BULL. AUSTRAL. MATH. SOC. 53A99, 53C42, 53D15
VOL. 65 (2002) [183–189]

PARALLEL SURFACES IN THE REAL SPECIAL LINEAR GROUP $SL(2,\mathbb{R})$

MOHAMED BELKHELFA, FRANKI DILLEN AND JUN-ICHI INOGUCHI

Dedicated to Professor Koichi Ogiue on his sixtieth birthday

We show that the only parallel surfaces in $SL(2,\mathbb{R})$ are rotational surfaces with constant mean curvature.

1. INTRODUCTION

It is well known that hypersurfaces of real space forms with parallel second fundamental form are spherical or products of spherical submanifolds, see [5] for a recent survey. On the other hand, there are few results on submanifolds with parallel second fundamental form, or simply parallel submanifolds, in arbitary Riemannian manifolds. In [2] parallel surfaces in the 3-dimensional Heisenberg group H_3 with canonical left invariant metric are classified. The Heisenberg group H_3 is a typical example of a naturally *reductive homogeneous 3-manifold or a Sasakian space form*. The special linear group $SL(2,\mathbb{R})$ is another such example. In this paper we shall classify parallel surfaces in $SL(2,\mathbb{R})$.

2. SPECIAL LINEAR GROUP $SL(2,\mathbb{R})$

We start with recalling fundamental properties of the real special linear group $SL(2,\mathbb{R})$. Let G denote the 2×2 real special linear group defined by

$$G = SL(2,\mathbb{R}) = \{g \in GL(2,\mathbb{R})\ ;\ \det g = 1\}.$$

It is well-known that any element g of G can be decomposed uniquely as

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Received 26th April, 2001.
The first author was supported by the Fund for Scientific Research - Flanders (Belgium)(F.W.O.) through project G.0366.98. The third author was partially supported by Grand-in-Aid for Encouragement of Young Scientists 12740051, Japan Society for Promotion of Science.

for some $x \in \mathbb{R}$, $y \in \mathbb{R}^+$ and $\theta \in S^1$ (the *Iwasawa decomposition* of g). We may consider (x, y, θ) as a global coordinate system of G. Let us denote by $\mathfrak{g}$ the Lie algebra of G

$$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) = \{X \in \mathfrak{gl}(2, \mathbb{R});\ \ \mathrm{tr} X = 0\}.$$

We define an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by $\langle X, Y \rangle = \mathrm{tr}\ ({}^t XY)/2$, where ${}^t X$ denotes the transposed matrix of X. Let us denote by ds_G^2 the left-invariant Riemannian metric induced by $\langle \cdot, \cdot \rangle$. The metric ds_G^2 is written as

$$ds_G^2 = \left(\frac{dx}{2y}\right)^2 + \left(\frac{dy}{2y}\right)^2 + \left(\frac{dx}{2y} + d\theta\right)^2$$

with respect to the global coordinate system (x, y, θ). The Lie group G has a compact, connected subgroup $K = SO(2)$. The homogeneous space G/K is diffeomorphic to the upper half-plane $\mathbf{H}^2$. The metric ds_G^2 induces the Poincaré metric of constant Gauss curvature -4 on $\mathbf{H}^2$. The natural projection $\pi : G \to \mathbf{H}^2$ is a Riemannian submersion with totally geodesic fibres.

Let $\{\omega^1, \omega^2, \omega^3\}$ be an orthonormal coframe field defined by

$$\omega^1 = \frac{dx}{2y},\quad \omega^2 = \frac{dy}{2y},\quad \omega^3 = \frac{dx}{2y} + d\theta.$$

The dual orthonormal frame field $\{e_1, e_2, e_3\}$ of $\{\omega^1, \omega^2, \omega^3\}$ is given by

$$e_1 = 2y\frac{\partial}{\partial x} - \frac{\partial}{\partial \theta},\quad e_2 = 2y\frac{\partial}{\partial y},\quad e_3 = \frac{\partial}{\partial \theta}.$$

Note that $\{\omega^i\}$ and $\{e_i\}$ are globally defined on G. Furthermore ω^3 is a *contact form* on G. Namely $d\omega^3 \wedge \omega^3 \neq 0$ on G.

Let us denote the Levi-Civita connection and Riemannian curvature tensor of (G, ds_G^2) by $\overline{\nabla}$ and $\overline{R}$ respectively. Then we have

$$\begin{array}{lll} \overline{\nabla}_{e_1} e_1 = 2e_2, & \overline{\nabla}_{e_1} e_2 = -2e_1 - e_3, & \overline{\nabla}_{e_1} e_3 = e_2, \\ \overline{\nabla}_{e_2} e_1 = e_3 & \overline{\nabla}_{e_2} e_2 = 0, & \overline{\nabla}_{e_2} e_3 = -e_1, \\ \overline{\nabla}_{e_3} e_1 = e_2, & \overline{\nabla}_{e_3} e_2 = -e_1, & \overline{\nabla}_{e_3} e_3 = 0, \\ [e_1, e_2] = -2e_1 - 2e_3, & [e_1, e_3] = 0, & [e_2, e_3] = 0. \end{array} \tag{1}$$

$$\begin{array}{lll} \overline{R}(e_1, e_2)e_1 = 7e_2, & \overline{R}(e_1, e_2)e_2 = -7e_1, & \overline{R}(e_1, e_3)e_1 = -e_3, \\ \overline{R}(e_1, e_3)e_3 = e_1, & \overline{R}(e_2, e_3)e_2 = -e_3, & \overline{R}(e_2, e_3)e_3 = e_2. \end{array} \tag{2}$$

For more information on properties of $(SL(2, \mathbb{R}), ds_G^2)$, especially on the Sasakian structure, we refer to [**4**] and [**3**].

To close this section, we recall the notion of *rotational surface* in G.

DEFINITION: According to [**4**], an immersed surface M in G is said to be *rotational* if it is invariant under a right K-action.

A rotational surface M is parametrised as $(x, y, \theta) = (x(t), y(t), \theta)$. The curve $(x(t), y(t))$ in $\mathbf{H}^2$ is called the *generating curve* of M.

3. Parallel surfaces in $SL(2,\mathbb{R})$

In this section we shall classify parallel surfaces in $SL(2,\mathbb{R})$.

Theorem. *The only parallel surfaces in the real special linear group $SL(2,\mathbb{R})$ are rotational surfaces of constant mean curvature. The generating curve is a Riemannian circle. Furthermore such surfaces are flat.*

Let M be a surface in G with unit normal vector field $\mathbf{n}$. Denote by ∇ the Levi-Civita connection of M. Then the second fundamental form h of M is determined by the *Gauss formula*,

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X, Y \in \mathfrak{X}(M).$$

For any $X, Y, Z \in \mathfrak{X}(M)$, the normal component of the curvature $\overline{R}(X,Y)Z$ is described by the *Codazzi equation*,

$$\left(\overline{R}(X,Y)Z\right)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z).$$

Here the covariant derivative $\overline{\nabla}h$ is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where $\nabla^{\perp}$ is the normal connection of M in G. Recall that a surface M in G is said to be *parallel* if $\overline{\nabla}h = 0$.

Now we take a moving frame $\{X_1, X_2, X_3 = \mathbf{n}\}$ and its dual coframe $\{\theta^1, \theta^2, \theta^3\}$ such that X_1, X_2 are tangent to M. Since θ^3 is dual to $\mathbf{n}$, the restriction of θ^3 on M satisfies $\theta^3 = 0$. We can write $\theta^3 = a\omega^1 + b\omega^2 + c\omega^3$. Since $\{\omega^i\}$ is orthonormal, $a^2 + b^2 + c^2 = 1$.

The vector fields $u_1 = ce_1 - ae_3$ and $u_2 = ce_2 - be_3$ are tangent to M. In particular if $c \neq 0$, $\{u_1, u_2\}$ is a frame field of M. To prove our main theorem we need the following.

Lemma. *Let M be a surface in G. Then the normal component of the curvature $\overline{R}$ is given by*

$$\left(\overline{R}(u_1,u_2)u_1\right)^{\perp} = 8bc^3\mathbf{n}, \quad \left(\overline{R}(u_1,u_2)u_2\right)^{\perp} = -8ac^3\mathbf{n}.$$

Proof of Lemma: By direct computations

$$\overline{R}(u_1,u_2)u_1 = c^3\overline{R}(e_1,e_2)e_1 - bc^2\overline{R}(e_1,e_3)e_1 + abc\overline{R}(e_1,e_3)e_3 + a^2c\overline{R}(e_3,e_2)e_3 - ac^2\left(\overline{R}(e_1,e_2)e_3 + \overline{R}(e_3,e_2)e_1\right).$$

Next by using the formula (2) we have

$$\overline{R}(u_1,u_2)u_1 = abc\, e_1 + (7c^3 - a^2c)e_2 + bc^2 e_3.$$

Since $\mathbf{n}$ is expressed as $\mathbf{n} = ae_1 + be_2 + ce_3$, we get

$$\left(\overline{R}(u_1,u_2)u_1\right)^{\perp} = 8bc^3\mathbf{n}.$$

The second formula can be proved similarly. ∎

PROOF OF THEOREM: Assume that M is parallel. Then the Codazzi equation implies

$$\big(\overline{R}(u_1,u_2)u_1\big)^\perp = (\overline{\nabla}_{u_1}h)(u_2,u_1) - (\overline{\nabla}_{u_1}h)(u_1,u_1) = 0.$$

Similarly we have

$$\big(\overline{R}(u_2,u_1)u_2\big)^\perp = 0.$$

First we assume $c \neq 0$. Then the Lemma yields $a = b = 0$. Hence $\theta^3 = \omega^3$. This implies that M is an integral surface of the distribution defined by $\omega^3 = 0$. However this is impossible, since ω^3 is a contact form on G. Thus M satisfies $c = 0$. Therefore θ^3 has the form $\theta^3 = a\omega^1 + b\omega^2$. Since $a^2 + b^2 = 1$, the orthonormal vector fields $v_1 = be_1 - ae_2$, $v_2 = e_3$ give a tangent orthonormal basis of the distribution $\theta^3 = 0$. Using (1) we obtain the following

$$\begin{aligned} \overline{\nabla}_{v_1}v_1 &= 2b\mathbf{n} + v_1(b)e_1 - v_1(a)e_2,\\ \overline{\nabla}_{v_1}v_2 &= \mathbf{n},\\ \overline{\nabla}_{v_2}v_1 &= v_2(b)e_1 - v_2(a)e_2 + \mathbf{n},\\ \overline{\nabla}_{v_2}v_2 &= 0. \end{aligned} \tag{3}$$

By using the relation

$$e_1 = bv_1 + a\mathbf{n},\ e_2 = -av_1 + b\mathbf{n},$$

we have

$$\overline{\nabla}_{v_2}v_1 = \big\{av_2(b) - bv_2(a) + 1\big\}\mathbf{n}.$$

By the Gauss formula, we obtain the induced connection ∇ and the second fundamental form h

$$\nabla_{v_1}v_1 = 0,\quad \nabla_{v_1}v_2 = 0,\quad \nabla_{v_2}v_2 = 0, \tag{4}$$

$$\begin{aligned} &h(v_1,v_1) = (2b + av_1(b) - bv_1(a)\,)\mathbf{n},\\ &h(v_1,v_2) = \mathbf{n},\quad h(v_2,v_1) = \big\{1 + av_2(b) - bv_2(a)\big\}\mathbf{n},\\ &h(v_2,v_2) = 0. \end{aligned} \tag{5}$$

In particular by the symmetry of h, we have

$$av_2(b) - bv_2(a) = 0. \tag{6}$$

Since $a^2 + b^2 = 1$, we may write $a = \cos\phi$, $b = \sin\phi$. Then (6) becomes

$$v_2(\phi) = 0. \tag{7}$$

The formulae (4) imply that M is flat. The mean curvature H of M is computed by (5),

$$2H = 2b + av_1(b) - bv_1(a). \tag{8}$$

Without loss of generality, we may assume $H \geqslant 0$. Moreover, since M is parallel, the mean curvature H is constant on M.

Note that the formulae in (4) imply that the parallel surface M is parametrised by θ and a coordinate t such that

$$\frac{\partial}{\partial t} = v_1.$$

The equation (7) is rewritten as

$$\frac{\partial}{\partial \theta}\phi = 0, \tag{9}$$

such that ϕ depends only on t. The equation (8) becomes

$$2\sin\phi + \frac{d\phi}{dt} \equiv 2H. \tag{10}$$

The formulae (3) and (4) involve actually somewhat more. In fact (4) implies that both t-coordinate curves and θ-coordinate curves are geodesics on M. Note that the θ-coordinate curves (integral curves of $v_2 = e_3$) are geodesics in G. Let $\gamma(t)$ be a t-coordinate curve (that is, an integral curve of v_1). Then the formulae (3) imply that γ is a Frenet curve of osculating order 3, see [**1**, p. 137]. The principal normal and binormal vector fields of γ are $N = \mathbf{n}$ and $B = -e_3$. The Frenet-Serret formula of γ with respect to $(T = \gamma', N, B)$ is

$$\overline{\nabla}_{\gamma'}(T, N, B) = (T, N, B)\begin{pmatrix} 0 & -2H & 0 \\ 2H & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This formula says every t-coordinate curve on M is a helix in G with constant curvature $2H$ and constant torsion 1. In particular every t-coordinate curve is a *Legendre curve* (that is, $\omega^3(\gamma') = 0$). Equivalently, γ is a horizontal curve with respect to the Riemannian submersion π. Thus the parallel surface M is generated by Legendre curves of constant curvature $2H$ and integral curves of e_3.

Let us denote by $\overline{\gamma} := \pi \circ \gamma$ the projection of γ onto $\mathbf{H}^2$. Then we notice

$$\pi_* v_1 = 2y\sin\phi\frac{\partial}{\partial x} - 2y\cos\phi\frac{\partial}{\partial y}.$$

On the other hand, since v_1 is the tangent vector field of γ, we have

$$\pi_* v_1 = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}.$$

Comparing these we get

$$\frac{dx}{dt} = 2y(t)\sin\phi(t),\ \frac{dy}{dt} = -2y(t)\cos\phi(t). \tag{11}$$

The formulae (11) imply that t is the arclength parameter of $\overline{\gamma}$. Recall that the curvature $\overline{\kappa}$ of $\overline{\gamma}$ is given by $(x'y'' - x''y')/(4y^2) + x'/y$. Direct computations using (10) and (11) show that $\overline{\kappa} = 2H$. Hence $\overline{\gamma}$ is a (part of) Riemannian circle of constant curvature $2H$ in $\mathbf{H}^2$. In particular the case $H = 0$, $\overline{\gamma}$ is a geodesic in $\mathbf{H}^2$.

Recall that the surface M is parametrised by t and θ. And the t-coordinate curves are horozontal lifts of Riemannian circles with curvature $2H$ in $\mathbf{H}^2$. Thus we concluded that M is a rotational surface over a Riemannian circle in $\mathbf{H}^2$. In particular, M is parametrised by $\big(x(t), y(t), \theta\big)$, see [**4**, Proposition 4.2].

We consider these Riemannian circles more in detail, see also [**4**, Proposition 4.3].

First we investigate the case ϕ is constant. Then by (10), we obtain $0 \leqslant H = \sin\phi \leqslant 1$. Thus $\cos\phi = \pm\sqrt{1 - H^2}$. If $H = 1$, then (11) implies $y = y_0 =$ constant, so $\overline{\gamma}$ is a horizontal line.

If $H \neq 1$, then (11) is solved as follows

$$x = x_0 \pm \frac{H}{\sqrt{1 - H^2}}y.$$

Thus $\overline{\gamma}$ is a straight line.

Next, if $\phi' \neq 0$, then we can proceed as in [**3**, p. 156-157] and solve (11) explicitly. From (10) and (11) we obtain

$$\frac{dy}{d\phi} = \frac{-y\cos\phi}{H - \sin\phi},$$

which can be solved as follows

$$y = r(H - \sin\phi)$$

for some non zero constant r. Similarly we obtain

$$\frac{dx}{d\phi} = r\sin\phi,$$

such that

$$\big(x(t), y(t)\big) = \Big(-r\cos\phi(t) + x_0, r\big(H - \sin\phi(t)\big)\Big),$$

for some constant $x_0 \in \mathbf{R}$. Hence $\overline{\gamma}$ is (a part of) circle

$$(x - x_0)^2 + (y - rH)^2 = r^2.$$

Note that $\overline{\gamma}$ is closed if and only if $H > 1$, see [**3**, p. 156] or [**4**, Corollary 4.4].

Conversely it is easy to check that every rotational surface of constant mean curvature has parallel second fundamental form. ▯

Note that our Theorem implies that there are no extrinsic spheres (that is, totally umbilical surfaces with parallel mean curvature vector field), in particular, no totally geodesic surfaces in $\big(SL(2,\mathbb{R}), ds_G^2\big)$.

References

[1] C. Baikoussis and D. E. Blair, 'On Legendre curves in contact 3-manifolds', *Geom. Dedicata* **49** (1994), 135–142.

[2] M. Belkhelfa and F. Dillen, 'Parallel surfaces in Heisenberg space' (to appear).

[3] J. Inoguchi, T. Kumamoto, N. Ohsugi and Y. Suyama, "Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces III', *Fukuoka Univ. Sci. Rep.* **30** (2000), 131–160.

[4] M. Kokubu, 'On minimal surfaces in the real special linear group $SL(2, \mathbf{R})$', *Tokyo J. Math.* **20** (1997), 287-297.

[5] Ü. Lumiste, 'Submanifolds with parallel fundamental form', in *Handbook of Differential Geometry* I (North-Holland, Amsterdam, 2000), pp. 779–864.

Department of Mathematics
Katholieke Universiteit Leuven
Celestijnenlaan 200B
B-3001 Heverlee
Belgium
e-mail: mohamed.belkhelfa@wis.kuleuven.ac.be

Department of Mathematics
Katholieke Universiteit Leuven
Celestijnenlaan 200B
B-3001 Heverlee
Belgium
e-mail: franki.dillen@wis.kuleuven.ac.be

Departement of Applied Mathematics
Fukuoka University
Fukuoka 814-0180
Japan
e-mail: inoguchi@bach.sm.fukuoka-u.ac.jp