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BANACH SPACE OPERATORS WITH A BOUNDED H^∞ FUNCTIONAL CALCULUS

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Abstract

In this paper, we give a general definition for $f(T)$ when T is a linear operator acting in a Banach space, whose spectrum lies within some sector, and which satisfies certain resolvent bounds, and when f is holomorphic on a larger sector.

We also examine how certain properties of this functional calculus, such as the existence of a bounded H^∞ functional calculus, bounds on the imaginary powers, and square function estimates are related. In particular we show that, if T is acting in a reflexive L^p space, then T has a bounded H^∞ functional calculus if and only if both T and its dual satisfy square function estimates. Examples are given to show that some of the theorems that hold for operators in a Hilbert space do not extend to the general Banach space setting.

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1. Introduction and notation

Operators whose spectrum lies in some sector of the complex plane, and whose resolvents satisfy certain bounds, have been extensively studied, both in abstract settings and for their applications to differential equations. For example the m-accretive and m-sectorial operators studied in [12] fall into this class. An extensive list of examples of such operators may be found in [17], which also includes a good description of some of the applications: diffusion semigroups, Stokes' operators, etc. Many problems in analysis depend on finding bounds on certain functions of such an operator, such as square function estimates, or bounds on the imaginary powers (see, for example, [5, 8, 10, 16, 17, 18]). Similar estimates and bounds have been

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proved by transplanting techniques from harmonic analysis, for special cases of such operators, particularly for generators of contraction semigroups (see [3, 4, 19]), but it has not been clarified to what extent these results depend on the particular properties of the operators used.

If T is a closed operator in a Banach space, with non-empty resolvent set, then the Riesz functional calculus allows us to form the bounded operator $f(T)$ for functions f which are bounded and holomorphic on some neighbourhood of the spectrum of T, including a neighbourhood of ∞ when T is unbounded. See, for example, [6, p. 600]. In many of the above applications however, the spectrum of T belongs to a sector of the complex plane with vertex at zero, while the function f is holomorphic on the interior of a larger sector, but not in a neighbourhood of zero or a neighbourhood of ∞. The most common examples in applications occur when the functions f_s are defined by $f_s(z) = z^{is}$, where s is real, in which case $f(T) = T^{is}$.

Our first aim in this paper is to define $f(T)$ for such holomorphic functions of T. The functions f do not need to be bounded at zero or ∞. However it is of interest to know whether $f(T)$ is a bounded operator whenever f is a bounded function. In the case when T is acting in a Hilbert space, necessary and sufficient conditions on T for this to be so have been given by McIntosh [15]. See also [16].

Our main aim is to examine the relationships between the various types of estimates that arise naturally in applications, and the boundedness of $f(T)$ for bounded holomorphic f, when T is acting in a general Banach space, or, more specifically, in an L^p space. It is hoped that by considering these matters in this more general framework, we may be able to cast light on the connections between many of the results in this area.

We remark that, for particular operators, it can be quite difficult to determine whether $f(T)$ is bounded when f is bounded, or, more particularly, whether T^{is} is bounded when s is real.

Throughout, X denotes a complex Banach space. By an *operator in* X we shall mean a linear mapping $T : \mathscr{D}(T) \to X$ whose *domain* $\mathscr{D}(T)$ is a linear subspace of X. The *range* of T is denoted by $\mathscr{R}(T)$ and the *nullspace* by $\mathscr{N}(T)$. The *norm* of T is the (possibly infinite) number

$$\|T\| = \sup\{\|Tu\| : u \in \mathscr{D}(T),\ \|u\| = 1\}.$$

We say that T is *bounded* if $\|T\| < \infty$ and defined *on* X if $\mathscr{D}(T) = X$. The algebra of all bounded operators on X is denoted by $\mathscr{L}(X)$. We call T *densely-defined* if $\mathscr{D}(T)$ is dense in X and *closed* if its graph $\{(u, Tu) : u \in \mathscr{D}(T)\}$ is a closed subspace of $X \times X$. The *spectrum* and *resolvent set* of T are denoted by $\sigma(T)$ and $\rho(T)$ respectively. The former set is the complement of the latter, which is the set of all complex λ for which there exists a bounded operator, called the *resolvent* and denoted $R_T(\lambda)$, such that $(\lambda I - T)R_T(\lambda)$ is the identity operator I, and $R_T(\lambda)(\lambda I - T)$ is the identity operator on $\mathscr{D}(T)$.

For θ in $[0, \pi)$, we define the open and closed sectors of angle θ, and the corresponding strips, in the complex plane $\mathbf{C}$:

$$\begin{aligned} S_\theta^0 &= \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \theta\}, \\ S_\theta &= \{z \in \mathbf{C} \setminus \{0\} : |\arg z| \le \theta\} \cup \{0\}, \\ \Sigma_\theta^0 &= \{z \in \mathbf{C} : |\operatorname{Im} z| < \theta\}, \\ \Sigma_\theta &= \{z \in \mathbf{C} : |\operatorname{Im} z| \le \theta\}. \end{aligned}$$

DEFINITION 1.1. An operator T in X is said to be of *type* ω, where $\omega \in [0, \pi)$, if T is closed, $\sigma(T) \subseteq S_\omega$, and for each θ in (ω, π),

$$\left\|(T - zI)^{-1}\right\| \le C\,|z|^{-1} \qquad \forall z \in \mathbf{C} \setminus S_\theta.$$

Our aim in this paper is to examine the holomorphic functional calculus possessed by an operator of type ω. If $0 < \mu \le \pi$, we denote by $H(S_\mu^0)$ the space of all holomorphic functions on S_μ^0. For the moment, let ψ denote the rational function $\zeta \mapsto \zeta/(1+\zeta)^2$. We shall employ the following subspaces of $H(S_\mu^0)$:

$$H^\infty(S_\mu^0) = \left\{f \in H(S_\mu^0) : \|f\|_\infty < \infty\right\},$$

where $\|f\|_\infty = \sup\left\{|f(z)| : z \in S_\mu^0\right\}$,

$$\Psi(S_\mu^0) = \left\{f \in H(S_\mu^0) : \exists s \in \mathbf{R}^+,\ f\psi^{-s} \in H^\infty(S_\mu^0)\right\},$$

$$\Phi(S_\mu^0) = \left\{f \in H^\infty(S_\mu^0) : \exists a, b \in \mathbf{R},\ f - a - b(1+\cdot)^{-1} \in \Psi(S_\mu^0)\right\},$$

and

$$\mathscr{F}(S_\mu^0) = \left\{f \in H(S_\mu^0) : \exists s \in \mathbf{R}^+,\ f\psi^{s} \in H^\infty(S_\mu^0)\right\},$$

so that

$$\Psi(S_\mu^0) \subset \Phi(S_\mu^0) \subset H^\infty(S_\mu^0) \subset \mathscr{F}(S_\mu^0) \subset H(S_\mu^0).$$

In Section 2, following [15], we describe a functional calculus for an operator T of type ω, which is one-to-one and has dense domain and dense range. If $\mu > \omega$, then $f(T)$ is defined for any f in $\mathscr{F}(S_\mu^0)$. It is clear from the definitions that $f(T)$ is bounded if $f \in \Psi(S_\mu^0)$. It is of interest to examine when the H^∞ functional calculus is bounded, that is, when $f(T)$ is automatically bounded for every f in $H^\infty(S_\mu^0)$, and various results on this question make up the bulk of this paper. More precisely, in Section 3, we consider operators of type ω whose domain or range is not dense, and we examine when there are restriction and quotient operators with dense domain and dense range. In Section 4, we consider the H^∞ functional calculus proper. We establish a condition on T, called $(W(\psi))$, a type of weak square function estimate, which is equivalent to the functional calculus of T being bounded. We also show that more classical functional calculi, such as L^p multiplier theory on $\mathbf{R}^n$, fit into our general framework. Section 5 is devoted to consideration of the purely imaginary powers T^{is}. In the Hilbert space case, the boundedness of T^{is} for all real s is equivalent to T having a bounded H^∞ functional calculus, but in general, as we show here, this is not true. In Section 6, we consider square functions (also known as g-functions), and show (for L^p spaces) that square function bounds are equivalent to a bounded H^∞ functional calculus.

We use here the "variable constant convention", according to which C, C_1, ..., denote constants (in $\mathbf{R}^+$) which may vary from one occurrence to the next. In a given formula, the constant does not depend on variables expressly quantified after the formula, but it may depend on variables quantified (implicitly or explicitly) before. Thus, in Definition 1.1, C may depend on X, T, ω, and θ, but not on z.

The Fourier transformation and its inverse are denoted $\hat{}$ and $\check{}$ respectively.

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2. A functional calculus for operators of type ω

In this section, we review a number of known results on functional calculus for operators of type ω. We start by recalling the definition of $r(T)$, where T is a one-to-one operator of type ω with dense domain and dense range, and r is a rational function with no poles in $\sigma(T)$ (from [6, VII.9]). In [15], McIntosh showed that when X is a Hilbert space, then such an operator T has a natural functional calculus for the functions $\mathscr{F}(S^0_\mu)$ where $0 \leq \omega < \mu < \pi$. Here we sketch this theory and observe that it extends to Banach spaces. Finally, we recall one of the key theorems of [15], on equivalent conditions for an operator on a Hilbert space to have an H^∞ functional calculus. In sections 4 to 6, we will examine analogues of these conditions on Banach spaces and consider the implications between them which continue to hold in this more general context.

Let T be a one-to-one operator of type ω with dense domain and dense range in a Hilbert or Banach space. If p is a polynomial, then $p(T)$ may be defined in a natural way, and, because the resolvent of T is nontrivial, $p(T)$ is densely defined. If q denotes a polynomial with no zeros in $\sigma(T)$, and $r = p/q$, then $r(T)$ is defined by $r(T) = p(T)q(T)^{-1}$. This too is a densely defined operator, which is independent of the choice of p and q used to represent r. Its domain is $\mathscr{D}(T^n)$, where $n = \max\{0, \deg p - \deg q\}$. If T is of type ω, then so is T^{-1}, so $r(T)$ is also well-defined when r is a rational function which is holomorphic at infinity and has no poles in $\sigma(T) \setminus \{0\}$. Combining these facts, $r(T)$ is well-defined, with dense domain, if r is rational with no poles in $\sigma(T) \setminus \{0\}$.

This functional calculus may be extended to $\mathscr{F}(S_\mu^0)$ as follows. For ψ in $\Psi(S_\mu^0)$, $\psi(T)$ is defined by a contour integral. More specifically, suppose that $\omega < \theta < \mu$ and that γ is the contour defined thus:

$$\gamma(t) = \begin{cases} -te^{-i\theta}, & \text{if } -\infty < t \le 0, \\ te^{i\theta}, & \text{if } 0 \le t < \infty. \end{cases}$$

Then $\psi(T)$ in $\mathscr{L}(X)$ is defined by

$$\psi(T) = \frac{1}{2\pi i}\int_\gamma (\zeta I - T)^{-1}\psi(\zeta)\,d\zeta. \tag{2.1}$$

This integral converges absolutely in the norm topology of $\mathscr{L}(X)$. It may be shown that the definition is independent of θ in (ω, μ), and that, if ψ is a rational function, then this definition is consistent with the previous one. Further, $(\psi\psi')(T) = \psi(T)\psi'(T)$, for ψ, ψ' in $\Psi(S_\mu^0)$. As a consequence, if $\phi \in \Phi(S_\mu^0)$, then we may define $\phi(T)$ by the rule

$$\phi(T) = aI + b(I+T)^{-1} + \left[\phi - a - b(1+\cdot)^{-1}\right](T),$$

where $\phi - a - b(1+\cdot)^{-1} \in \Psi(S_\mu^0)$. This definition is consistent with the previous definitions, and always defines a bounded operator. In particular, if $\omega < \pi/2$, then by taking $\phi(z) = e^{-tz}$, where $t \in \mathbf{R}^+$, we obtain the well known result that $-T$ generates a bounded semigroup.

Suppose now that $f \in \mathscr{F}(S_\mu^0)$, so that $f\psi^s$ is bounded for some positive s, where $\psi(\zeta) = \zeta/(1+\zeta)^2$. Choose an integer k larger than s. Then ψ^k is a rational function with no zeros in $\sigma(T) \setminus \{0\}$, and $f\psi^k \in \Psi(S_\mu^0)$, so we may define $f(T)$ by

$$f(T) = \psi^{-k}(T)(f\psi^k)(T).$$

(The domain of $f(T)$ is the set of all u for which $(f\psi^k)(T)u$ is in $\mathscr{D}(\psi^{-k}(T))$). We remark that the operator $\psi^k(T)$ is one-to-one with dense range. Thus $f(T)$ is a closed operator which is densely defined, because its domain includes $\mathscr{R}(\psi^k(T))$, as may be seen by noting that

$$\begin{aligned} f(T)\psi^k(T) &= \psi(T)^{-k}(f\psi^k)(T)\psi^k(T) \\ &= \psi(T)^{-k}\psi^k(T)(f\psi^k)(T) \\ &= (f\psi^k)(T). \end{aligned}$$

It is not difficult to show that this definition is consistent with those above. Moreover if $f, f_1 \in \mathscr{F}(S_\mu^0)$ and $\alpha \in \mathbf{C}$, then

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(i) $$\alpha(f(T)) + f_1(T) = (\alpha f + f_1)(T)|_{\mathscr{D}(f(T))\cap\mathscr{D}(f_1(T))},$$
(ii) $$f_1(T)f(T) = (f_1 f)(T)|_{\mathscr{D}(f(T))\cap\mathscr{D}(f_1 f(T))}.$$

It should be noted that $f(T)$ may be unbounded even if f is bounded and it is an interesting problem to show that for certain operators T, the operators $f(T)$ are bounded for all f in $H^\infty(S^0_\mu)$.

It should also be noted that the above procedure is just one way of defining a functional calculus for these functions. It is possible to show however that any other functional calculus for $\mathscr{F}(S^0_\mu)$, subject to the requirement that $p_k(T) = T^k$ for all nonnegative integers k (where $p_k(\zeta) = z^k$ for all complex numbers z), and a minimal continuity condition, agrees with the one just described. The proof requires the repeated use of the following result, whose proof copies that when X is a Hilbert space [15].

LEMMA 2.1. (Convergence Lemma.) *Suppose that T is a one-to-one operator of type ω with dense domain and dense range in X, and that $\mu > \omega$. Let $\{f_\alpha\}$ be a uniformly bounded net of functions in $\Psi(S^0_\mu)$ which converges to a function f in $H^\infty(S^0_\mu)$ uniformly on compact subsets of S^0_μ. Suppose further that the operators $f_\alpha(T)$ are uniformly bounded on X. Then $f_\alpha(T)u$ converges to $f(T)u$ for all u in X and consequently $f(T)$ is a bounded linear operator on X, and $\|f(T)\| \le \sup_\alpha \|f_\alpha(T)\|$.*

The Convergence Lemma is a useful technical tool. In many cases, it allows us to prove that a formula which holds for analytic functions also holds for functions of an operator. Many authors have worked hard to show just this in particular cases. In Theorem 5.1 an example of this use may be found. Another use is in proving the following result.

COROLLARY 2.2. *Suppose that T is a one-to-one operator of type ω with dense domain and dense range in X, and that $\mu > \omega$. Suppose that*

$$\|f(T)\| \le C\,\|f\|_\infty \qquad \forall f \in \Psi(S^0_\mu).$$

Then $f(T) \in \mathscr{L}(X)$ and

$$\|f(T)\| \le C\,\|f\|_\infty \qquad \forall f \in H^\infty(S^0_\mu).$$

PROOF. It suffices to find a uniformly bounded net of functions in $\Psi(S^0_\mu)$ which converges to 1 uniformly on compact subsets of S^0_μ. One such net is $\{f_{\varepsilon,N}\}$, where

$$f_{\varepsilon,N}(z) = \left(1 + \frac{z}{N}\right)^{-1} - \left(1 + \frac{z}{\varepsilon}\right)^{-1} \qquad \forall z \in S^0_\mu,\ \forall \varepsilon, N \in \mathbf{R}^+,$$

where $\varepsilon \to 0$ and $N \to \infty$.

One use for this result is in proving boundedness theorems by transference methods. For example, Theorem 1 of [4] is best proved by applying the Coifman–Weiss transference theorem [3] to show that $f(T)$ is bounded on $L^p(M)$ for all f in $\Psi(S_\mu^0)$, and then using the above corollary to deal with general f in $H^\infty(S_\mu^0)$.

We end this section by summarizing some of the related Hilbert space results.

THEOREM 2.3. *Let T be a one-to-one operator of type ω in a Hilbert space. Then T has dense domain and dense range.*

The proof of this will be given in the next section.

THEOREM 2.4. *Let T be a one-to-one operator of type ω in a Hilbert space $\mathscr{H}$. Then the following statements are equivalent:*

(i) *for every μ in (ω, π), T has a bounded $H^\infty(S_\mu^0)$ functional calculus, that is, for all f in $H^\infty(S_\mu^0)$, $f(T) \in \mathscr{L}(X)$ and*

$$\|f(T)\| \le C\,\|f\|_\infty \qquad \forall f \in H^\infty(S_\mu^0);$$

(ii) *there exists μ in (ω, π) such that T has a bounded $H^\infty(S_\mu^0)$ functional calculus;*

(iii) *$\{T^{is} : s \in \mathbf{R}\}$ is a C^0 group, and for every μ in (ω, π),*

$$\left\|T^{is}\right\| \le Ce^{\mu|s|} \qquad \forall s \in \mathbf{R};$$

(iv) *if A and B are non-negative self-adjoint operators and U and V are isometries such that $T = UA$ and $T^* = VB$, then for all α in $(0, 1)$, $\mathscr{D}(T^\alpha) = \mathscr{D}(A^\alpha)$, $\mathscr{D}(T^{*\alpha}) = \mathscr{D}(B^\alpha)$, and*

$$C^{-1}\,\|A^\alpha u\| \le \|T^\alpha u\| \le C\,\|A^\alpha u\| \qquad \forall u \in \mathscr{D}(T^\alpha)$$

and

$$C^{-1}\,\|B^\alpha v\| \le \left\|T^{*\alpha} v\right\| \le C\,\|B^\alpha v\| \qquad \forall v \in \mathscr{D}(T^{*\alpha});$$

(v) *for every μ in (ω, π), and every ψ in $\Psi(S_\mu^0)$,*

$$C^{-1}\,\|u\| \le \left[\int_0^\infty \|\psi(tT)u\|^2\,\frac{dt}{t}\right]^{1/2} \le C\,\|u\| \qquad \forall u \in \mathscr{H}$$

and

$$C^{-1}\,\|u\| \le \left[\int_0^\infty \|\psi(tT^*)u\|^2\,\frac{dt}{t}\right]^{1/2} \le C\,\|u\| \qquad \forall u \in \mathscr{H};$$

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(vi) *there exist μ in (ω, π) and ψ, $\tilde{\psi}$ in $\Psi(S_\mu^0)$ such that $\psi(x)$, $\tilde{\psi}(x) > 0$ whenever $x > 0$, and*

$$\left[\int_0^\infty \|\psi(tT)u\|^2 \frac{dt}{t}\right]^{1/2} \leq C\,\|u\| \qquad \forall u \in \mathscr{H},$$

and

$$\left[\int_0^\infty \|\tilde{\psi}(tT^*)u\|^2 \frac{dt}{t}\right]^{1/2} \leq C\,\|u\| \qquad \forall u \in \mathscr{H}.$$

Theorem 2.4 is essentially proved in [15], which is based on earlier work of Yagi [21] and many others before him.

Even in a Hilbert space, there exist invertible closed operators of type ω which do not have a bounded $H^\infty(S_\mu^0)$ functional calculus. Examples are given in [16] (for which $\omega > 0$) and [1] (for which $\omega = 0$).

3. Decompositions and dual pairs

The functional calculus described in the last section requires that T is a one-to-one operator with dense domain and dense range. In this section we shall discuss how one may deal with operators of type ω for which these conditions do not hold.

An important concept in what follows will be duality. We say that the Banach spaces $\langle X, Y\rangle$ form a *dual pair* if there is a bilinear form $\langle\cdot,\cdot\rangle$ on $X \times Y$ such that

$$|\langle u, v\rangle| \leq C_0\,\|u\|\,\|v\| \qquad \forall u \in X,\ \forall v \in Y,$$
$$\|u\| \leq C_1 \sup\{|\langle u, v\rangle| : v \in Y,\ \|v\| = 1\} \qquad \forall u \in X,$$
$$\|v\| \leq C_2 \sup\{|\langle u, v\rangle| : u \in X,\ \|u\| = 1\} \qquad \forall v \in Y.$$

It is clear that X and Y form a dual pair if and only if Y and X do. If X_0 and Y_0 are closed subspaces of X and Y respectively, and if $\langle X, Y\rangle$ and $\langle X_0, Y_0\rangle$ form a dual pair, with the same bilinear form, then we call $\langle X_0, Y_0\rangle$ a dual subpair of the dual pair $\langle X, Y\rangle$.

EXAMPLE 3.1. From the classical theory of Banach spaces, $\langle \ell^p, \ell^{p'}\rangle$ (where $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$) and $\langle c_0, \ell^1\rangle$, equipped with the bilinear form $\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{k=1}^\infty u_k v_k$, form dual pairs. In these cases we may take $C_0 = C_1 = C_2 = 1$.

More generally, in the case when X is a Hilbert space, then we may take $Y = X$ and use the inner product for the duality. (The fact that the inner product is conjugate linear rather than linear in the second variable causes no problems.) When $X = L^p(\Omega)$, $1 \leq p \leq \infty$, for some measure space Ω, then we may take the usual dual pairing with the conjugate space $L^{p'}(\Omega)$.

It is clear from our definitions that if $\langle X, Y\rangle$ form a dual pair, then Y is isomorphic to a closed linear subspace of X^*, the Banach space dual of X (and of course X is isomorphic to a closed subspace of Y^*). Indeed, the Hahn-Banach Theorem (which of course requires the Axiom of Choice in general) implies that $\langle X, X^*\rangle$ always forms a dual pair of Banach spaces. Another consequence of the Hahn-Banach Theorem is that if $\langle X, Y\rangle$ form a dual pair and X is reflexive, then Y must actually be isomorphic to X^*.

DEFINITION 3.2. Suppose that $\langle X, Y\rangle$ form a dual pair and that T and S are operators in X and Y respectively. Then T and S are said to be *dual operators,* or S is said to be dual to T, if both T and S are closed and

$$\langle Tu, v\rangle = \langle u, Sv\rangle \qquad \forall u \in \mathscr{D}(T),\ \forall v \in \mathscr{D}(S).$$

PROPOSITION 3.3. *Suppose that $\langle X, Y\rangle$ form a dual pair, and that T and T' are dual operators in X and Y respectively. If $T \in \mathscr{L}(X)$, then $\|T'\| \le C_2C_0\|T\|$, where the constants C_0 and C_2 are those in the definition of dual pairs.*

PROOF. This is routine.

(We cannot conclude that $T' \in \mathscr{L}(Y)$ in Proposition 3.3, because $\mathscr{D}(T')$ may be a proper closed subspace of Y.)

This proposition raises the questions of the existence and uniqueness of dual operators. If $T \in \mathscr{L}(X)$, the existence of T' in $\mathscr{L}(Y)$ such that T and T' are dual operators is not guaranteed. Consider for example the operator T on ℓ^1 defined by $T\mathbf{u} = (\sum_{j=1}^{\infty} u_j, 0, \dots)$. This has a dual operator $T'\mathbf{v} = (v_1, v_1, \dots)$ on ℓ^∞ but no dual operator on c_0. However, if a dual operator T' exists in $\mathscr{L}(Y)$, then it is unique.

LEMMA 3.4. *Suppose that T, T' are dual operators in the dual pair $\langle X, Y\rangle$. Then T^2, T'^2 are dual operators.*

PROOF. Since $\mathscr{D}(T^2) \subseteq \mathscr{D}(T)$ and $\mathscr{D}(T'^2) \subseteq \mathscr{D}(T')$,

$$\left\langle T^2u, v\right\rangle = \left\langle Tu, T'v\right\rangle = \left\langle u, T'^2\right\rangle \qquad \forall u \in \mathscr{D}(T^2),\ \forall v \in \mathscr{D}(T'^2).$$

The same methods clearly enable us to prove the following.

PROPOSITION 3.5. *Suppose that T, T' are dual operators in the dual pair $\langle X, Y\rangle$ and that p is a polynomial. Then $p(T)$, $p(T')$ are dual operators.*

PROPOSITION 3.6. *Suppose that T, T' are dual operators in the dual pair $\langle X, Y\rangle$ and that $\lambda \in \rho(T) \cap \rho(T')$. Then $R_T(\lambda)$, $R_{T'}(\lambda)$ are dual operators.*

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PROOF. Clearly $(\lambda I - T)$ and $(\lambda I - T')$ are dual operators. Further, $R_T(\lambda)$ and $R_{T'}(\lambda)$ exist and have domains X and Y respectively. Thus, for all u in X and v in Y,

$$\begin{aligned}\langle R_T(\lambda)u, v\rangle &= \big\langle R_T(\lambda)u, (\lambda I - T')R_{T'}(\lambda)\big\rangle \\ &= \langle(\lambda I - T)R_T(\lambda)u, R_{T'}(\lambda)\rangle \\ &= \langle u, R_{T'}(\lambda)\rangle .\end{aligned}$$

It is a corollary of this result and the above remarks that T may have at most one dual operator T' for which $\rho(T) \cap \rho(T')$ is nonvoid. In particular, an operator T of type ω may have at most one dual operator T' of type ω. (We are not assuming that T has dense domain.)

COROLLARY 3.7. *Suppose that T, T' are dual operators in the dual pair $\langle X, Y\rangle$, and that both T and T' are of type ω, where $0 \le \omega < \pi$. If $\mu > \omega$, and $\phi \in \Phi(S^0_\mu)$, then $\phi(T)$ and $\phi(T')$ are dual.*

PROOF. Suppose first that $\psi \in \Psi(S^0_\mu)$. Then (2.1) (the definition of $\psi(T)$) and Proposition 3.6 show that

$$\begin{aligned}\langle \psi(T)u, v\rangle &= \frac{1}{2\pi i}\int_\gamma \psi(\lambda)\,\langle R_T(\lambda)u, v\rangle\, d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma \psi(\lambda)\,\langle u, R_{T'}(\lambda)v\rangle\, d\lambda \\ &= \big\langle u, \psi(T')v\big\rangle \qquad \forall u \in X,\ \forall v \in Y.\end{aligned}$$

In general, $\phi(\cdot) = \phi(\infty) + [\phi(0) - \phi(\infty)](\cdot + 1)^{-1} + \psi(\cdot)$, where $\psi \in \Psi(S^0_\mu)$, and so

$$\begin{aligned}\langle \phi(T)u, v\rangle &= \langle(\phi(\infty)I - [\phi(0) - \phi(\infty)]R_T(-1) + \psi(T))u, v\rangle \\ &= \big\langle u, (\phi(\infty)I - [\phi(0) - \phi(\infty)]R_{T'}(-1) + \psi(T'))v\big\rangle \\ &= \big\langle u, \phi(T')v\big\rangle \qquad \forall u \in X,\ \forall v \in Y.\end{aligned}$$

Our aim in the remainder of this section is to show that, at least in reflexive spaces, one may always factor off the nullspace of an operator of type ω. Similar results in slightly different situations may be found in [11] and [13]. See the discussion after the proof of the following theorem.

THEOREM 3.8. *Let T be an operator of type ω in X, and define X_0, X_{00}, and X_∞ thus:*

$$X_0 = \left\{u \in X : \lim_{n\to\infty} (I + nT)^{-1}u \quad \textit{exists}\right\},$$
$$X_{00} = \left\{u \in X : \lim_{n\to\infty} (I + nT)^{-1}u = 0\right\},$$
$$X_\infty = \left\{u \in X : \lim_{n\to\infty} (I + n^{-1}T)^{-1}u = u\right\}.$$

Then these are all closed linear subspaces of X, and $X_0 = \mathscr{N}(T) \oplus \overline{\mathscr{R}(T)}$, $X_{00} = \overline{\mathscr{R}(T)}$ and $X_\infty = \overline{\mathscr{D}(T)}$.

Let T_0 be the restriction of T to the domain $\mathscr{D}(T) \cap X_0$, and define T_{00} and T_∞ analogously. Then these are operators of type ω in X_0, X_{00} and X_∞ respectively, and T_{00} is one-to-one with dense range, and T_∞ has dense domain. Moreover, the restriction $\widetilde{T}$ of T to the domain $\mathscr{D}(\widetilde{T}) = \mathscr{D}(T_{00}) \cap \mathscr{D}(T_\infty)$ is a one-to-one operator of type ω with dense domain and dense range in $\widetilde{X} = X_{00} \cap X_\infty$.

If X is reflexive, then $X = X_0 = X_\infty$.

PROOF. Recall that since T is of type ω,

$$\|(I + nT)^{-1}\| = \frac{1}{n}\left\|\left(-\frac{1}{n}I - T\right)^{-1}\right\| \le \frac{1}{n}\frac{C}{1/n} = C. \tag{3.1}$$

This implies that X_0, X_{00}, and X_∞ are closed subspaces of X; we shall give the details of this for X_0 only. Suppose that u lies in the closure of X_0. Then, for any v in X_0, and positive integers n, n',

$$\begin{aligned}(I + nT)^{-1}u - (I + n'T)^{-1}u = {} & \left[(I + nT)^{-1}u - (I + nT)^{-1}v\right] \\ & + \left[(I + nT)^{-1}v - (I + n'T)^{-1}v\right] \\ & + \left[(I + n'T)^{-1}v - (I + n'T)^{-1}u\right].\end{aligned}$$

Since the norms of the terms in the first and last square parentheses on the right hand side may be made small, uniformly in n and n', by taking v close enough to u, and the middle term may then be made small by taking n and n' large enough, the sequence $(I + nT)^{-1}u$ is Cauchy, and so converges, that is, u lies in X_0.

Let $P : X_0 \to X$ be the linear map given by

$$Pu = \lim_{n\to\infty} (I + nT)^{-1}u.$$

Then $\|P\| \le C$, from (3.1). We shall first show that $\mathscr{R}(P) = \mathscr{N}(T) = \mathscr{N}(I - P)$ (whence $P^2 = P - (I - P)P = P$), and then that $\mathscr{N}(P) = \overline{\mathscr{R}(T)}$.

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The required set inclusions will follow from repeated use of the identity

$$(I+nT)^{-1} = I - nT(I+nT)^{-1}.$$

If $u \in \mathcal{N}(T)$, then $(I+nT)u = u$ and $u = (I+nT)^{-1}u$ for all n, whence

$$Pu = \lim_{n\to\infty}(I+nT)^{-1}u = \lim_{n\to\infty} u = u,$$

so $u \in \mathcal{R}(P)$. Suppose that $w \in \mathcal{R}(P)$ and $w = Pu = \lim_{n\to\infty}(I+nT)^{-1}u$. Then

$$T(I+nT)^{-1}u = \frac{1}{n}u - \frac{1}{n}(I+nT)^{-1}u \to 0,$$

as $n \to \infty$. Hence $(I+nT)^{-1}u \to w$ and $T(I+nT)^{-1}u \to 0$. Since T is closed, this implies that $w \in \mathcal{D}(T)$ and that $Tw = 0$. Thus $\mathcal{N}(T) = \mathcal{R}(P)$.

Suppose next that $w \in \mathcal{N}(T)$. This implies, as above, that $Pw = w$. In other words, $\mathcal{N}(T) \subseteq \mathcal{N}(I-P)$. Conversely, if $w \in \mathcal{N}(I-P)$, then $nT(I+nT)^{-1}w \to 0$ as $n \to \infty$, and another argument using the fact that T is closed shows that $w \in \mathcal{N}(T)$.

It now follows that P is a bounded projection on X_0, and that $X_0 = \mathcal{R}(P) \oplus \mathcal{N}(P)$.

We now proceed to show that $\mathcal{N}(P) = \overline{\mathcal{R}(T)}$. If $w \in \mathcal{N}(P)$, that is, $(I+nT)^{-1}u \to 0$ as $n \to \infty$, then $T(T+n^{-1}I)^{-1}u \to u$, and so $u \in \overline{\mathcal{R}(T)}$. On the other hand, if $w \in \mathcal{R}(T)$, with $w = Tu$ say, then

$$(I+nT)^{-1}w = \frac{1}{n}u - \frac{1}{n}(I+nT)^{-1}u \to 0$$

as $n \to \infty$, and so $w \in \mathcal{N}(P)$. Since $\mathcal{N}(P)$ is closed, it follows that $\overline{\mathcal{R}(T)} = \mathcal{N}(P)$.

A similar (but easier) calculation shows that $X_\infty = \overline{\mathcal{D}(T)}$.

It is straightforward to verify that T_0, T_{00}, and T_∞ are of type ω in the Banach spaces X_0, X_{00}, and X_∞, by using the facts already established about domains and ranges. So we may define the subspace $(X_0)_0$ thus:

$$(X_0)_0 = \left\{u \in X_0 : \lim_{n\to\infty}(I+nT)^{-1}u \text{ exists}\right\},$$

and define similarly $(X_{00})_{00}$ and $(X_\infty)_\infty$. Clearly $(X_0)_0 = X_0$, $(X_{00})_{00} = X_{00}$, and $(X_\infty)_\infty = X_\infty$. The statements about domains and ranges follow.

If X is reflexive, then X is locally weakly sequentially compact, by the Eberlein–Šmulian theorem (see [22, p. 141]). Then for any u in X, the sequence $(I+nT)^{-1}u$ has a weakly convergent subsequence, that is, there exist v in X and an increasing sequence of integers $\{n_k\}$ such that $(I+n_kT)^{-1}u \xrightarrow{w} v$ in X. (The symbol $\xrightarrow{w}$ indicates weak convergence.) Then $n_kT(I+n_kT)^{-1}u \xrightarrow{w} u-v$. Now $n_kT(I+n_kT)^{-1}u$ is a bounded sequence in the closed *subspace* $\overline{\mathscr{R}(T)}$, so $u - v \in \overline{\mathscr{R}(T)} \subset X_0$. We shall now show that $v \in \mathscr{N}(T) = \mathscr{N}(I - P)$. For any positive integer m,

$$mT(I + mT)^{-1}v = v - (I + mT)^{-1}v.$$

Now bounded operators are weak-weak continuous, so for all y in X^*, by the resolvent identity,

$$\begin{aligned}
&\langle mT(I + mT)^{-1}v, y\rangle \\
&\quad = \langle v, y\rangle - \lim_{k\to\infty} \langle (I + mT)^{-1}(I + n_kT)^{-1}u, y\rangle \\
&\quad = \langle v, y\rangle - \lim_{k\to\infty} \left\langle \frac{n_k}{n_k - m}(I + n_kT)^{-1}u, y\right\rangle + \lim_{k\to\infty} \left\langle \frac{m}{n_k - m}(I + mT)^{-1}u, y\right\rangle \\
&\quad = \langle v, y\rangle - \langle v, y\rangle + 0 \\
&\quad = 0.
\end{aligned}$$

It follows then that $u \in X_0$.

Similarly, if X is reflexive, then for all u in X there exists v in X and an increasing sequence of integers $\{n_k\}$ such that $(I + n_k^{-1}T)^{-1}u \xrightarrow{\text{w}} v$ as $k \to \infty$. Note that $(I + n_k^{-1}T)^{-1}u$ is in $\mathscr{D}(T)$, so, as before, $v \in \overline{\mathscr{D}(T)} = X_\infty$. It follows from weak-weak continuity that, if $\lambda > 0$, then

$$(T + \lambda I)^{-1}\left(I + \frac{1}{n_k}T\right)^{-1} u \xrightarrow{\text{w}} (T + \lambda I)^{-1}v$$

as $k \to \infty$. Now, by using the resolvent identity, we see that

$$(T+\lambda I)^{-1}\left(I + \frac{1}{n_k}T\right)^{-1} - (T+\lambda I)^{-1} = \frac{1}{n_k}\left[\lambda(T + \lambda I)^{-1} - I\right]\left(I + \frac{1}{n_k}T\right)^{-1} \to 0$$

in $\mathscr{L}(X)$. Thus

$$(T + \lambda I)^{-1}\left(I + \frac{1}{n_k}T\right)^{-1} u \to (T + \lambda I)^{-1}u.$$

This implies that $(T + \lambda I)^{-1}u = (T + \lambda I)^{-1}v$, and so $u = v$. Thus $u \in \overline{\mathscr{D}(T)}$ and so $X_\infty = X$.

One particular case of the above theorem states that every operator T of type ω in a Hilbert space satisfies $\overline{\mathscr{D}(T)} = X$ and $\mathscr{N}(T) \oplus \overline{\mathscr{R}(T)} = X$. This proves Theorem 2.3. Another special case is when $\langle X, Y\rangle$ is a pair $\langle L^p(\Omega), L^{p'}(\Omega)\rangle$ where $1 < p < \infty$. Again we get that $\overline{\mathscr{D}(T)} = X$ and $\mathscr{N}(T) \oplus \overline{\mathscr{R}(T)} = X$. The result that $\mathscr{D}(T)$ is dense when X is reflexive is essentially due to [11]. We have extended the technique to deal with the range of T as well.

Theorem 3.8 shows how one may deal with operators which are not one-to-one. Suppose that T and T' are dual operators of type ω in $\langle X, Y\rangle$, and that X is reflexive. Then $X = \mathscr{N}(T)\oplus\overline{\mathscr{R}(T)}$ and $Y = \mathscr{N}(T')\oplus\overline{\mathscr{R}(T')}$. Further the annihilator of $\overline{\mathscr{R}(T)}$ in Y is $\mathscr{N}(T')$, and the annihilator of $\overline{\mathscr{R}(T')}$ in X is $\mathscr{N}(T)$. Thus $\overline{\mathscr{R}(T')}$ is isomorphic (though not necessarily isometric) to the dual space of $\overline{\mathscr{R}(T)}$, and *vice versa*, that is, $\langle\overline{\mathscr{R}(T)}, \overline{\mathscr{R}(T')}\rangle$ form a dual subpair of the dual pair $\langle X, Y\rangle$. If we define operators $\widetilde{T} = T|_{\overline{\mathscr{R}(T)}}$ and $\widetilde{T}' = T'|_{\overline{\mathscr{R}(T')}}$, then $\widetilde{T}$ and $\widetilde{T}'$ are densely defined dual operators of type ω in the dual pair $\langle\overline{\mathscr{R}(T)}, \overline{\mathscr{R}(T')}\rangle$. The results of this section then, show that at least in reflexive spaces, there is no essential loss of generality if we assume that a dual pair of operators are one-to-one and have dense range, provided that we are happy to work with dual subpairs.

The following examples illustrate these theorems.

EXAMPLE 3.9. Suppose that $X = C[0, 1]$ and that $Tu(s) = su(s)$. Suppose that $Tu = 0$. Then $su(s) = 0$ for all s and so $u(s) = 0$ for all $s \neq 0$. Since u is continuous, it follows that $u = 0$ and that T is one-to-one. However it is easy to see that T is not onto since every function in the image of T vanishes at 0. Thus in this case $X \supsetneq X_0 = \overline{\mathscr{R}(T)}$.

EXAMPLE 3.10. This example illustrates the idea behind the proof of Theorem 3.8. Suppose that $X = \mathbf{C}^n$ and that

$$T = \begin{pmatrix} 0 & & & & \\ & \ddots & & 0 & \\ & & 0 & & \\ & & & \lambda_1 & \\ & 0 & & & \lambda_2 \\ & & & & & \ddots \end{pmatrix}$$

where $\lambda_j > 0$ for each j. Then as $n \to \infty$,

$$(I + nT)^{-1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & & & \frac{1}{1+n\lambda_1} & \\ & 0 & & & \frac{1}{1+n\lambda_2} \\ & & & & & \ddots \end{pmatrix} \to \begin{pmatrix} I & O \\ O & O \end{pmatrix} = P,$$

say. This constructs a projection P for which $\mathscr{R}(P) = \mathscr{N}(T)$ and $\mathscr{N}(P) = \overline{\mathscr{R}(T)}$.

4. H^∞ functional calculus and the condition $(W(\psi))$

We shall return now to the discussion of the relationships between some of the many properties of functional calculi for operators in a Banach space. We also relate our functional calculus to more traditional functional calculi.

We assume throughout this section that $0 \le \omega < \mu < \pi$ and that T is an operator of type ω in X, which is one-to-one, with dense domain and dense range. We shall say that T has a bounded $H^\infty(S_\mu^0)$ functional calculus if it satisfies the following condition:

$$(F_\mu) \qquad \begin{cases} b(T) \in \mathscr{L}(X) & \forall b \in H^\infty(S_\mu^0), \\ \|b(T)\| \le C\,\|b\|_\infty & \forall b \in H^\infty(S_\mu^0). \end{cases}$$

Note that if $b \in H^\infty(S_\mu^0)$, the functions ψ_s, defined by the formula

$$\psi_s(z) = b(z)\frac{z^s}{(1+z)^{2s}} \qquad \forall z \in S_\mu^0,\ \forall s \in \mathbf{R}^+,$$

are in $\Psi(S_\mu^0)$ and satisfy

$$\|\psi_s\|_\infty \le \sup_{z\in S_\mu^0}\left|\frac{z}{(1+z)^2}\right|^s \|b\|_\infty \qquad \forall s \in \mathbf{R}^+.$$

Furthermore, as s tends to 0, the functions ψ_s converge to b uniformly on compact subsets of S_μ^0. It follows from the Convergence Lemma that, in order to show that T satisfies (F_μ), it suffices to show that $\|\psi(T)\| \le C\,\|\psi\|_\infty$ for all ψ in $\Psi(S_\mu^0)$.

THEOREM 4.1. *Suppose that T, T' are a dual pair of operators of type ω which are one-to-one, with dense domain and dense range, acting in the dual pair of Banach spaces $\langle X, Y\rangle$. If T satisfies (F_μ) then T' also satisfies (F_μ) and $b(T)' = b(T')$ for all $b \in H^\infty(S_\mu^0)$.*

PROOF. The fact that T' satisfies (F_μ) follows immediately from the above remark and Corollary 3.7. Suppose that $b \in H^\infty(S_\mu^0)$, and define ψ_s in $\Psi(S_\mu^0)$ as above. Using the Convergence Lemma and Corollary 3.7 we get

$$\begin{aligned} \langle b(T)u, v\rangle &= \lim_{s\to 0+} \langle \psi_s(T)u, v\rangle \\ &= \lim_{s\to 0+} \big\langle u, \psi_s(T')v\big\rangle \\ &= \big\langle u, b(T')v\big\rangle \qquad \forall u \in X,\ \forall v \in Y. \end{aligned}$$

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Suppose now that $\langle X, Y\rangle$ is a dual pair of Banach spaces, and that $\psi \in \Psi(S^0_\mu)$. Consider the following condition which the operator T might satisfy:

$$(W(\psi)) \qquad \int_0^\infty |\langle \psi(tT)u, v\rangle| \frac{dt}{t} \le C\,\|u\|\,\|v\| \qquad \forall u \in X,\ \forall v \in Y.$$

Our aim in this section is to show that the conditions (F_μ) and $(W(\psi))$ may be regarded as being almost equivalent. To show that (F_μ) implies $(W(\psi))$ is relatively straightforward, and we do this first.

THEOREM 4.2. *Suppose that T is a one-to-one operator of type ω in a Banach space X, with dense domain and dense range. If T satisfies (F_μ) and $\psi \in \Psi(S^0_\mu)$ then T satisfies $(W(\psi))$.*

PROOF. Fix u in X, v in Y and positive real numbers ε (small) and N (large). It is easy to see that there exists a Borel function h of modulus 1 such that

$$\int_\varepsilon^N |\langle \psi(tT)u, v\rangle| \frac{dt}{t} = \int_\varepsilon^N \langle \psi(tT)u, v\rangle\, h(t, u, v) \frac{dt}{t}.$$

Define $b : S^0_\mu \to \mathbf{C}$ by the rule

$$b(\zeta) = \int_\varepsilon^N \psi(t\zeta) h(t, u, v) \frac{dt}{t}.$$

Then, since $\psi \in \Psi(S^0_\mu)$, there exist positive C and s such that

$$\|b\|_\infty \le \sup_{\zeta \in S^0_\mu} \int_\varepsilon^N |\psi(t\zeta)| \frac{dt}{t} \le C \sup_{\zeta \in S^0_\mu} \int_0^\infty \left| \frac{\zeta t}{(1+\zeta t)^2} \right|^s \frac{dt}{t} \le C_\psi.$$

Note that although b depends on u, v, ε and N, the bound on $\|b\|_\infty$ depends only on the function ψ and the angle μ. We have then that

$$\begin{aligned}\int_\varepsilon^N |\langle \psi(tT)u, v\rangle| \frac{dt}{t} &= \langle b(T)u, v\rangle \le \|b(T)\|\,\|u\|\,\|v\| \\ &\le C\,\|b\|_\infty\,\|u\|\,\|v\| \le C_\psi\,\|u\|\,\|v\|.\end{aligned}$$

Taking limits as $\varepsilon \to 0$ and $N \to \infty$ gives the result.

We need to impose some extra conditions on the function ψ to ensure that $(W(\psi))$ is strong enough to imply (F_μ).

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An important tool in our work is the following variant of the Mellin transform, $\hat{\psi}_e$. Suppose that $\psi \in \Psi(S^0_\mu)$. The function ψ_e is defined to be the function $\psi \circ \exp$ on the strip Σ^0_μ. The Fourier transform of ψ_e thus satisfies the condition that

$$\begin{aligned}\hat{\psi}_e(\lambda) &= \int_{-\infty}^{\infty} e^{-i\lambda x}\psi_e(x)\,dx = \int_0^\infty \tau^{-i\lambda}\psi(\tau)\frac{d\tau}{\tau} \\ &= \int_0^\infty (\tau e^{i\theta})^{-i\lambda}\psi(\tau e^{i\theta})\frac{d\tau}{\tau} = e^{\theta\lambda}\int_0^\infty \tau^{-i\lambda}\psi(\tau e^{i\theta})\frac{d\tau}{\tau} \qquad \forall\lambda\in\mathbf{R},\ \forall\theta\in(-\mu,\mu).\end{aligned}$$

Thus

$$\left|\hat{\psi}_e(\lambda)\right| \le e^{\theta\lambda}\int_0^\infty \left|\psi(\tau e^{i\theta})\right|\frac{d\tau}{\tau} \le Ce^{\theta\lambda} \qquad \forall\lambda\in\mathbf{R}.$$

In particular,

$$\left|\hat{\psi}_e(\lambda)\right| \le Ce^{-\mu|\lambda|} \qquad \forall\lambda\in\mathbf{R}.$$

A similar proof shows that

$$\left|\frac{d}{d\lambda}\hat{\psi}_e(\lambda)\right| \le Ce^{-\mu|\lambda|} \qquad \forall\lambda\in\mathbf{R}.$$

DEFINITION 4.3. Suppose that $0 < \nu < \pi$. Then

$$\Psi_\nu(S^0_\mu) = \left\{\psi\in\Psi(S^0_\mu) : e^{-\nu|\cdot|}\left|\hat{\psi}_e(\cdot)\right|^{-1}\in L^\infty(\mathbf{R})\right\} \qquad \forall\mu\in(0,\nu),$$

and

$$\Psi_\nu(S^0_\nu-) = \bigcap_{\mu\in(0,\nu)}\Psi_\nu(S^0_\mu).$$

THEOREM 4.4. *Suppose that T is a one-to-one operator of type ω in a Banach space X, with dense domain and dense range, that $\omega < \mu < \nu < \pi$, and that $2\nu - \mu < \eta < \pi$. Suppose also that $\psi \in \Psi_\nu(S^0_\mu)$. If T satisfies $(W(\psi))$, that is, if*

$$\int_0^\infty |\langle\psi(tT)u, v\rangle|\frac{dt}{t} \le C\,\|u\|\,\|v\| \qquad \forall u\in X,\ \forall v\in Y,$$

then T satisfies (F_η).

PROOF. Choose α such that $2\nu - \mu < \alpha < \eta$, and b in $H^\infty(S^0_\eta)$. We must find a bound on $\|b(T)\|$.

We define $\gamma : \mathbf{R}^+ \to \mathbf{C}$ as follows: $\gamma\circ\exp = \gamma_e$, and

$$\hat{\gamma}_e(\lambda) = \frac{1}{\hat{\psi}_e(\lambda)\cosh(\alpha\lambda)} \qquad \forall\lambda\in\mathbf{R}. \tag{4.1}$$

Clearly we have

$$\left|\hat{\gamma}_e(\lambda)\right| \leq Ce^{(\nu-\alpha)|\lambda|} \qquad \forall\lambda \in \mathbf{R}.$$

Furthermore,

$$\left|\hat{\gamma}_e'(\lambda)\right| = \left|-\frac{\hat{\psi}_e'(\lambda)\cosh(\alpha\lambda) + \hat{\psi}_e(\lambda)\alpha\sinh(\alpha\lambda)}{\hat{\psi}_e^2(\lambda)\cosh^2(\alpha\lambda)}\right| \leq Ce^{(2\nu-\mu-\alpha)|\lambda|} \qquad \forall\lambda \in \mathbf{R}.$$

It follows that $\left|\hat{\gamma}_e\right|$ and $\left|\hat{\gamma}_e'\right|$ are both bounded by $Ce^{(2\nu-\mu-\alpha)|\cdot|}$. Now

$$(1+ix)\gamma_e(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}(\hat{\gamma}_e(\lambda) - \hat{\gamma}_e'(\lambda))\,d\lambda \qquad \forall x \in \mathbf{R},$$

so

$$\begin{aligned}
\int_0^\infty |\gamma(\tau)|\,\frac{d\tau}{\tau} &= \int_{-\infty}^{\infty} |\gamma_e(x)|\,dx \\
&\leq \left(\int_{-\infty}^{\infty} |(1+ix)\gamma_e(x)|^2\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty} |(1+ix)|^{-2}\,dx\right)^{1/2} \\
&\leq C\left(\int_{-\infty}^{\infty} \left|\hat{\gamma}_e(\lambda) - \hat{\gamma}_e'(\lambda)\right|^2 d\lambda\right)^{1/2} \\
&\leq C\left(\int_{-\infty}^{\infty} e^{2(2\nu-\mu-\alpha)|\lambda|}\,d\lambda\right)^{1/2} \\
&< \infty
\end{aligned}$$

because $\alpha > 2\nu - \mu$.

We now define functions $\beta^+, \beta^- : \mathbf{R}^+ \to \mathbf{C}$ by

$$\hat{\beta}^+(\lambda) = \frac{1}{2}\hat{\gamma}_e(\lambda)\hat{b}_e(\lambda)e^{-\alpha\lambda} \qquad \forall\lambda \in \mathbf{R}$$

and

$$\hat{\beta}^-(\lambda) = \frac{1}{2}\hat{\gamma}_e(\lambda)\hat{b}_e(\lambda)e^{\alpha\lambda} \qquad \forall\lambda \in \mathbf{R}.$$

Since $\gamma_e \in L^1(\mathbf{R})$ and

$$\left|(\hat{b}_e e^{-\alpha\cdot})^\vee(\tau)\right| = |b_e(\tau + i\alpha)| \leq \|b\|_\infty \qquad \forall\tau \in \mathbf{R},$$

β^+ (and similarly β^-) are in $L^\infty(\mathbf{R}^+)$ and $\left\|\beta^\pm\right\|_\infty \leq C\,\|b\|_\infty$.

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Note that

$$(4.2)\qquad \begin{aligned}&\hat{\beta}^+(\lambda)\hat{\psi}_e(\lambda)+\hat{\beta}^-(\lambda)\hat{\psi}_e(\lambda)\\ &\quad=\frac{\hat{b}_e(\lambda)e^{-\alpha\lambda}}{2\hat{\psi}_e(\lambda)\cosh(\alpha\lambda)}\hat{\psi}_e(\lambda)+\frac{\hat{b}_e(\lambda)e^{\alpha\lambda}}{2\hat{\psi}_e(\lambda)\cosh(\alpha\lambda)}\hat{\psi}_e(\lambda)\\ &\quad=\hat{b}_e(\lambda)\qquad \forall\lambda\in\mathbf{R}.\end{aligned}$$

Inverting the Fourier transforms gives

$$b_e(z)=\int_{-\infty}^{\infty}\beta_e^+(\tau)\psi_e(z-\tau)\,d\tau+\int_{-\infty}^{\infty}\beta_e^-(\tau)\psi_e(z-\tau)\,d\tau\qquad \forall z\in\mathbf{R},$$

and by analytic continuation, this continues to hold for z in Σ_μ^0. By changing variables ($t=e^\tau$ and $\zeta=e^z$) we deduce that

$$(4.3)\qquad b(\zeta)=\int_0^\infty \beta^+(t)\psi\left(\frac{\zeta}{t}\right)\frac{dt}{t}+\int_0^\infty \beta^-(t)\psi\left(\frac{\zeta}{t}\right)\frac{dt}{t}\qquad \forall\zeta\in S_\mu^0.$$

For small positive ε and large postive N, we define $b_{\varepsilon,N}$ by the formula

$$b_{\varepsilon,N}(\zeta)=\int_\varepsilon^N \beta^+(t)\psi\left(\frac{\zeta}{t}\right)\frac{dt}{t}+\int_\varepsilon^N \beta^-(t)\psi\left(\frac{\zeta}{t}\right)\frac{dt}{t}\qquad \forall\zeta\in S_\mu^0.$$

As ε and N tend to 0 and ∞, $b_{\varepsilon,N}$ tends to b uniformly on compact subsets of S_μ^0, and the functions $b_{\varepsilon,N}$ are uniformly bounded. Moreover,

$$\begin{aligned}|\langle b_{\varepsilon,N}(T)u,v\rangle|&=\left|\int_\varepsilon^N \beta^+(t)\left\langle\psi\left(\frac{1}{t}T\right)u,v\right\rangle\frac{dt}{t}+\int_\varepsilon^N \beta^-(t)\left\langle\psi\left(\frac{1}{t}T\right)u,v\right\rangle\frac{dt}{t}\right|\\ &\le C\,\|b\|_\infty\left(\int_0^\infty|\langle\psi(sT)u,v\rangle|\frac{ds}{s}+\int_0^\infty|\langle\psi(sT)u,v\rangle|\frac{ds}{s}\right)\\ &\le C\,\|b\|_\infty\,\|u\|\,\|v\|\qquad \forall u\in X,\ \forall v\in Y,\end{aligned}$$

because T satisfies $(W(\psi))$. Therefore, by the Convergence Lemma, $b(T)\in\mathscr{L}(X)$ and $\|b(T)\|\le C\,\|b\|_\infty$.

The following corollary is immediate.

COROLLARY 4.5. *Suppose that T is a one-to-one operator of type ω in a Banach space X, with dense domain and dense range. If T satisfies $(W(\psi))$ for some ψ in $\Psi_\nu(S_\nu^0-)$, where $0<\nu<\pi$, then T satisfies (F_η) for any η in (ν,π).*

There is a problem in applying Theorem 4.4 and its corollary, namely that if we use a function ψ for which $\left|\hat{\psi}_e(\lambda)\right| \geq Ce^{-\nu|\lambda|}$ for all real λ, we cannot obtain (F_η) for any η in $(0, \nu)$, no matter how small ω is. For instance, if $\psi(z) = ze^{-z}$ for all z in S^0_μ, then, as shown below, $\nu = \pi/2$. The next result presents a way to get around this limitation. (The idea of introducing the extra parameter θ may be found in [2].)

THEOREM 4.6. *Suppose that T is a one-to-one operator of type ω in a Banach space X, with dense domain and dense range, that $\omega < \mu < \nu < \pi$, that $0 \leq \theta < \mu - \omega$, and that $2\nu - \mu - \theta < \eta < \pi$. Suppose also that $\psi \in \Psi_\nu(S^0_\mu)$. If*

$$\int_0^\infty \left|\langle \psi(te^{\varepsilon i\theta}T)u, v\rangle\right| \frac{dt}{t} \leq C\,\|u\|\,\|v\| \qquad \forall u \in X,\ \forall v \in Y,\ \forall \varepsilon \in \{\pm 1\},$$

then T satisfies (F_η).

PROOF. Some straightforward modifications to the proof of Theorem 4.4 establish this more general result. We list these here, and leave the reader to fill in the details.

(i) γ is redefined: (4.1) is replaced by

$$\hat{\gamma}_e(\lambda) = \frac{1}{\hat{\psi}_e(\lambda)\cosh((\alpha+\theta)\lambda)} \qquad \forall \lambda \in \mathbf{R},$$

so that $\gamma \in L^1(\mathbf{R})$ as before;

(ii) β^+ and β^- are unchanged, and so (4.2) is replaced by

$$e^{-\theta\lambda}\hat{\beta}^+(\lambda)\hat{\psi}_e(\lambda) + e^{\theta\lambda}\hat{\beta}^-(\lambda)\hat{\psi}_e(\lambda) = \hat{b}_e(\lambda) \qquad \forall \lambda \in \mathbf{R};$$

(iii) the integral expression for $|\langle b(T)u, v\rangle|$ involves the integrals involved in this variant of condition $(W(\psi))$. Equality (4.3) is replaced by

$$b(\zeta) = \int_0^\infty \beta^+(t)\psi\left(\frac{1}{t}e^{i\theta}\zeta\right)\frac{dt}{t} + \int_0^\infty \beta^-(t)\psi\left(\frac{1}{t}e^{-i\theta}\zeta\right)\frac{dt}{t} \qquad \forall \zeta \in S^0_\mu.$$

Next we give a few examples of how this theorem may be used. We begin by looking at some functions in the class $\Psi_\nu(S^0_\mu)$. In particular we obtain some of the results from [2].

EXAMPLE 4.7. Recall that $0 < \omega < \mu$, choose Z in $S^0_\pi \setminus S_\mu$, and consider the rational function ψ:

$$\psi(z) = \frac{Z}{z - Z} - \frac{\bar{Z}}{z - \bar{Z}} = \frac{(Z - \bar{Z})z}{(z - Z)(z - \bar{Z})} \qquad \forall z \in S^0_\mu.$$

Clearly $\psi \in \Psi(S^0_\mu)$. Further,

$$\hat{\psi}_e(\lambda) = \int_{-\infty}^{\infty} f(\tau)\,d\tau \qquad \forall \lambda \in \mathbf{R},$$

where

$$f(\tau) = \frac{(Z - \bar{Z})e^{-i\tau\lambda}e^{\tau}}{(e^{\tau} - Z)(e^{\tau} - \bar{Z})} \qquad \forall \tau \in \mathbf{R}.$$

As is easily checked, $f(\cdot + 2\pi i) = e^{2\pi\lambda} f(\cdot)$. Now

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(\tau)\,d\tau - \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(\tau + 2\pi i)\,d\tau = \operatorname{Res}(f, \log Z) + \operatorname{Res}(f, \log \bar{Z})$$

(where the branch of the logarithm has imaginary part in the range $[0, 2\pi)$). Thus

$$\begin{aligned}\int_{-\infty}^{\infty} f(\tau)\,d\tau &= \frac{2\pi i}{1 - e^{2\pi\lambda}}\left[\operatorname{Res}(f, \log Z) + \operatorname{Res}(f, \log \bar{Z})\right] \\ &= \frac{2\pi i}{1 - e^{2\pi\lambda}}\left[e^{-i\lambda \log Z} - e^{-i\lambda \log \bar{Z}}\right] \\ &= \frac{2\pi i e^{-i\lambda \log|Z|}}{1 - e^{2\pi\lambda}}\left[e^{\lambda \arg Z} - e^{(2\pi - \arg Z)\lambda}\right] \\ &= \frac{2\pi i e^{-i\lambda \log|Z|}\sinh((\pi - \arg Z)\lambda)}{\sinh(\pi\lambda)}.\end{aligned}$$

It follows then that

$$\left|\hat{\psi}_e(\lambda)\right| = \frac{2\pi \sinh((\pi - \arg Z)\lambda)}{\sinh(\pi\lambda)} \qquad \forall \lambda \in \mathbf{R},$$

so if $\nu = \arg Z$, $\psi \in \Psi_\nu(S^0_\nu -)$.

Corollary 4.5 shows that if T satisfies $(W(\psi))$ for this choice of ψ, then T satisfies (F_η) when $\eta > \arg Z$.

Similar calculations show that the properties of ψ^2 are similar to those of ψ. The key fact is that

$$\hat{\psi}^2_e(\lambda) = -2\pi|Z|^{-i\lambda}\left[\frac{\lambda\cosh((\pi - \arg Z)\lambda)\sin(\arg Z) + \sinh((\pi - \arg Z)\lambda)\cos(\arg Z)}{\sinh(\pi\lambda)\sin(\arg Z)}\right].$$

EXAMPLE 4.8. Define $\psi(z) = ze^{-z}$, for all z in $S^0_{\pi/2}$. If $\mu < \pi/2$, then $\psi \in \Psi(S^0_\mu)$. Further,

$$\hat{\psi}_e(\lambda) = \int_0^{\infty} \tau^{-i\lambda}e^{-\tau}\,d\tau = \Gamma(1 - i\lambda) \qquad \forall \lambda \in \mathbf{R},$$

and since $|\Gamma(1 - i\lambda)| \geq Ce^{-\pi|\lambda|/2}$ for all λ in $\mathbf{R}$, $\psi \in \Psi_{\pi/2}(S^0_{\pi/2}-)$.

Suppose that T is an operator of type ω, where $\omega < \pi/2$. If

$$\int_0^\infty \left|\left\langle Te^{-tT}u, v\right\rangle\right| dt \le C\,\|u\|\,\|v\| \qquad \forall u \in X,\ \forall v \in Y,$$

then Corollary 4.5 shows that T has a bounded $H^\infty(S_\eta^0)$ functional calculus whenever $\eta > \pi/2$.

More generally, suppose $\theta \in (0, \pi/2 - \omega)$. Note that by a simple change of variables, and a little simplification,

$$\int_0^\infty \left|\left\langle \psi(te^{i\theta}T)u, v\right\rangle\right| \frac{dt}{t} = \int_0^\infty \left|\left\langle Te^{-te^{i\theta}T}u, v\right\rangle\right| dt.$$

Thus, if

$$\int_0^\infty \left|\left\langle Te^{-te^{\varepsilon i\theta}T}u, v\right\rangle\right| dt \le C\,\|u\|\,\|v\| \qquad \forall u \in X,\ \forall v \in Y,\ \forall \varepsilon \in \{\pm 1\},$$

then by Theorem 4.6, taking μ to be smaller than, but close to $\pi/2$, T has a bounded $H^\infty(S_\eta^0)$ functional calculus for any η greater than $\pi/2 - \theta$.

Similar reasoning shows that if $\psi(z) = z^2e^{-2z}$, then $\psi \in \Psi_{\pi/2}(S_{\pi/2}^0-)$.

EXAMPLE 4.9. Suppose that $\psi(z) = z/(1+z)^2$, for all z in S_π^0. Again it is clear that when $0 < \mu < \pi$, $\psi \in \Psi(S_\mu^0)$. Now

$$\hat{\psi}_e(\lambda) = \int_0^\infty \tau^{-i\lambda}\frac{1}{(1+\tau)^2}\,d\tau = \frac{\lambda\pi}{\sinh(\lambda\pi)} \qquad \forall \lambda \in \mathbf{R}.$$

Thus $\left|\hat{\psi}_e(\lambda)\right| \ge Ce^{-\pi|\lambda|}$ and so $\psi \in \Psi_\pi(S_\pi^0-)$.

Again suppose that T is an operator of type ω, where $\omega < \pi$, and that $0 \le \theta < \pi - \omega$. As above, a simple change of variables shows that

$$\int_0^\infty \left|\left\langle \psi(te^{i\theta}T)u, v\right\rangle\right| \frac{dt}{t} = \int_0^\infty \left|\left\langle T(t + e^{i\theta}T)^{-2}u, v\right\rangle\right| dt.$$

If we can find a pair of estimates of the form

$$\int_0^\infty \left|\left\langle T(t + e^{\pm i\theta}T)^{-2}u, v\right\rangle\right| dt \le C\,\|u\|\,\|v\| \qquad \forall u \in X,\ \forall v \in Y,$$

then by Theorem 4.6, T has a bounded $H^\infty(S_\eta^0)$ functional calculus whenever $\eta > \pi - \theta$. In particular, one may establish a bounded functional calculus for a small sector S_η^0, if one can establish these estimates for large values of θ.

Examples 4.8 and 4.9 show that our methods give a different approach to the results of [2], and put the results there in a more general light.

It is interesting to compare our H^∞ functional calculus with more classical theories. Hörmander's multiplier theorem, applied to radial functions, tells us that if $m : \mathbf{R}^+ \to \mathbf{C}$ satisfies the conditions

$$\left|\frac{d^k}{d\xi^k}m(\xi)\right| \le C|\xi|^{-k} \qquad \forall \xi \in \mathbf{R}^+, \tag{4.4}$$

whenever $0 \le k \le [\![n/2]\!] + 1$ ($[\![x]\!]$ denoting the integer part of x), then $m(\Delta)$ is a bounded map on $L^p(\mathbf{R}^n)$ whenever $1 < p < \infty$. This result may be improved slightly, but all variants require a little more than $n\,|1/p - 1/2|$ derivatives behaving well.

With a view towards proving Hörmander type theorems using our H^∞ functional calculus, we establish a connection between the two types of calculus. To obtain a reasonably precise statement, we need another definition. Given a function m on $\mathbf{R}^+$, we write m_e for the function on $\mathbf{R}$ obtained by composing with the exponential, that is, $m_e = m \circ \exp$.

For any positive real number α, let $\Lambda^\alpha_{\infty,1}(\mathbf{R}^+)$ be the set of all bounded continuous functions m on $\mathbf{R}^+$ such that $\|m\|_{\Lambda^\alpha_{\infty,1}} < \infty$, where

$$\|m\|_{\Lambda^\alpha_{\infty,1}} = \sum_{n\in\mathbf{Z}} 2^{|n|\alpha} \left\|m_e * \check{\phi}_n\right\|_\infty .$$

Here, for all ξ in $\mathbf{R}$,

$$\begin{aligned}\phi_0(\xi) &= (2 - 2|\xi|)_+ - (1 - 2|\xi|)_+\\ \phi_1(\xi) &= (1 - 2|\xi - 1|)_+ + (1/2 - |\xi - 3/2|)_+\end{aligned}$$

and

$$\phi_{n\varepsilon}(\xi) = \phi_1(2^{1-n}\varepsilon\xi) \qquad \forall n \in \mathbf{Z}^+,\ \forall \varepsilon \in \{\pm 1\}.$$

This space is sometimes called a Lipschitz space, and sometimes a Besov space. It is not hard to check, using Fourier analysis, that if condition (4.4) holds when $k = 0, 1, 2, \dots, K$, then m is in $\Lambda^\alpha_{\infty,1}(\mathbf{R}^+)$ when $\alpha < K$. The key to doing this is the observation that, if $\mathscr{D}$ denotes differentiation and $\mathscr{I}$ integration on $\mathbf{R}$ (that is, $\mathscr{I} f(x) = \int_{-\infty}^x f(t)dt$), then

$$m_e * \check{\phi}_n = \mathscr{D}^K m_e * \mathscr{I}^K \check{\phi}_n,$$

coupled with the estimate that

$$\left\|\mathscr{I}^K \check{\phi}_n\right\|_1 \le C_K 2^{-|n|K}.$$

THEOREM 4.10. *Suppose that T is a one-one operator of type 0. Then the following conditions are equivalent:*

(i) *T admits a bounded $H^\infty(S^0_\mu)$-functional calculus for all positive μ, and there exist α and A in $\mathbf{R}^+$ such that*

$$\|m(T)\| \le A\mu^{-\alpha}\|m\|_\infty \qquad \forall m \in H^\infty(S^0_\mu),\ \forall \mu \in \mathbf{R}^+;$$

(ii) *T admits a bounded $\Lambda^\alpha_{\infty,1}(\mathbf{R}^+)$ functional calculus.*

PROOF. We first show that (i) implies (ii). Take m in $\Lambda^\alpha_{\infty,1}$ and write

$$m_e = \sum_{n\in\mathbf{Z}} m_e * \check{\phi}_n = \sum_{n\in\mathbf{Z}} m_{(n)} \circ \exp,$$

say.

By the Paley-Wiener theorem, $m_e * \check{\phi}_n$ continues analytically to an entire function $\mathscr{E}(m_e * \check{\phi}_n)$, and, for any b in $\mathbf{R}^+$,

$$\sup_{\substack{x\in\mathbf{R}\\ |y|\le b}} \left|\mathscr{E}(m_e * \check{\phi}_n)(x+iy)\right| \le e^{b2^{|n|+1}} \left\|m_e * \check{\phi}_n\right\|_\infty .$$

Thus, $m_{(n)}$ has a bounded analytic extension $\mathscr{E}_\mu m_{(n)}$ into S^0_μ for each positive μ, and taking μ to be $2^{-|n|}$, we have the estimate

$$\left\|\mathscr{E}_{2^{-|n|}} m_{(n)}\right\|_\infty \le e^2 \left\|m_e * \check{\phi}_n\right\|_\infty .$$

Since T has a bounded $H^\infty(S^0_{2^{-|n|}})$-functional calculus, and

$$\left\|\mathscr{E}_{2^{-|n|}} m_{(n)}(T)\right\| \le A(2^{-|n|})^{-\alpha} \left\|\mathscr{E}_{2^{-|n|}} m_{(n)}\right\|_\infty ,$$

we may define $m_{(n)}(T) = \mathscr{E}_\mu m_{(n)}(T)$, for any positive μ, and we then have that

$$\left\|m_{(n)}(T)\right\| \le e^2 A 2^{|n|\alpha} \left\|m_e * \check{\phi}_n\right\|_\infty ,$$

so we may sum the operators $m_{(n)}(T)$ to obtain the desired result.

To prove the converse, we show that if m is in $H^\infty(S^0_\mu)$, then $m|_{\mathbf{R}^+}$ lies in $\Lambda^\alpha_{\infty,1}$, and

$$\|m|_{\mathbf{R}^+}\|_{\Lambda^\alpha_{\infty,1}} \le A\mu^{-\alpha}\|m\|_\infty .$$

With this result, if (ii) holds, then (i) follows.

Suppose that m is in $H^\infty(S^0_\mu)$, and observe that the proof of the Paley–Wiener theorem implies that, if $n \ne 0$, then

$$\left\|m_e * \check{\phi}_n\right\|_\infty \le \exp(-\mu 2^{|n|-2}) \|m\|_\infty .$$

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Now it follows that

$$\sum_{n\in\mathbf{Z}} 2^{|n|\alpha} \left\| m_e * \check{\phi}_n|_{\mathbf{R}^+} \right\|_\infty \le \|m\|_\infty + \sum_{\substack{n\in Z\\ n\neq 0}} 2^{|n|\alpha} \exp(-\mu 2^{|n|-2}) \|m\|_\infty$$
$$\le A\mu^{-\alpha} \|m\|_\infty ,$$

as required.

REMARK 1. In 1979, Coifman remarked to one of the authors that there were similarities between analyticity in all sectors and Hörmander-type conditions. This quantifies his comments.

REMARK 2. Theorem 4.10 has been used by X. T. Duong to reduce the number of derivatives required for a Hörmander type multiplier theorem for sub-Laplacians on nilpotent Lie groups [7].

5. Operators with bounded imaginary powers

There are quite a few interesting examples of operators of type ω for which the imaginary powers are bounded. When this happens, the growth of the norms of T^{is} as s goes to infinity is sometimes polynomial and sometimes exponential. We consider only the latter possibility in this paper. We say that T satisfies condition (E_μ) if $T^{is} \in \mathscr{L}(X)$ for all real s, and

$$(E_\mu) \qquad \left\| T^{is} \right\| \le C\, e^{|s|\mu} \qquad \forall s \in \mathbf{R}.$$

A consequence of the Convergence Lemma is that (T^{is}) is then a C^0-group. (We remark that T^{is} is defined to be $f_s(T)$, where $f_s(z) = z^{is}$ for all z in S^0_π, as in Section 2).

One important consequence of condition (E_μ) for an operator T in X is that the domains of the fractional powers T^θ may be characterised by the complex method of interpolation: precisely, one has

$$\mathscr{D}(T^\theta) = [X, \mathscr{D}(T)]_\theta \qquad \forall \theta \in (0,1).$$

This is because the bounds (E_μ) allow the three lines theorem, the basic result underlying the complex method, to be applied.

It is immediate that (F_μ) implies (E_μ), and for operators in a Hilbert space these two conditions are equivalent. However, as we shall see, this equivalence does not hold in more general spaces.

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In the last section, we saw that, under appropriate restrictions on ψ and μ, $(W(\psi))$ implies (F_μ). *A fortiori*, $(W(\psi))$ implies (E_μ). We give here a brief alternative proof of this fact.

THEOREM 5.1. *Suppose that $\langle X, Y\rangle$ is a dual pair of Banach spaces, that T is a one-to-one operator of type ω with dense domain and dense range in X, and that $\omega < \mu < \nu < \pi$. Then for ψ in $\Psi_\nu(S^0_\mu)$, $(W(\psi))$ implies (E_ν).*

PROOF. For any real λ, we have that

$$1 = 1/\hat{\psi}_e(\lambda) \int_0^\infty t^{-i\lambda}\psi(t)\frac{dt}{t}$$

and so, from the Convergence Lemma,

$$T^{i\lambda} = 1/\hat{\psi}_e(\lambda) \int_0^\infty t^{-i\lambda}\psi(tT)\frac{dt}{t}.$$

It follows that, for any u in X and v in Y,

$$\begin{aligned} \left|\langle T^{i\lambda}u, v\rangle\right| &= \left|1/\hat{\psi}_e(\lambda) \int_0^\infty t^{-i\lambda}\, \langle\psi(tT)u, v\rangle \,\frac{dt}{t}\right| \\ &\le 1/\left|\hat{\psi}_e(\lambda)\right| \int_0^\infty |\langle\psi(tT)u, v\rangle|\,\frac{dt}{t} \\ &\le Ce^{\nu|\lambda|}\,\|u\|\,\|v\|, \end{aligned}$$

by assumption. Hence, T satisfies (E_ν).

(It may be of interest to note that deriving Theorem 5.1 from Theorem 4.4 yields a weaker result: one can show only that T satisfies condition (E_η), where $\eta > 2\nu - \mu$.)

EXAMPLE 5.2. An interesting example in this theory is based on the celebrated theorem of Stečkin. This example shows that, unlike in the Hilbert space case, the bounds on the norms of the complex powers do not guarantee the existence of a bounded H^∞ functional calculus, that is, (E_μ) does not imply (F_μ). Recall that Stečkin's theorem states that if $b \in L^\infty(\mathbf{R})$ and has bounded total variation, and $1 < p < \infty$, then the Fourier multiplier operator $F_b : f \mapsto (b\hat{f})^\vee$, initially defined as a bounded operator on $L^2(\mathbf{R})$ by means of the Plancherel theorem, extends to a bounded operator on $L^p(\mathbf{R})$. In particular, if b is bounded and differentiable, and b' is integrable, then $\|F_b\|_p \le C_p(\|b\|_\infty + \|b'\|_1)$.

Fix p in $(1, \infty)$. It is an immediate consequence of Stečkin's theorem that if $t \in \mathbf{R}^+$ and S_t is the Fourier multiplier operator on $L^p(\mathbf{R})$ given by the rule

$$(S_t f)\hat{}\,(\xi) = e^{-te^\xi} \hat{f}(\xi) \qquad \forall \xi \in \mathbf{R},$$

then $\|S_t\|_p \le 2C_p$. It is also easy to see that $(S_t)_{t>0}$ is a bounded semigroup of class C_0, with infinitesimal generator T such that

$$(Tf)\hat{}\,(\xi) = e^\xi \hat{f}(\xi) \qquad \forall \xi \in \mathbf{R},$$

and that the resolvent operator $(\lambda I - T)^{-1}$ extends holomorphically into $\mathbf{C} \setminus S_0$. Since

$$\begin{aligned} \sup_{\xi \in \mathbf{R}} \left|(\lambda - e^\xi)^{-1}\right| + \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial \xi}(\lambda - e^\xi)^{-1} \right| d\xi &= \sup_{t \in \mathbf{R}^+} \left|(\lambda - t)^{-1}\right| + \int_0^\infty \frac{dt}{|\lambda - t|^2} \\ &\le 5 \operatorname{dist}(\lambda, \mathbf{R}^+)^{-1} \qquad \forall \lambda \in \mathbf{C} \setminus S_0, \end{aligned}$$

the semigroup is holomorphic and T is of type 0. Moreover, T is a one-to-one operator with dense domain and dense range in $L^p(\mathbf{R})$.

It is clear that $(T^{iu} f)\hat{}\,(\xi) = e^{i\xi u} \hat{f}(\xi)$, that is, T^{iu} is the operator of translation by u. In particular,

$$\left\|T^{iu}\right\|_p = 1 \qquad \forall u \in \mathbf{R},\ \forall p \in (1, \infty).$$

LEMMA 5.3. *Suppose that $p \neq 2$. Then the operator T just described is a one-to-one operator of type* 0 *with dense domain and dense range in $L^p(\mathbf{R})$, which satisfies (E_0), but T does not admit a bounded $H^\infty(S^0_\theta)$ functional calculus for any positive θ.*

PROOF. We argue by *reductio ad absurdum.* We make the hypothesis that, for some p in $(1, \infty) \setminus \{2\}$ and some θ in $(0, \pi)$, T has a bounded H^∞ functional calculus for the sector S^0_θ. For m in $H^\infty(S^0_\theta)$ and f in $L^p(\mathbf{R})$,

$$(m(T)f)\hat{}\,(\xi) = m(e^\xi)\hat{f}(\xi),$$

that is, $m(T)$ is the Fourier multiplier operator associated with $m \circ \exp$. It follows that, for any bounded holomorphic function b in the strip Σ^0_θ, the Fourier multiplier operator F_b is bounded on $L^p(\mathbf{R})$.

Observe that if $a : \mathbf{Z} \to \mathbf{C}$ is any bounded function, then the function b_a, defined thus

$$b_a(z) = \sum_{n \in \mathbf{Z}} a(n) e^{-(z-n)^2} \qquad \forall z \in \Sigma^0_\theta,$$

is bounded and holomorphic; our hypothesis guarantees that b_a is a bounded Fourier multiplier for $L^p(\mathbf{R})$. By K. de Leeuw's theorem [14], $b_a|_\mathbf{Z}$ is then a bounded Fourier multiplier for $L^p(\mathbf{T})$ (where $\mathbf{T}$ is the circle). It is clear that $b_a|_\mathbf{Z} = a + h * a$, where

$$h * a(n) = \sum_{k \in \mathbf{Z} \setminus \{0\}} a(n-k) e^{-k^2} \qquad \forall n \in \mathbf{Z}.$$

Since

$$\sum_{k\in\mathbf{Z}\setminus\{0\}} e^{-k^2} = 2(e^{-1}+e^{-4}+e^{-9}+\cdots) < 1,$$

if $b : \mathbf{Z} \mapsto \mathbf{C}$ is a given bounded function, then setting

$$a = (b - h*b + h*h*b - h*h*h*b + \cdots),$$

we have produced a bounded function such that $b_a|_{\mathbf{Z}} = b$. Thus every bounded function $b : \mathbf{Z} \to \mathbf{C}$ is a Fourier multiplier of $L^p(\mathbf{T})$. But this is possible only if $p = 2$. See for example, R. E. Edwards and G. I. Gaudry [9].

A positive result in this direction is the following proposition, which states that if T has a bounded H^∞ functional calculus on a big sector S_θ^0 and satisfies bounds on its imaginary powers on a smaller sector S_μ^0, then it must have a bounded H^∞ functional calculus for all sectors bigger than S_μ^0.

THEOREM 5.4. *Suppose that* $0 \le \omega \le \mu < \nu \le \theta < \pi$. *If* T *is a one-to-one operator of type* ω *with dense domain and dense range in* X, *which admits a bounded* H^∞ *functional calculus for the sector* S_θ^0, *and if*

$$\|T^{iu}\| \le Ce^{\mu|u|} \qquad \forall u \in \mathbf{R},$$

then T *admits a bounded* $H^\infty(S_\nu^0)$ *functional calculus. In other words, if* T *satisfies* (F_θ) *and* (E_μ), *then* T *satisfies* (F_ν).

PROOF. Suppose that $b \in H^\infty(S_\nu^0)$. Our aim is to decompose b thus:

$$b(z) = \sum_{k\in\mathbf{Z}} z^{ik} b_k(z) \qquad \forall z \in S_\nu^0,$$

where each $b_k \in H^\infty(S_\theta^0)$, and

$$\sum_{k\in\mathbf{Z}} e^{\mu|k|} \|b_k\|_{H^\infty(S_\theta^0)} \le C \|b\|_{H^\infty(S_\nu^0)} .$$

If we can do this then the sum $\sum_{k\in\mathbf{Z}} T^{ik} b_k(T)$ converges in $\mathscr{L}(X)$ and, by the Convergence Lemma, is equal to $b(T)$. Furthermore $\|b(T)\| \le C \|b\|_\infty$.

We effect this decomposition by changing variables ($z = e^w$), thereby transforming from sectors to strips. We must now show that any n in $H^\infty(\Sigma_\nu^0)$ may be expressed in the form

$$n(w) = \sum_{k\in\mathbf{Z}} e^{ikw} n_k(w) \qquad \forall w \in \Sigma_\nu^0,$$

where $n_k \in H^\infty(\Sigma_\theta^0)$ and

$$\sum_{k\in\mathbf{Z}} e^{\mu|k|} \|n_k\|_{H^\infty(\Sigma_\theta^0)} \le C \|n\|_{H^\infty(\Sigma_\nu^0)} .$$

We shall do this by using Fourier analysis.

Let ϕ be any $C_c^\infty(\mathbf{R})$ function such that

(i) $\operatorname{supp}(\phi) \subseteq [-1, 1]$,

(ii) $0 \le \phi(\xi) \le 1 \quad \forall \xi \in \mathbf{R}$,

(iii) $\sum_{k\in\mathbf{Z}} \phi(\xi - k) = 1 \quad \forall \xi \in \mathbf{R}$.

Define n_k by the formula

$$n_k(w) = \int_{-\infty}^{\infty} \check{\phi}(w - t) n(t) e^{-ikt} dt \qquad \forall w \in \Sigma_\theta^0.$$

By the Paley–Wiener Theorem, $\check{\phi}$ is an entire function, whose decay properties ensure that the integral defining $n_k(w)$ converges for all complex w. Fix w in Σ_θ^0 and k in $\mathbf{Z}$. Contour shifting allows us to write

$$n_k(w) = \int_{-\infty}^{\infty} \check{\phi}(w - t - i\varepsilon\nu) n(t + i\varepsilon\nu) e^{-i(t+i\varepsilon\nu)k} dt,$$

for any ε in $(-1, +1)$. Choosing ε to have the opposite sign to k shows that

$$\|n_k\|_\infty \le C_{\theta,\nu} e^{-\nu|k|} \|n\|_\infty \qquad \forall k \in \mathbf{Z},$$

where

$$C_{\theta,\nu} = \sup_{|y|<\theta+\nu} \int_{-\infty}^{\infty} \left|\check{\phi}(x + iy)\right| dx < \infty.$$

A routine calculation shows that if $u_k(\xi) = \phi(\xi - k)\hat{n}(\xi)$ for all ξ in $\mathbf{R}$, then

$$\check{u}_k(w) = \int_{-\infty}^{\infty} \check{\phi}(w - t) n(t) e^{i(w-t)k} dt = e^{ikw} n_k(w) \qquad \forall w \in \Sigma_\theta^0.$$

Clearly, $\sum_{k\in\mathbf{Z}} u_k(\xi) = \hat{n}(\xi)$, so by taking the inverse transforms we get the required decomposition.

In Theorem 2.4 we saw that if T is a one-to-one operator of type ω in a Hilbert space which satisfies (F_μ), where $\mu > \omega$, then T satisfies (F_ν) for all ν in (ω, ν). It would be interesting to know whether this is the case for one-to-one operators of type ω with dense domain and dense range in a Banach space X.

We next present an example of a Banach space X and a one-to-one operator in $\mathscr{L}(X)$ which is of type ω, and which has a bounded H^∞ functional calculus on one sector without having bounded imaginary powers on any smaller sector. We note however that the Banach space is not reflexive, and the operator does not have dense range.

EXAMPLE 5.5. In this example we shall show that for an operator of type ω, where $\omega < \pi$, one may have a bounded H^∞ functional calculus for one sector without having bounded imaginary powers for any smaller sector.

Suppose that $A(D^0)$ is a unital Banach subalgebra of $H^\infty(D^0)$ for some open subset D^0 of $\mathbf{C}$. Define the operator T in $A(D^0)$ by

$$Tf(z) = zf(z) \qquad \forall z \in D^0.$$

Under appropriate conditions on $A(D^0)$, T is closed and densely defined. Spectral theory for T is then the study of multiplication of functions in $A(D^0)$. More precisely, given m in $H^\infty(D^0)$, $m(T)$ is the multiplication operator $f \mapsto mf$, and if $m \in A(D^0)$, then

$$\|mf\|_A \le \|m\|_A \, \|f\|_A \qquad \forall f \in A(D^0),$$

so $m(T) \in \mathscr{L}(A(D^0))$, and $\|m(T)\| \le \|m\|_A$. On the other hand, $m = m(T)1$, so if $m(T) \in \mathscr{L}(A(D^0))$, then $m \in A(D^0)$ and $\|m\|_A \le \|m(T)\|$.

Here is a particular example. For ω in $(0, \pi)$, we denote the set $\{z \in S^0_\omega : |z| < 1\}$ and its usual closure by D^0_ω and D_ω. If $0 \le \omega < \theta < \pi$, let $A(D^0_\omega)$ be the set of all functions f in $H^\infty(D^0_\omega)$ which admit representations of the form $f = f_1|_{D^0_\omega} + f_2|_{D^0_\omega}$, where f_1 is continuous on D_ω, holomorphic on D^0_ω, and vanishes at 0, and $f_2 \in H^\infty(D^0_\theta)$. We set

$$\|f\|_A = \inf\left\{\|f_1\|_\infty + \|f_2\|_\infty\right\},$$

where the infimum is taken over all such decompositions. Then it is easy to check that $A(D^0_\omega)$ is a unital Banach algebra. If $\nu > \omega$, and $\lambda \in \mathbf{C} \setminus S_\nu$, then

$$(\lambda - z)^{-1} = \left[(\lambda - z)^{-1} - \frac{|\lambda|}{\lambda}(|\lambda| + z)^{-1}\right] + \left[\frac{|\lambda|}{\lambda}(|\lambda| + z)^{-1}\right] \qquad \forall z \in \mathbf{C} \setminus \{\lambda, -|\lambda|\},$$

which shows that the functions $R_\lambda : z \mapsto (\lambda - z)^{-1}$ belong to $A(D^0_\omega)$, and further that

$$\|R_\lambda\|_A \le C\,|\lambda|^{-1} \qquad \forall \lambda \in \mathbf{C} \setminus S_\nu.$$

Thus T is of type ω. It is straightforward to check also that T is one-to-one. However T does not have dense range (see Example 3.9).

Moreover, $H^\infty(S^0_\theta)$ acts on $A(D^0_\omega)$ by definition. We shall see that T does not admit a bounded H^∞ functional calculus for any smaller sector. Indeed, if it did, then for some ϕ in (ω, θ), we would have the inequality

$$\left\|T^{is}\right\| \le C \sup_{z \in S^0_\phi} \left|z^{is}\right| = Ce^{\phi|s|} \qquad \forall s \in \mathbf{R}.$$

PROPOSITION 5.6. *For the operator T just defined,*

$$\left\|T^{is}\right\| = e^{\theta|s|} \qquad \forall s \in \mathbf{R}.$$

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PROOF. It is trivial that $\|T^{is}\| \le e^{\theta|s|}$. It suffices to show that if $f_1 \in C_0(D_\omega)$, $f_2 \in H^\infty(D^0_\theta)$, $s \in \mathbf{R}$, and

$$(f_1|_{D^0_\omega} + f_2|_{D^0_\omega})(z) = z^{is} \qquad \forall z \in D^0_\omega,$$

then

$$\|f_2\|_\infty \ge \exp(\theta|s|).$$

Let Δ^0_θ denote the half-strip $\{x + iy \in \Sigma^0_\theta : x < 0\}$, and Δ_θ denote its closure. By composing with exponentials, it suffices to show that, if $g_1 \in C_0(\Delta_\omega)$, $g_2 \in H^\infty(\Delta^0_\theta)$, and

$$(g_1|_{\Delta^0_\omega} + g_2|_{\Delta^0_\omega})(z) = \exp(isz) \qquad \forall z \in \Delta^0_\omega,$$

then

$$\|g_2\|_\infty \ge \exp(\theta|s|).$$

For $j = 1, 2$, let $h_j(z) = \exp(-isz)g_j(z)$, for all z in $\mathscr{D}(g_j)$. Then

$$h_1|_{\Delta^0_\omega} + h_2|_{\Delta^0_\omega} = 1.$$

Since $h_1 \in C_0(\Delta_\omega)$, $h_2(x + i0)$ tends to 1 as $x \to -\infty$ in $\mathbf{R}$. By a simple variant of Montel's theorem (see for example Titchmarsh [20, 5.23], we conclude that $h_2(x+iy)$ tends to 1 as $x \to -\infty$ in $\mathbf{R}$, uniformly for y in any closed subinterval of $(-\theta, \theta)$. It follows that

$$\|f_2\|_\infty = \|g_2\|_\infty \ge \exp(\theta|s|).$$

6. Square-function estimates

In this section we shall examine the relationship of the above conditions with square-function estimates. Here we restrict our attention to function spaces such as L^p. An operator T acting in a function space X is said to satisfy a square-function estimate if it satisfies

$$(S(\psi)) \qquad \left\| \left[\int_0^\infty |\psi(tT)u(\cdot)|^2 \frac{dt}{t} \right]^{1/2} \right\| \le C\,\|u\| \qquad \forall u \in X.$$

In L^2, this statement about T (for a suitable ψ), together with a dual statement, is equivalent to the statements (F_μ), (E_μ) and $(W(\psi))$ (see Theorem 2.4). Our aim here is to examine the relationship between these statements for more general Banach spaces. Our first theorem will describe the sense in which $(S(\psi))$ is stronger than $(W(\psi))$.

In what follows, Ω denotes a σ-finite measure space. As usual, we shall use p' to denote the conjugate index to p; that is, $1/p + 1/p' = 1$.

Recall from Section 3 that if T and T' are dual operators in $\langle L^p(\Omega), L^{p'}(\Omega)\rangle$ of type ω, and $1 < p < \infty$, then we may pass to a dual subpair $\langle X, Y\rangle$ in which T and T' are one-to-one. Further, in this dual subpair, T and T' have dense domain and range.

THEOREM 6.1. *Suppose that T and T' are dual one-to-one operators of type ω in the dual subpair $\langle X, Y\rangle$ of $\langle L^p(\Omega), L^{p'}(\Omega)\rangle$, where $1 < p < \infty$. Suppose also that $\mu > \omega$ and $\psi, \tilde{\psi} \in \Psi(S^0_\mu)$. If T satisfies $(S(\psi))$ and T' satisfies $(S(\tilde{\psi}))$, then T satisfies $W(\psi\tilde{\psi})$.*

PROOF. Suppose that

$$\left\| \left[\int_0^\infty |\psi(tT)u(\cdot)|^2 \frac{dt}{t} \right]^{1/2} \right\|_p \leq C_1 \|u\|_p \qquad \forall u \in X,$$

and

$$\left\| \left[\int_0^\infty \left|\tilde{\psi}(tT')v(\cdot)\right|^2 \frac{dt}{t} \right]^{1/2} \right\|_{p'} \leq C_2 \|v\|_{p'} \qquad \forall v \in Y.$$

Then, from Corollary 3.7 and standard inequalities,

$$\begin{aligned}
\int_0^\infty |\langle \psi\tilde{\psi}(tT)u, v\rangle| \frac{dt}{t} &= \int_0^\infty \left|\langle \psi(tT)u, \tilde{\psi}(tT')v\rangle\right| \frac{dt}{t} \\
&\leq \int_\Omega \int_0^\infty \left|\psi(tT)u(x)\,\tilde{\psi}(tT')v(x)\right| \frac{dt}{t}\, dx \\
&\leq \int_\Omega \left[\int_0^\infty |\psi(tT)u(x)|^2 \frac{dt}{t}\right]^{1/2} \left[\int_0^\infty \left|\tilde{\psi}(tT')v(x)\right|^2 \frac{dt}{t}\right]^{1/2} dx \\
&\leq \left\| \left[\int_0^\infty |\psi(tT)u(\cdot)|^2 \frac{dt}{t}\right]^{1/2} \right\|_p \left\| \left[\int_0^\infty \left|\tilde{\psi}(tT')v(\cdot)\right|^2 \frac{dt}{t}\right]^{1/2} \right\|_{p'} \\
&\leq C_1 C_2 \|u\|_p \|v\|_{p'} \qquad \forall u \in X,\ \forall v \in Y,
\end{aligned}$$

and hence T satisfies $W(\psi\tilde{\psi})$.

By Corollary 4.5, it follows that if T and T' satisfy suitable square function estimates, then T must have a bounded H^∞ functional calculus for some sector. As an example of this we give the following result.

COROLLARY 6.2. *Suppose that $\omega < \pi/2$, that $1 < p < \infty$, and that T and T' are dual one-to-one operators of type ω in in the dual subpair $\langle X, Y\rangle$ of $\langle L^p(\Omega), L^{p'}(\Omega)\rangle$. Suppose also that*

$$\left\| \left[\int_0^\infty \left| T e^{-tT} u(\cdot) \right|^2 dt \right]^{1/2} \right\|_p \le C_1 \|u\|_p \qquad \forall u \in X$$

and

$$\left\| \left[\int_0^\infty \left| T' e^{-tT'} v(\cdot) \right|^2 dt \right]^{1/2} \right\|_{p'} \le C_2 \|v\|_{p'} \qquad \forall v \in Y.$$

Then T satisfies (F_η) whenever $\eta > \pi/2$.

PROOF. The hypotheses imply that T and T' satisfy $(S(\psi))$ when $\psi(z) = ze^{-z}$ for all z in $S^0_{\pi/2}$. By the theorem then, T satisfies $W(\psi^2)$. Now $\psi^2 : z \mapsto z^2 e^{-2z}$ was shown in Example 4.8 to lie in $\Psi_{\pi/2}(S^0_{\pi/2}-)$. Thus by Corollary 4.5, T satisfies (F_η) provided that $\eta > \pi/2$.

We shall now show that if T has a bounded H^∞ functional calculus, then it satisfies square function estimates. The main reason that we have restricted ourselves to the case of L^p is that we require the following well known randomization lemma [23, V.8].

LEMMA 6.3. *Suppose that $1 < p < \infty$ and that $\{u_k\}_{k=1}^\infty$ is a sequence of functions in $L^p(\Omega)$. Then*

$$\left\| \left[\sum_k |u_k|^2 \right]^{1/2} \right\|_p \le C \sup_{|a_k| \le 1} \left\| \sum_k a_k u_k \right\|_p .$$

PROOF. Let $\{r_k\}$ denote the Rademacher functions on $[0, 1]$. Then, using the properties of the Rademacher system,

$$\begin{aligned} \left\| \left[\sum_k |u_k|^2 \right]^{1/2} \right\|_p^p &= \int_\Omega \left[\sum_k |u_k|^2 \right]^{p/2} dx = \int_\Omega \left[\int_0^1 \left| \sum_k r_k(t) u_k(x) \right|^2 dt \right]^{p/2} dx \\ &\le C \int_\Omega \left[\int_0^1 \left| \sum_k r_k(t) u_k(x) \right|^p dt \right] dx = C \int_0^1 \left\| \sum_k r_k(t) u_k \right\|_p^p dt \\ &\le C \sup_{|a_k| \le 1} \left\| \sum_k a_k u_k \right\|_p^p . \end{aligned}$$

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Suppose then that T is a one-to-one operator of type ω in a closed subspace X of $L^p(\Omega)$. For μ in $(0,\pi)$, define the new norm $\|\cdot\|_{p,\mu}$ by

$$\|u\|_{p,\mu} = \sup\left\{\|b(T)u\|_p : b \in H^\infty(S^0_\mu),\ \|b\|_\infty = 1\right\} \qquad \forall u \in X.$$

It is clear that if $0 < \nu < \mu$, then

$$\|u\|_p \le \|u\|_{p,\mu} \le \|u\|_{p,\nu} \qquad \forall u \in X.$$

If T possesses a bounded $H^\infty(S^0_\mu)$ functional calculus, then $\|u\|_{p,\mu} \le C\,\|u\|_p$ and so the norms $\|\cdot\|_p$ and $\|\cdot\|_{p,\mu}$ are equivalent.

LEMMA 6.4. *Suppose that T is a one-to-one operator of type ω in a closed subspace X of $L^p(\Omega)$, where $1 < p < \infty$, and that $0 < \nu < \pi$. Suppose also that $\{b_j\}$ is a sequence of functions in $H^\infty(S^0_\nu)$. Then*

$$\left\|\left[\sum_j \left|b_j(T)u(\cdot)\right|^2\right]^{1/2}\right\|_p \le C \sup_{\zeta\in S^0_\nu} \sum_j \left|b_j(\zeta)\right| \|u\|_{p,\nu}\,.$$

PROOF. By Lemma 6.3,

$$\begin{aligned}\left\|\left[\sum_j \left|b_j(T)u(\cdot)\right|^2\right]^{1/2}\right\|_p &\le C \sup_{|a_j|\le 1}\left\|\sum_j a_j b_j(T)u\right\|_p \\ &\le C \sup_{|a_j|\le 1}\sup_{\zeta\in S^0_\nu}\left|\sum_j a_j b_j(\zeta)\right| \|u\|_{p,\nu} \\ &\le C \sup_{\zeta\in S^0_\nu}\sum_j \left|b_j(\zeta)\right| \|u\|_{p,\nu}\,.\end{aligned}$$

Now choose a nonnegative function h in $C_0^\infty(\mathbf{R})$, supported in $[-2,2]$, such that $\sum_{k=-\infty}^{\infty} h_k^2 = 1$, where $h_k = h(\cdot - k)$. Suppose $\psi \in \Psi(S^0_\mu)$ and define $b_{jk} : S^0_\nu \to \mathbf{C}$ thus:

$$b_{jk}(\zeta) = \int_{-\infty}^{\infty} h_k(s)\hat{\psi}_e(s)e^{-isj}\zeta^{is}\,ds \qquad \forall \zeta \in S^0_\nu.$$

LEMMA 6.5. *Suppose that $\nu < \mu$. Then*

$$\sum_k \sup_{\zeta\in S^0_\nu}\sum_j \left|b_{jk}(\zeta)\right| < \infty.$$

PROOF. We apply the change of variables $e^z = \zeta$. Define $\beta_{jk} : \Sigma_\nu^0 \to \mathbf{C}$ to be $b_{jk} \circ \exp$. Then

$$\begin{aligned}|\beta_{jk}(z)| &= \left|\int_{-\infty}^{\infty} h_k(s)\hat{\psi}_e(s)e^{-is(j-z)}ds\right| \le \int_{k-2}^{k+2} \left|\hat{\psi}_e(s)\right| e^{\nu|s|}ds \\ &\le C\int_{k-2}^{k+2} e^{(\nu-\mu)|s|}ds \le C_1 e^{(\nu-\mu)|k|} \qquad \forall z \in \Sigma_\nu^0,\end{aligned}$$

and similarly

$$\begin{aligned}|\beta_{jk}(z)| &= \left|\frac{1}{(j-z)^2}\int_{-\infty}^{\infty} \frac{d^2}{ds^2}[h_k(s)\hat{\psi}_e(s)]e^{-is(j-z)}ds\right| \\ &\le C\frac{1}{|j-z|^2}\int_{k-2}^{k+2}\left[\left|\hat{\psi}_e(s)\right| + \left|\hat{\psi}_e'(s)\right| + \left|\hat{\psi}_e''(s)\right|\right]e^{\nu|s|}ds \\ &\le C_2\frac{e^{(\nu-\mu)|k|}}{|j-z|^2} \qquad \forall z \in \Sigma_\nu^0,\end{aligned}$$

where the constants C_1 and C_2 depend only on ψ, μ, and ν. Thus

$$\begin{aligned}\sup_{z\in\Sigma_\nu^0}\sum_j |\beta_{jk}(z)| &= \sup_{z\in\Sigma_\nu^0}\left(\sum_{|j-z|<1}|\beta_{jk}(z)| + \sum_{|j-z|\ge 1}|\beta_{jk}(z)|\right) \\ &\le \sup_{z\in\Sigma_\nu^0}\left(2C_1e^{(\nu-\mu)|k|} + C_2\sum_{j\ne 0}\frac{e^{(\nu-\mu)|k|}}{j^2}\right) \\ &\le Ce^{(\nu-\mu)|k|}.\end{aligned}$$

It follows that

$$\sum_k \sup_{\zeta\in S_\nu^0}\sum_j |b_{jk}(\zeta)| \le C\sum_k e^{(\nu-\mu)|k|} < \infty.$$

We may now progress to the main result. This says that we always get a square function estimate for T if we use the norm $\|\cdot\|_{p,\nu}$.

THEOREM 6.6. *Suppose that $0 \le \omega < \nu < \mu < \pi$ and $1 < p < \infty$. Let T be a one-to-one operator of type ω in a closed subspace X of $L^p(\Omega)$. If $\psi \in \Psi(S_\mu^0)$, then*

$$\left\|\left[\int_0^\infty |\psi(tT)u(\cdot)|^2\frac{dt}{t}\right]^{1/2}\right\|_p \le C\|u\|_{p,\nu} \qquad \forall u \in X.$$

PROOF. Again we remark that T automatically has dense domain and dense range. Note that

$$\begin{aligned}\left[\int_0^\infty |\psi(tT)u|^2\,\frac{dt}{t}\right]^{1/2} &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{\psi}_e(s)T^{is}u\right|^2 ds\right]^{1/2}\\ &= \left[\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\left|h_k(s)\hat{\psi}_e(s)T^{is}u\right|^2 ds\right]^{1/2}\\ &\le \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left|h_k(s)\hat{\psi}_e(s)T^{is}u\right|^2 ds\right]^{1/2}.\end{aligned}$$

Thus

$$\begin{aligned}\left\|\left[\int_0^\infty |\psi(tT)u|^2\,\frac{dt}{t}\right]^{1/2}\right\|_p &\le C\sum_{k=-\infty}^{\infty}\left\|\left[\int_{-\infty}^{\infty}\left|h_k(s)\hat{\psi}_e(s)T^{is}u\right|^2 ds\right]^{1/2}\right\|_p\\ &= C\sum_{k=-\infty}^{\infty}\left\|\left[\sum_j\left|b_{jk}(T)u\right|^2\right]^{1/2}\right\|_p,\end{aligned}$$

where

$$b_{jk}(\zeta) = \int_{-\infty}^{\infty} h_k(s)\hat{\psi}_e(s)e^{-isj}\zeta^{is}ds \qquad \forall \zeta \in S_\nu^0,$$

as in Lemma 6.5. But by Lemmas 6.4 and 6.5,

$$\sum_{k=-\infty}^{\infty}\left\|\left[\sum_j\left|b_{jk}(T)u\right|^2\right]^{1/2}\right\|_p \le C\sum_k \sup_{\zeta\in S_\nu^0}\sum_j\left|b_{jk}(\zeta)\right|\,\|u\|_{p,\nu} \le C\,\|u\|_{p,\nu},$$

and so the result follows.

COROLLARY 6.7. *Under the conditions of Theorem 6.6, if T also satisfies (F_ν), then T satisfies $(S(\psi))$; that is*

$$\left\|\left[\int_0^\infty |\psi(tT)u(\cdot)|^2\,\frac{dt}{t}\right]^{1/2}\right\|_p \le C\,\|u\|_p.$$

PROOF. This follows immediately from Theorem 6.6 since, as was mentioned above, if T satisfies (F_ν), then $\|u\|_{p,\nu} \le \|u\|_p$ for all u in X.

By combining Theorem 4.1, Theorem 6.1, and Corollary 6.7, we may regard the possession of a bounded H^∞ functional calculus by T as being equivalent to both T and T' satisfying square function estimates. The existence of ψ as in (ii) of the following corollary is shown in Examples 4.7 and 4.8.

COROLLARY 6.8. *Suppose that* $0 \leq \omega < \nu < \mu < \pi$ *and* $1 < p < \infty$. *Let* T *and* T' *be dual one-to-one operators of type* ω *in the dual subpair* $\langle X, Y \rangle$ *of* $\langle L^p(\Omega), L^{p'}(\Omega)\rangle$.

(i) *If* T *satisfies* (F_ν) *then* T *and* T' *both satisfy* $(S(\psi))$, *for all* ψ *in* $\Psi(S_\mu^0)$.
(ii) *If* T *and* T' *both satisfy* $(S(\psi))$ *for some* ψ *in* $\Psi(S_\nu^0)$ *for which* $\psi^2 \in \Psi_\nu(S_\nu^0-)$, *then* T *satisfies* (F_μ).

We conclude this section by noting that attempts to use square functions of the form

$$\left[\int_0^\infty \|\psi(tT)u(\cdot)\|^2 \frac{dt}{t}\right]^{1/2}$$

rather than

$$\left\|\left[\int_0^\infty |\psi(tT)u(\cdot)|^2 \frac{dt}{t}\right]^{1/2}\right\|$$

lead to the interesting theory of Besov spaces (also known as Lipschitz spaces). However, this development deviates somewhat from our focus in this paper. In particular, for operators T such as the Laplacian in $L^p(\mathbf{R}^n)$, inequalities such as

$$\left[\int_0^\infty \|\psi(tT)u(\cdot)\|_p^2 \frac{dt}{t}\right]^{1/2} \leq C\,\|u\|_p \qquad \forall u \in L^p(\mathbf{R}^n)$$

may fail. For instance, if $n = 1$, $\psi(z) = z\big/(1+z)^2$ for all z in S_π^0, and $\hat{u}(\xi) = |\xi|^{-1/2}\big/(1+|\xi|^2)$ for all ξ in $\mathbf{R}$, then $u \in L^p(\mathbf{R})$ whenever $p > 2$, but

$$\left[\int_0^\infty \|\psi(tT)u(\cdot)\|_p^2 \frac{dt}{t}\right]^{1/2} = \infty.$$

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