6.1 into the remaining two subsections of the current Section 6. In particular, we prove the "moreover" claim from Theorem 6.1 in Subsection 6.3, and then we finish the proof of Theorem 6.1 in Subsection 6.4.

So, from now on, we work with the notation and the assumptions from Theorem 6.1.

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6.3 Proof of the "moreover" claim from Theorem 6.1.

In this subsection, we show that if either one of conditions (1)–(3) from the conclusion of Theorem 6.1 holds, then for each $\lambda \in \overline{L}$, we have that α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

We argue case by case, as follows.

(1) For any $\gamma \in \overline{\mathbb{F}}_p$, using equations (6.1.1) and (6.1.2), we get that for each $\lambda \in \overline{L}$, we have that γ is preperiodic under the action of f_λ if and only if $\lambda \in \overline{\mathbb{F}}_p$. Therefore, if both α and β live in $\overline{\mathbb{F}}_p$, we have that for each $\lambda \in \overline{L}$, α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

(2) Now, assume $d = p^\ell$ for some positive integer ℓ. Then a simple induction on n shows that for each $\gamma \in L$, we have (see equations (6.1.1) and (6.1.2)) that

$$P_{n,\gamma}(\lambda) = f_\lambda^n(\gamma) = \gamma^{p^{n\ell}} + \sum_{i=0}^{n-1} \lambda^{p^{i\ell}}. \tag{6.3.1}$$

Now, assume also that $\alpha - \beta \in \overline{\mathbb{F}}_p$; more precisely, we assume $\nu := \alpha - \beta \in \mathbb{F}_{p^{\ell m}}$ for some positive integer m. Then equation (6.3.1) yields that for each $n \geq 1$, we have

$$f_\lambda^n(\alpha) - f_\lambda^n(\beta) = \nu^{p^{n\ell}}; \tag{6.3.2}$$

moreover, the elements $\left\{\nu^{p^{n\ell}}\right\}_{n\geq 1}$ cycle among the values

$$\nu, \nu^{p^\ell}, \nu^{p^{2\ell}}, \cdots, \nu^{p^{(m-1)\ell}} \text{ (since } \nu^{p^{m\ell}} = \nu).$$

Therefore, α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ, as desired in the "moreover" claim from the conclusion of Theorem 6.1.

(3) Finally, if $\alpha^d = \beta^d$, we note that for each $\lambda \in \overline{L}$, we have that $f_\lambda(\alpha) = f_\lambda(\beta)$, and therefore, α is preperiodic under the action of f_λ if and only if β is preperiodic under the action of f_λ.

This concludes our proof that whenever one of the conditions (1)–(3) from the conclusion of Theorem 6.1 holds, then for each $\lambda \in \overline{L}$, we have that α is preperiodic for f_λ if and only if β is preperiodic under the action of f_λ. Furthermore, according to Proposition 6.3, we know there exist infinitely many $\lambda \in \overline{L}$ such that α (and therefore, also β) is preperiodic under the action of f_λ. Therefore, either one of the conditions (1)–(3) from Theorem 6.1 yields the existence of infinitely many $\lambda_n \in \overline{L}$ such that both α and β are preperiodic under the action of f_{λ_n}. Clearly (see also Remark 3.9), for each such $\lambda_n \in \overline{L}$, we have

$$\widehat{h}_{f_{\lambda_n}}(\alpha) = \widehat{h}_{f_{\lambda_n}}(\beta) = 0. \tag{6.3.3}$$

In particular, equation (6.3.3) allows us to establish the converse statement in Theorem 6.1. So, it remains to prove the direct implication from the statement of Theorem 6.1; this is done in Subsection 6.4.

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6.4 Conclusion of the proof of Theorem 6.1

In this subsection, we prove the last remaining statement from Theorem 6.1; that is, that if there exist infinitely many $\lambda_n \in \overline{L}$ such that $\lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\alpha) = \lim_{n\to\infty} \widehat{h}_{f_{\lambda_n}}(\beta) = 0$, then at least one of the conditions (1)–(3) from the conclusion of Theorem 6.1 must hold.

Theorem 4.1 shows that condition (6.0.1) yields that for each $\lambda \in \overline{L}$ and for each place $v \in \Omega = \Omega_L$, we have that

$$\widehat{h}_{v,\lambda}(\alpha) = \widehat{h}_{v,\lambda}(\beta) \tag{6.4.1}$$

(i.e., hypothesis (5.0.1) from Proposition 5.1 is met). Then Proposition 5.1 yields that

(1) either both α and β live in $\overline{\mathbb{F}}_p$;
(2) or $d = p^\ell$ for some positive integer ℓ;
(3) or $\alpha^d = \beta^d$.

So, it remains to prove that when $d = p^\ell$, then we *must* have that also $\alpha - \beta \in \overline{\mathbb{F}}_p$. Indeed, we will see that the existence of a *single* $\lambda_1 \in \overline{L}$ such that both α and β are preperiodic under the action of

$$f_{\lambda_1}(x) = x^d + \lambda_1 = x^{p^\ell} + \lambda_1 \tag{6.4.2}$$

would give that $\alpha - \beta \in \overline{\mathbb{F}}_p$. Now, the existence of such $\lambda_1 \in \overline{L}$ comes from the fact that once α is preperiodic under the action of some f_{λ_1} (and there are infinitely many such parameters $\lambda_1 \in \overline{L}$ according to Proposition 6.3), then actually α is preperiodic under the action of $f_{\sigma(\lambda_1)}$ for any $\sigma \in \mathrm{Gal}(L^{\mathrm{sep}}/L)$ (since $\sigma(f^n_{\lambda_1}(\alpha)) = f^n_{\sigma(\lambda_1)}(\alpha)$ for any n, because $\alpha \in L$ is fixed by σ). But then (see Remark 3.9), we have

$$\widehat{h}_{v,\sigma(\lambda_1)}(\alpha) = 0 \text{ for each } v \in \Omega \text{ and for each } \sigma \in \mathrm{Gal}(L^{\mathrm{sep}}/L). \tag{6.4.3}$$

Equations (6.4.3) and (6.4.1) yield that $\widehat{h}_{v,\sigma(\lambda_1)}(\beta) = 0$ for each $v \in \Omega$ and for each $\sigma \in \mathrm{Gal}(L^{\mathrm{sep}}/L)$. Using equations (2.0.6), (3.3.1), and (3.5.1) along with Lemma 3.10, we conclude that $\widehat{h}_{f_{\lambda_1}}(\beta) = 0$. But then, as noted in Remark 3.9, also the converse holds: we must have that β must be preperiodic under the action of f_{λ_1}.

So, knowing that both α and β are preperiodic under the action of f_{λ_1}, we argue as in Subsection 6.3 (see equations (6.3.1), (6.3.2), and (6.4.2)) and writing $\gamma = \alpha - \beta$, then we get that the sequence $\left\{\gamma^{p^{\ell n}}\right\}_{n\geq 1}$ must consist of finitely many distinct elements (because its elements are the differences of the elements in the orbits of α and, respectively, of β under the action of f_{λ_1}). This can only happen if $\gamma \in \overline{\mathbb{F}}_p$, as desired.

This concludes our proof of Theorem 6.1.

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