ON THE COMPLETENESS OF GALOIS THEORIES

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1. Introduction

Let E be a given field and G some (not necessarily the total) group of automorphisms of E. We introduce a topology in G by saying that a net of elements T_α in G converges to T in G if for each $x \in E$, $T_\alpha(x)$ coincides with $T(x)$ ultimately. We shall call this the Krull topology on G.

DEFINITION 1. We say that (E, K, G) is a Krull Galois ($K.G.$) system (or that E/K allows a $K.G.$ theory) if G is a group of automorphisms of E with fixed field K; and if there exists a one-one 'Galois' correspondence between all the closed subgroups (under the Krull topology) of G on the one hand and all the intermediate fields of E/K on the other.

The 'Galois correspondence' is one which associates with each intermediate field F, the subgroup Γ of all automorphisms of E in G fixing each element of F; and to each subgroup Γ of G the subfield F consisting of all elements left invariant by each element of Γ. It is easy to see in this case — 'if Γ is the closed subgroup associated with the intermediate field F, then F is the intermediate field associated with Γ'; and conversely.

The classical Krull Galois theory [2] asserts that if E is a separable algebraic normal extension of a field K, then E/K allows a $K.G.$ theory with G as the total group of automorphisms of E/K.

The question arises whether this or an analogous result would continue to be true for other types of extensions. In this note we show that such a Galois theory does not hold good for any other extension field E of K. We also obtain the analogous result for the usual Galois theory of finite separable normal extensions.

2. Preliminary results

We start with a series of propositions.

PROPOSITION 1. *If (E, K, G) is a $K.G.$ system and E' is an intermediate field in E/K and G' the sub-group of G associated with E', then (E, E', G') is a $K.G.$ system.*

PROOF. This follows from the facts that G' is a closed subgroup of G and that the topology induced in G' by G coincides with the Krull topology in G' as a group of automorphisms. So every 'closed' subgroup of G' will correspond to an intermediate subfield of E/E' and vice-versa.

PROPOSITION 2. *Let (E, K, G) be a K.G. system. If E is not algebraic over K then there exists an intermediate field A such that A is algebraically closed in E, E is of transcendence degree* 1 *over A and E/A allows a K.G. theory.*

PROOF. Let $B = (x, x_1, \cdots)$ be a transcendence base of E over K. Consider the subfield $\Sigma = K(x_1, \cdots) = K(\{B - x\})$. Let A be the algebraic closure of Σ in E. Over this field E is of transcendence degree 1, A is algebraically closed in E and E/A allows a $K.G.$ theory by proposition 1.

PROPOSITION 3. *A simple transcendental extension of a field does not allow a K.G. system.*

PROOF. Let, if possible, (E, K, G) be a $K.G.$ system with $E = K(x)$, x transcendental over K. We will get a contradiction. Consider the subfield $E_1 = K(x^3 + x^2)$ and let G_1 be the corresponding subgroup. Then, E_1 is the fixed field of G_1. Every automorphism of E over K is of the form $x \to ax + b/cx + d$, $ad - bc \neq 0$. It is easy to verify that the only automorphism which leaves E_1 fixed is the identity so that G_1 consists of the identity alone. Then the fixed field of G_1 is the whole field E but not E_1. This is a contradiction, and our result follows.

PROPOSITION 4. *Let (E, K, G) be a K.G. system and E_1 an intermediate field. Then E_1 is left invariant (setwise) by all the automorphisms of G if and only if the associated subgroup G_1 of E_1 is normal in G.*

The proof, being along obvious and well-known lines, is omitted.

PROPOSITION 5. *If E/K allows a K.G. theory (E, K, G) and if E_1 is left invariant (setwise) by all automorphisms of E over K in G, then E_1/K allows a K.G. theory.*

PROOF. Let G_1 be the associated group of E_1. Then by Proposition 4 G_1 is normal in G; let $Q = G/G_1$. Then each element of Q can be identified with an automorphism of E_1 in a natural way and we can speak of the Krull-closed subgroups of Q. It is easily seen that these are precisely the images of the closed subgroups of G containing G_1; and that this makes (E_1, K, Q) a $K.G.$ system.

If S is a closed subgroup of G containing G_1 and $\bar{S}$ is its image in the quotient group $Q = G/G_1$, we assert that $\bar{S}$ is closed in the Krull topology of Q considered as automorphisms of E_1 the fixed field of G_1. We note that we have only to show that, whenever T_α is a net of transformations in S, converging pointwise, (for all points of E_1) to a transformation T in G, then T itself is in S. This follows immediately on noting that the given closed subgroup S is paired with a subfield F of E_1 such that a transformation is in S if and only if it fixes each point of F. Since $T_\alpha \to T$ for all points in E_1 and T_α fix F pointwise, so does T.

On the other hand it is easy to verify that the canonical mapping from G to Q is continuous when the two groups are given the concerned Krull topologies. Thus there is a natural $1-1$ correspondence between the closed sub-groups of Q and those of G containing G_1. This is easily seen to be a Krull Galois theory for E_1 over K.

PROPOSITION 6. *Let E be a field and G some group of automorphisms with fixed field K. If G is locally finite (i.e., for each $x \in E$ the set $\{g(x)|g \in G\}$ is finite) then E is algebraic separable normal over K.*

We omit the proof, fairly straight forward. The result has also been stated without proof in [1].

PROPOSITION 7. *If (E, K, G) is a K.G. system then E is normal separable over every intermediate field over which it is algebraic.*

PROOF. Let F be a subfield over which it is algebraic. Then, if G_1 is the associated group of F, F is the fixed field of G_1. Since E is algebraic over F each element of E can be taken into only a finite number of elements by G_1 and so G_1 is locally finite. So by Proposition 6 E is normal separable over F.

PROPOSITION 8. *Let E be an extension field of K of transcendence degree* 1. *Then E/K does not allow a K.G. theory.*

PROOF. If possible let (E, K, G) be a $K.G.$ system. Without loss of generality we can suppose that K is algebraically closed in E. (Otherwise if K_1 is the algebraic closure of K in E, E/K_1 allows a $K.G.$ theory by Proposition 1 and we can consider E/K_1). Now E is algebraic over $K(\theta)$ for every θ not in K since E/K is of transcendence degree 1.

Case (*i*). Let E be of characteristic $p \neq 0$. Let x be an element not in K and consider $K(x^p)$. E is algebraic over $K(x^p)$ and so by Proposition 7 E is separable normal over $K(x^p)$. But x satisfies $z^p - x^p = 0$, an irreducible equation over $K(x^p)$ and hence is inseparable. This contradiction yields the desired result.

Case (*ii*). Let E be of characteristic 0. Let again x be an element not in K. Consider the subfields $K(x^2)$ and $K(x^2+x)$. $K(x)$ is normal over each since it is a quadratic extension of each. Let G_2, G_3 be the closed subgroups of G corresponding to $K(x^2)$, $K(x^2+x)$ respectively. Each element of $G_2(G_3)$ induces an automorphism of $K(x)$ over $K(x^2)(K(x^2+x))$ since $K(x)$ is normal over $K(x^2)(K(x^2+x))$.

Let us now consider the *closure* of the group $(G_2 \vee G_3)$ generated by G_2 and G_3. This can easily be verified to be a group. Let us call this closed group G_1. If F_1 is the subfield corresponding to G_1 then F_1 is the fixed field of G_1. Since G_1 contains G_2 and G_3, F_1 is contained in $K(x^2)$ and $K(x^2+x)$ since these are the fixed fields of G_2 and G_3. We assert that $F_1 = K$ itself. For there is an automorphism $\sigma : x \to -x$ of $K(x)/K(x^2)$ and an automorphism $\tau : x \to -(x+1)$ of $K(x)/K(x^2+x)$. Since F_1 is contained in $K(x^2) \cap K(x^2+x)$ both σ and τ fix F_1 and so $\tau\sigma : x \to x+1$ fixes F_1. But the fixed field of this automorphism is K only. Hence $F_1 = K$.

Since G_1 corresponds to $F_1 (= K)$ we get $G_1 = G$. Each element of the union $(G_2 \vee G_3)$ induces an automorphism of $K(x)$. If T is a limit of a net of elements from $(G_2 \vee G_3)$ then T has the same effect on any element of $K(x)$ as the members of the net ultimately and hence T also induces an automorphism of $K(x)$. Hence we get that $K(x)$ is invariant under all automorphisms of G, and by Proposition 5 $K(x)/K$ has *K.G.* theory which gives a contradiction to Proposition 3. This establishes our proposition.

3. The main theorem

We can now state the main result of our paper.

THEOREM 1. *The following properties of an extension field E of a field K are equivalent*:

(i) *E is an algebraic separable normal extension of K.*

(ii) *(E, K, G) is a K.G. system in the sense of Definition* 1 *of* § 1 *where G is the group of all automorphisms of E/K.*

(iii) *(E, K, G) is a K.G. system in the sense of Definition* 1 *of* § 1 *where G is some group of automorphisms of E for which the fixed field is K.*

PROOF. That (i) $\Rightarrow$ (ii) is the content of the classical result of Krull [2]. (ii) $\Rightarrow$ (iii) is obvious. We shall show that (iii) $\Rightarrow$ (i).

Let (E, K, G) be a *K.G.* system. If E/K is algebraic, Proposition 7 shows that it is separable and normal. *If* E/K is not algebraic, by Proposition 2, we can find an intermediate field I such that (a) E/I allows a *K.G.* theory and (b) E/I is of dimension 1 — which contradicts Proposition 8. This establishes our theorem.

4. The finite theory

In this section we attempt to prove an analogous result for the case of finite separable normal extensions. We start with slightly stronger assumptions than in § 1 which are suitable for our purpose.

Let E be an arbitrary field and G a group of automorphisms with fixed field K. We say there exists a full Galois theory for E if there exists a one to one 'Galois' correspondence between all the sub-fields of E containing the fixed field K on the one hand and all the subgroups of G on the other, such that the subfield F_1 corresponds to the subgroup Γ_1, if and only if F_1 is the set of all elements left fixed by (each element of) Γ_1; and Γ_1 is the set of all automorphisms fixing (each point of) F_1. It follows immediately that G consists of all automorphisms of E over K and K is the fixed field of G.

We shall also indicate this by saying that the system (E, K, G) is a Galois system (*G.S.*) or also that (E, K) allows a full Galois theory.

It is our aim to prove

THEOREM 2. *An extension field E of a field K allows a full Galois theory (E, K) if and only if E is finite, separable and normal over K.*

In view of Galois theory [5] we need prove only the 'only if' part.

PROOF OF THE THEOREM. We wish to observe first that the analogous statements of our Propositions 1, 2, 3 and 4, 6 and 7 hold in this case also by practically similar proofs. As the analogue of Proposition 5 may not hold, we proceed differently.

Let (E, K, G) be a full Galois system. We prove first that E is separable normal over K. In case E is algebraic over K the proof is the same as in proposition 7. If E is not algebraic over K we can assume that K is algebraically closed in E and E is of transcendence degree 1 over K (by proposition 2).

If E is of characteristic p, the proof is the same as in proposition 8. Considering characteristic zero, along the same lines as in proposition 8, we can prove that every element of G leaves $K(x)$ invariant for any x in $E-K$. We will show that this is a contradiction. For consider $K(x^3+x)$. Let its associated group be G_1. Then $K(x^3+x)$ is the fixed field of G_1. Also E is normal separable over $K(x^3+x)$ with group G_1 and $K(x)$ remains invariant under all automorphisms. So it follows that $K(x)$ is normal separable over $K(x^3+x)$, i.e. $K(x^3+x)$ is the fixed field of all automorphisms of $K(x)$ over $K(x^3+x)$. This is a contradiction since $K(x)$ is easily verified to have only the identity automorphism over $K(x^3+x)$.

Hence we have proved that E is separable normal over K. If E is not finite then by [2] there always exists a 'nonclosed' subgroup of G contrary to the assumption that (E, K, G) is a full Galois system.

Hence E is finite separable normal over K. Q.E.D.

5. Normal extensions

We can now restate our results as characterizations of normal extensions.

THEOREM 3. *Let Δ be an extension of the field K. Then the following are equivalent.*

(i) *Δ is finite, separable and normal over K.*

(ii) *There exists a finite group G of automorphisms of Δ with fixed field K.*

(iii) *There exists a group G with fixed field K such that G is the whole group of automorphisms of Δ over K and there exists a one-one 'Galois' correspondence between all intermediate fields and all sub-groups of G.*

PROOF. Proof of (i) $\Leftrightarrow$ (ii) can be found in [3], that of (i) $\Rightarrow$ (iii) in [5]. We have shown above that (iii) $\Rightarrow$ (i). Hence all the three are equivalent.

THEOREM 4. *Let Δ be an extension of the field K. Then the following conditions are equivalent.*

i) *Δ is separable and normal over K.*

ii) *There exists a locally finite group of automorphisms of Δ with fixed field K.*

iii) *There exists a group H of automorphisms with fixed field K such that there exists a one-one 'Galois' correspondence between all intermediate fields of Δ/K and all closed subgroups of G under the Krull topology of G.*

PROOF (i) $\Rightarrow$ (ii). If G is the group of all automorphisms of Δ over K, then G is locally finite. Since Δ is a normal extension, if $\alpha \in \Delta - K$, and α_1 is a conjugate of α we can find an automorphism of G taking α to α_1. Hence K is the fixed field of G.

(ii) $\Rightarrow$ (i) follows from proposition (6) of § 2. For (i) $\Rightarrow$ (iii), cf. [2]. We have shown (iii) $\Rightarrow$ (i). Hence all are equivalent.

References

[1] R. Baer, A Galois theory of linear systems over commutative fields, Amer. J. Math. 62 (1940), 571.

[2] W. Krull, Galoissche Theorie der unendlichen algebraischen Erweiterungen, Math. Ann. 100 (1928), 687–698.

[3] G. Pickert, Einführung in die Höhere Algebra (Göttingen, 1951).

[4] E. Steinitz, Algebraische Theorie der Körper (New York, 1951).

[5] B. L. van der Waerden, Modern Algebra, Vol. I (New York, 1950).

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