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LEXICOGRAPHIC DIRECT SUMS OF ELEMENTARY C^*-ALGEBRAS

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1. Introduction. Besides the simple ones, there are several other kinds of C^*-algebras which it has proved interesting to try to classify. For instance, a large body of results relates to the extensions of one given C^*-algebra, possibly simple, by another. Extrapolating in this direction, we have considered the class of C^*-algebras which can be decomposed in the strongest possible nontrivial sense in terms of their simple subquotients, and such that these simple subquotients in turn are as uncomplicated as possible.

We have found that the classification of these C^*-algebras, namely, the lexicographic direct sums of elementary C^*-algebras, is to a large degree tractable, and yet involves an interesting new invariant in the antiliminary case, which is the case of no minimal ideals. Even the postliminary case, which is the case that the ordered set of simple subquotients satisfies the decreasing chain condition, is not without interest as an extension of the case of finitely many simple subquotients, analysed in the earlier papers [**1**] and [**2**].

This new invariant, being a generalized integer (or a family of generalized integers), resembles in a striking way the invariant introduced by Glimm in [**13**] to classify uniformly hyperfinite C^*-algebras, which are simple and apparently unrelated to lexicographic direct sums of elementary C^*-algebras.

Lexicographic direct sums of elementary C^*-algebras are, it is easy to show, approximately finite-dimensional. Their dimension groups, as defined in [**9**], turn out to be lexicographic direct sums of copies of $\mathbf{Z}$ (2.8).

We recall that the lexicographic direct sum of an ordered family of ordered abelian groups $(G_x)_{x \in P}$, which we shall denote by

$$^{\text{lex}}\bigoplus_{x \in P} G_x,$$

is the direct sum group $\bigoplus_{x \in P} G_x$, with a nonzero element positive if for each $x \in P$ maximal such that the coordinate at x is not zero, that coordinate is positive in G_x. By 3.10 of [**10**], the lexicographic direct sum of an ordered family of dimension groups is a dimension group. (We shall use

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the definition of dimension group given in 3.10 of [**10**], namely, the ordered group direct limit of a net of finite ordered group direct sums of copies of $\mathbf{Z}$.)

In the case that the dimension group G_x is equal to $\mathbf{Z}$ for every $x \in P$, we shall write $\mathbf{Z}^{(P)}$ instead of $\bigoplus_{x \in P}\mathbf{Z}$, and $\mathbf{Z}^{(P)\mathrm{lex}}$ instead of ${}^{\mathrm{lex}}\bigoplus_{x \in P}\mathbf{Z}$. It is easily seen that the ideals of $\mathbf{Z}^{(P)\mathrm{lex}}$ are precisely the subgroups $\mathbf{Z}^{(R)}$ with R an ideal of P, and that this correspondence is an isomorphism of the ideal lattice of $\mathbf{Z}^{(P)\mathrm{lex}}$ and the ideal lattice of P (cf. Section 5.2 of [**10**]).

We recall that an ideal of a dimension group is a positively generated subgroup such that if $0 \leqq g \leqq h$ and h is in the subgroup then g is, too. The quotient of a dimension group by an ideal, ordered by the image of the positive part, is again a dimension group. A dimension group is prime if it is nonzero and does not have two nonzero ideals with intersection zero. A prime ideal is an ideal with respect to which the quotient is prime. We shall use similar terminology for ordered sets: an ideal is a subset such that if $x \leqq y$ and y belongs to the subset then x does, too; an ordered set is prime if it is nonempty and does not have two nonempty ideals with empty intersection; a prime ideal is an ideal with respect to which the quotient, i.e., the complement, is prime. Incidentally, we shall also use similar terminology for algebras, which is however completely standard, except that we shall say ideal instead of two-sided ideal.

The prime ideal spectrum of an ordered set or of a dimension group will mean the space of prime ideals with the Jacobson (or hull-kernel) topology, with the proviso that for dimension groups the kernel of a collection of ideals is the ideal whose positive part is the intersection of the positive parts of the ideals in the collection. (The intersection of the ideals themselves may fail to be positively generated, as shown in [**12**].) We shall denote the prime ideal spectrum of a dimension group G by Spec G.

While on the subject of ideals, we should like briefly to mention an error in [**5**]: in Theorem 6 the conditions given are insufficient for a topological space to be the primitive spectrum of a separable postliminary approximately finite-dimensional C^*-algebra; this was pointed out to the second author by B. Blackadar and L. Brown. Blackadar and Brown have proved sufficiency with the additional (necessary) condition that the space be almost Hausdorff.

As shown in [**9**], the additional information which determines a separable approximately finite-dimensional C^*-algebra, besides the dimension group, is the range of the dimension. This is an upward directed, generating subset of the positive part of the dimension group, such that if $0 \leqq g \leqq h$ and h is in the subset then g is, too. Following Handelman in [**15**], we shall refer to such a subset of a dimension group simply as an interval. By [**9**], at least in a countable dimension group, every interval arises as the dimension range of an approximately finite-dimensional C^*-algebra. The image of an interval in a quotient of a dimension group is an interval.

Our main results are as follows. We shall show that an interval in $\mathbf{Z}^{(P)\mathrm{lex}}$ is determined by knowing when its image in a prime quotient has a largest element, and also what that element is (3.5). We shall summarize this latter information by means of a function from P to $\mathbf{Z} \cup \{+\infty\}$, the defector of the interval, much as in the case of finite P considered in [**1**] and [**2**] (3.5).

We shall show that the dimension range of a lexicographic direct sum of elementary C^*-algebras is an interval with positive defector (a positive defector is one with positive values), with a largest element in any prime quotient where this is permitted by the defector (2.8, 3.6).

Conversely, we shall show that every such interval is the dimension range of a lexicographic direct sum of elementary C^*-algebras. Furthermore, we shall do this by a construction which is functorial, from the category of such intervals in lexicographic direct sums of copies of $\mathbf{Z}$, with isomorphisms, to the category of C^*-algebras, with isomorphisms (2.6, 2.9). By 4.3 of [**9**], it follows that every separable lexicographic direct sum of elementary C^*-algebras is isomorphic to one of the algebras given by this construction. This leads to a characterization of postliminary C^*-algebras in this class (5.5).

Finally, we shall introduce a divisor invariant of the defector of an interval, which is a function from certain prime ideals of the ordered set to generalized integers (4.12). We shall show that for intervals in $\mathbf{Z}^{(-\mathbf{N})\mathrm{lex}}$ with defector equal to 1 at 0, this invariant, a single generalized integer, is complete (4.13).

2. Lexicographic direct sum C^*-algebras.

2.1. *Definition.* Let P be an ordered set, and let $(B_x)_{x \in P}$ be a family of nonzero simple C^*-algebras. Let A be a C^*-algebra. We shall say that A is a *lexicographic direct sum* (or an *external lexicographic direct sum*) of the ordered family of simple C^*-algebras $(B_x)_{x \in P}$ if there exists a linearly independent family $(A_x)_{x \in P}$ of sub-C^*-algebras of A, with A_x isomorphic to B_x for each $x \in P$, with the following properties:

(i) If $x, y \in P$ are not comparable, then $A_xA_y = 0$.

(ii) If $x, y \in P$ and $x < y$ then $A_xA_y \subseteq A_x$ and A_x is the unique nonzero proper closed ideal of $A_x + A_y$.

(iii) $\sum_{x \in P} A_x$ is dense in A.

In these circumstances we shall also say that A is the *internal lexicographic direct sum* of the ordered family of simple sub-C^*-algebras $(A_x)_{x \in P}$. (We use the indefinite article in referring to an external lexicographic direct sum because this in general is not unique.)

2.2. THEOREM. *Let A be a C^*-algebra, let P be an ordered set, and suppose that A is the internal lexicographic direct sum of the ordered family of simple* *sub-C^*-algebras $(A_x)_{x \in P}$. It follows that the lattice of closed ideals of A is isomorphic to the lattice of ideals of P. More precisely, if R is an ideal of P then the closure of $\sum_{x \in R} A_x$ is an ideal of A, and the map*

$$R \mapsto (\textstyle\sum_{x \in R} A_x)^-$$

from the lattice of ideals of P to the lattice of closed ideals of A is an order isomorphism. Furthermore, if R is an ideal of P, then the restriction of the quotient map

$$A \to A/(\textstyle\sum_{x \in R} A_x)^-$$

to the sub-C^*-algebra $(\sum_{x \in P \setminus R} A_x)^-$ is an isomorphism.

Proof. Note first that by (i) and (ii), for any $x, y \in P$, $A_x A_y$ is contained either in A_x or in A_y. Thus, $\sum_{x \in Q} A_x$ is a subalgebra of A for any $Q \subseteq P$.

Let R be an ideal of P. Fix $x \in R$ and $y \in P \setminus R$. Then either x and y are not comparable, or $x < y$ (if $y \leqq x$ then $y \in R$). In the first case, by (i), $A_x A_y = 0$. In the second case, by (ii), $A_x A_y \subseteq A_x$. Thus, in either case, $A_x A_y \subseteq A_x$. It follows that $\sum_{x \in R} A_x$ is an ideal of $\sum_{y \in P} A_y$. Hence $(\sum_{x \in R} A_x)^-$ is an ideal of $(\sum_{y \in P} A_y)^-$, which by (iii) is equal to A.

Let R_1 and R_2 be ideals of P. Clearly, if $R_1 \subseteq R_2$, then

$$(\textstyle\sum_{x \in R_1} A_x)^- \subseteq (\textstyle\sum_{x \in R_2} A_x)^-.$$

Suppose that

$$(\textstyle\sum_{x \in R_1} A_x)^- \subseteq (\textstyle\sum_{x \in R_2} A_x)^-.$$

To show that $R_1 \subseteq R_2$, let us first show that the quotient map

$$A \to A/(\textstyle\sum_{x \in R_2} A_x)^-$$

is isometric on $\sum_{x \in P \setminus R_2} A_x$. This will also establish the last statement of the theorem. It is sufficient to show that for each $a \in \sum_{x \in P \setminus R_2} A_x$ and $b \in \sum_{x \in R_2} A_x$,

$$\|a\| \leqq \|a + b\|.$$

Passing to a finite subset Q of A such that both a and b belong to $\sum_{x \in Q} A_x$, to show that $\|a\| \leqq \|a + b\|$ we may suppose that P is finite. In this case, as we shall show below, $\sum_{x \in R_2} A_x$ and $\sum_{x \in P \setminus R_2} A_x$ are closed. Since the quotient map

$$A \to A/\textstyle\sum_{x \in R_2} A_x$$

with respect to the closed ideal $\sum_{x \in R_2} A_x$ is, by independence, injective on the sub-C^*-algebra $\sum_{x \in P \setminus R_2} A_x$, by 1.8.3 of [7] it is isometric there.

Hence

$$\|a\| = \|a + \textstyle\sum_{x \in R_2} A_x\| \leqq \|a + b\|.$$

It follows that

$$(\textstyle\sum_{x \in P \setminus R_2} A_x) \cap (\sum_{x \in R_2} A_x)^- = 0$$

(whether P is finite or not). Hence, as

$$\textstyle\sum_{x \in R_1} A_x \subseteq (\sum_{x \in R_2} A_x)^-,$$

$R_1 \subseteq R_2$ ($x \in R_1 \setminus R_2$ would imply $A_x = 0$).

In what precedes we stated that certain finite sums of A_x's are closed. We shall in fact prove that any finite sum of A_x's is closed (whether P is finite or not). Let Q be a finite subset of P. Choose y maximal in Q. By induction, we may suppose that $\sum_{x \in Q \setminus \{y\}} A_x$ is closed. It is then a closed ideal of the C^*-algebra $(\sum_{x \in Q} A_x)^-$. By 1.8.4 of [7], the sum of the closed ideal $\sum_{x \in Q \setminus \{y\}} A_x$ and the sub-C^*-algebra A_y is closed. Thus we see that $\sum_{x \in Q} A_x$ is closed.

Finally, let I be an arbitrary closed ideal of A, and let us show that

$$I = (\textstyle\sum_{x \in R} A_x)^-$$

for some ideal R of P. As A is the direct limit of the net of sub-C^*-algebras

$$(\textstyle\sum_{x \in Q} A_x)_{Q \subseteq P,\ Q \text{ finite}},$$

by the proof of 1.4 of [8] (see also 3.1 of [4]) we have

$$I = (\textstyle\bigcup_{Q \subseteq P,\ Q \text{ finite}} I \cap \sum_{x \in Q} A_x)^-.$$

From this we see that it is sufficient to show, for each finite subset Q of P, that $I \cap \sum_{x \in Q} A_x$ is equal to

$$\textstyle\sum_{x \in R(Q)} A_x$$

for some ideal $R(Q)$ of Q. (Then $R(Q_1) \subseteq R(Q_2)$ whenever $Q_1 \subseteq Q_2$, so $\bigcup_Q R(Q)$ is an ideal of P; clearly

$$I = (\textstyle\sum_{x \in \bigcup_Q R(Q)} A_x)^-.)$$

Thus, it suffices to consider the case that P is finite. Then the ordered set P has a finite composition series of ideals

$$\emptyset = P_0 \subseteq P_1 \subseteq \ldots \subseteq P_n = P$$

with each subquotient (i.e., difference) $P_{i+1} \setminus P_i$ a single point. Hence the sequence of ideals

$$0 \subseteq \textstyle\sum_{x \in P_1} A_x \subseteq \ldots \subseteq \sum_{x \in P_n} A_x = A$$

is a composition series for A, with subquotients isomorphic to

$$A_{P_1\setminus P_0}, A_{P_2\setminus P_1}, \ldots, A_{P_n\setminus P_{n-1}}$$

and therefore simple. It follows that the primitive spectrum of A is finite and in particular every proper closed ideal of A is contained in a maximal (proper) closed ideal. If $I = A$, then

$$I = \sum_{x\in P} A_x.$$

Assume that $I \neq A$. We shall show that I is contained in $\sum_{x\in Q} A_x$ for some proper subset Q of P; the desired conclusion then holds by induction on the number of elements of P. For this purpose we may replace I by a maximal closed ideal of A containing I (that such exists was shown above). Choose y maximal in P. Then

$$\sum_{x\in P\setminus\{y\}} A_x$$

is a maximal closed ideal of A (the quotient is isomorphic to the simple nonzero C^*-algebra A_y). Either the two maximal closed ideals I and $\sum_{x\in P\setminus\{y\}} A_x$ are equal, or their intersection is a maximal closed ideal of each of them. In the first case we are finished. In the second case, by induction there exists a maximal element z of $P\setminus\{y\}$ such that

$$I \cap \sum_{x\in P\setminus\{y\}} A_x = \sum_{x\in P\setminus\{y,z\}} A_x.$$

Hence we may pass to the quotient of A by the closed ideal

$$\sum_{x\in P\setminus\{y,z\}} A_x,$$

modulo which A_y and A_z are independent and satisfy conditions (i), (ii), and (iii) of 2.1 with $\{y, z\}$ in place of P. If y and z are incomparable, then $A_yA_z = 0$, so the only nonzero proper closed ideals of A (now equal to $A_y + A_z$) are A_y and A_z. If y and z are comparable, then $z < y$ (since y is maximal in P, and $z \in P\setminus\{y\}$); hence in this case, by (ii), A_z is the unique nonzero proper closed ideal of A.

2.3. *Remark*. In Definition 2.1, independence of the family $(A_x)_{x\in P}$ together with properties (i) and (ii) could be replaced by the following two properties:

(iv) For any subset Q of P and any ideal R of Q, $\sum_{x\in Q} A_x$ is a subalgebra of A and $\sum_{x\in R} A_x$ is an ideal of $\sum_{x\in Q} A_x$.

(v) For any subset Q of P, the map

$$R \mapsto \left(\sum_{x\in R} A_x\right)^-$$

from the lattice of ideals of Q to the lattice of closed ideals of $\left(\sum_{x\in Q} A_x\right)^-$ is a bijection.

That (iv) follows from 2.1 is immediate; that (v) follows from 2.1 is part of 2.2. Conversely, bearing in mind that A_x and A_y are simple, $x, y \in P$, we see that (i) of 2.1 is just a reformulation of (iv) and (v) with $Q = \{x, y\}$, x and y not comparable, and that (ii) of 2.1 is a reformulation of (iv) and (v) with $Q = \{x, y\}$, $x < y$. It remains only to show that (iv) and (v) imply independence of the family $(A_x)_{x \in P}$. Let Q be a finite subset of P and let y be maximal in Q. Then Q and $R = Q \setminus \{y\}$ are distinct ideals of Q, so by injectivity in (v), A_y is not contained in the ideal $(\sum_{x \in R} A_x)^-$ of $(\sum_{x \in Q} A_x)^-$. Since A_y is simple and $(\sum_{x \in R} A_x)^-$ is closed (actually, $\sum_{x \in R} A_x$ is itself closed), it follows that

$$A_y \cap (\textstyle\sum_{x \in R} A_x)^- = 0,$$

and in particular

$$A_y \cap \textstyle\sum_{x \in R} A_x = 0.$$

If we assume inductively that $(A_x)_{x \in R}$ is independent, it then follows that $(A_x)_{x \in Q}$ is independent.

It is clear from the proof that it is sufficient to have properties (iv) and (v) for finite subsets Q of P.

2.4. *Remark*. If A is the internal lexicographic direct sum of the ordered family of simple sub-C^*-algebras $(A_x)_{x \in P}$ then for any pair Q_1, Q_2 of disjoint subsets of P,

$$(\textstyle\sum_{x \in Q_1} A_x)^- \cap (\sum_{x \in Q_2} A_x)^- = 0,$$

$$(\textstyle\sum_{x \in Q_1} A_x)^- + (\sum_{x \in Q_2} A_x)^- = (\sum_{x \in Q_1 \cup Q_2} A_x)^-.$$

To see this, set

$$R = \{x \in Q_1;\ x < y \text{ for some } y \in Q_2\}.$$

Then R is an ideal of Q_1 and of $Q_1 \cup Q_2$, and Q_2 is an ideal of $(Q_1 \setminus R) \cup Q_2$. Hence, by three applications of the last statement of 2.2,

$$\begin{aligned}(\textstyle\sum_{x \in Q_1 \cup Q_2} A_x)^- &= (\textstyle\sum_{x \in R} A_x)^- \\ &\quad \dotplus ((\textstyle\sum_{x \in Q_1 \setminus R} A_x)^- \dotplus (\textstyle\sum_{x \in Q_2} A_x)^-) \\ &= (\textstyle\sum_{x \in Q_1} A_x)^- \dotplus (\textstyle\sum_{x \in Q_2} A_x)^-.\end{aligned}$$

2.5. *Remark*. As is well known, a lexicographic direct sum of the ordered family of simple C^*-algebras $(B_x)_{x \in P}$ is not necessarily unique. (Consider the case $P = \{x, y\}$ with $x < y$, B_x infinite elementary, i.e., isomorphic to the algebra of compact operators on a Hilbert space of infinite dimension, and $B_y = \mathbf{C}$. See also [**1**].) Furthermore, there may not even exist such a sum. (Consider P as before with $B_x = \mathbf{C}$, $B_y = \mathbf{C}$.)

If each B_x is an infinite elementary C^*-algebra, of order

$$c_x = \aleph_0 \operatorname{card}\{y \in P;\ y \geqq x\},$$

then the construction which follows shows that there does exist a lexicographic direct sum of the ordered family $(B_x)_{x \in P}$; in fact, a canonical one.

If for each $x \in P$ the set $\{y \in P; y \geqq x\}$ is countable, i.e., $c_x = \aleph_0$, then it seems reasonable to expect that a lexicographic direct sum of the family $(B_x)_{x \in P}$ is unique.

If P itself is countable, then, as follows from 2.8 below, together with 4.3 of [**9**], uniqueness holds. (In 2.8, if each A_x is infinite, necessarily $d(P) = \{+\infty\}$.) Uniqueness in the case that P is finite is a consequence of Theorem 3 of [**2**]. In the case that P is countable and every subset of P has at least one minimal element, and at most finitely many, uniqueness was shown in 5.2 of [**10**].

2.6. THEOREM. *There exists a functor $P \mapsto A_P$, from the category of ordered sets and injections which are order isomorphisms onto their images, to the category of C^*-algebras and injective morphisms, such that for each ordered set P, A_P is a lexicographic direct sum of elementary C^*-algebras of order*

$$c_x = \aleph_0 \operatorname{card}\{y \in P; y \geqq x\}, \quad x \in P.$$

Proof. We remark that for the category of totally ordered sets this was proved in [**3**]. (See 4.7.17 of [**7**] for the case $P = \mathbf{N}$.) Denote by $\mathbf{Z}^{(P)}$ the set of finitely nonzero functions from P into $\mathbf{Z}$. (We shall ignore for the time being the group structure of this direct sum group.) For each fixed $x \in P$, define a C^*-algebra A_x on the Hilbert space $l^2(\mathbf{Z}^{(P)})$ as follows.

For each $g \in \mathbf{Z}^{(P)}$ denote also by g the vector in $l^2(\mathbf{Z}^{(P)})$ which is 1 at g and 0 elsewhere. For each pair $g, g' \in \mathbf{Z}^{(P)}$ denote by $g^* \otimes g'$ the partial isometry of rank one $\xi \mapsto (\xi|g)g'$ on $l^2(\mathbf{Z}^{(P)})$. Denote by S_x the set of pairs (g, g') in $\mathbf{Z}^{(P)}$ with the following three properties:

$$g(y) = 0 = g'(y) \text{ unless } x \text{ and } y \text{ are comparable;}$$

$$g(x) \neq 0, \ g'(x) \neq 0;$$

$$g(y) = g'(y) \text{ for all } y < x.$$

Say that (g, g') and (h, h') are equivalent in S_x if $(g, g') \in S_x$, $(h, h') \in S_x$, and $g(y) = h(y)$, $g'(y) = h'(y)$ for all $y \geqq x$. For each pair $(g, g') \in S_x$, the orthogonal sum

$$\sum\{h^* \otimes h'; (h, h') \text{ is equivalent to } (g, g') \text{ in } S_x\}$$

is a partial isometry on $l^2(\mathbf{Z}^{(P)})$; denote it by $(g, g')_x$. If $(f, f') \in S_x$ and $(g, g') \in S_x$ then from

$$(f^* \otimes f')(g^* \otimes g') = \delta_{f',g} f^* \otimes g'$$

it follows that the product $(f, f')_x(g, g')_x$ is equal to $(f, h')_x$ if there exists (h, h') equivalent to (g, g') in S_x with $h = f'$, and otherwise is equal to 0. Furthermore, if $(g, g') \in S_x$ then

$$(g, g')_x^* = (g', g)_x.$$

This shows that the closure in norm of the linear span of the set of operators

$$\{(g, g')_x; (g, g') \in S_x\}$$

is a C^*-algebra. We shall denote this C^*-algebra of operators on $l^2(\mathbf{Z}^{(P)})$ by A_x. Clearly A_x is isomorphic to the C^*-algebra of compact operators on the Hilbert space

$$l^2(\mathbf{Z}^{(\{y \in P; y \geqq x\})}),$$

of dimension

$$c_x = \aleph_0 \operatorname{card}\{y \in P; y \geqq x\}.$$

Fix $x, y \in P$ with $x \not> y$, and let us show that $A_xA_y \subseteq A_x$. Let (f, f') be in S_x, and (g, g') be in S_y. If

$$(f, f')_x(g, g')_y \neq 0,$$

then there exist (h, h') equivalent to (f, f') in S_x and (k, k') equivalent to (g, g') in S_y such that $h' = k$. Since the support of h' contains x, and the support of k is comparable with y, this requires that x be comparable with y, which means, as $x \not> y$, that $x \leqq y$. It follows that $(h, k') \in S_x$. Furthermore,

$$(f, f')_x(g, g')_y = (h, h')_x(k, k')_y = (h, k')_x.$$

This shows that $A_xA_y \subseteq A_x$ (whenever $x \not> y$).

It follows that property 2.3 (iv) holds. Let us next verify property 2.3 (v). First fix x and y in P with $x < y$, and fix $g \in \mathbf{Z}^{(P)}$ with $(g, g) \in S_y$. Then $(g, g)_y$ is a projection in A_y,

$$(g, g)_yA_x(g, g)_y \subseteq A_x,$$

and, moreover, the family of finite sums of projections in the infinite orthogonal set

$$\{(h, h)_x; (h, h) \in S_x \cap S_y \text{ and } (h, h)_y = (g, g)_y\}$$

is an approximate unit for the C^*-algebra $(g, g)_yA_x(g, g)_y$. To see this, note that if $(h, h) \in S_x$ and $(h, h)_x(g, g)_y \neq 0$, then $(h, h) \in S_y$, (h, h) is equivalent to (g, g) in S_y, and hence

$$(h, h)_x(g, g)_y = (h, h)_x(h, h)_y = (h, h)_x.$$

As pointed out at the end of 2.3, to verify 2.3 (v) it suffices to do so in the case of a finite subset of P. We remark that, as shown before, it follows from 2.3 (iv) (by 1.8.4 of [7]) that $\sum_{x \in Q} A_x$ is closed in A for any finite subset Q of P. Let us show first that if Q is a finite subset of P, and R_1 and R_2 are ideals of Q such that

$$\sum_{x \in R_1} A_x \subseteq \sum_{x \in R_2} A_x,$$

then $R_1 \subseteq R_2$. For this we must show that if R is an ideal of Q and $y \in Q\backslash R$ then

$$A_y \not\subseteq \sum_{x\in R} A_x.$$

To show this it is sufficient to show that for fixed $g \in \mathbf{Z}^{(P)}$ with $(g, g) \in S_y$, and for fixed $a \in \sum_{x\in R} A_x$, there exists a nonzero projection $p \leqq (g, g)_y$ such that $\|pa\|$ is arbitrarily small. Choose z maximal in R; then $R\backslash\{z\}$ is also an ideal of Q. Write

$$a = \sum_{x\in R} a_x \quad \text{where } a_x \in A_x.$$

Suppose first that $z < y$. By the preceding paragraph, there exists $h \in \mathbf{Z}^{(P)}$ such that

$$(h, h) \in S_z, \quad (h, h)_z \in (g, g)_y A_z (g, g)_y,$$

and $\|(h, h)_z a_z\|$ $(= \|(h, h)_z (g, g)_y a_z\|)$ is arbitrarily small. By induction on the number of elements in R, we may suppose that there exists a nonzero projection $q \leqq (h, h)_z$ such that

$$\|q\sum_{x\in R\backslash\{z\}} a_x\|$$

is arbitrarily small. Since also $\|qa_z\|$ $(= \|q(h, h)_z a_z\|)$ is arbitrarily small, and

$$(h, h)_z \leqq (g, g)_y,$$

we have $q \leqq (g, g)_y$ and $\|qa\|$ is arbitrarily small.

Suppose next that z and y are not comparable. Then

$$(g, g)_y a_z = 0.$$

By induction on the number of elements in R, we may suppose that there exists a nonzero projection

$$q \leqq (g, g)_y$$

such that $\|q\sum_{x\in R\backslash\{z\}} a_x\|$ is arbitrarily small. Since

$$qa_z = q(g, g)_y a_z = 0,$$

$\|qa\|$ is arbitrarily small.

To finish verifying 2.3 (v), we must show that if Q is a finite subset of P and I is a closed ideal of $\sum_{x\in Q} A_x$, then

$$I = \sum_{x\in R} A_x$$

for some ideal R of Q. It is sufficient to consider the case that I is the closed ideal of $\sum_{x\in Q} A_x$ generated by a single element

$$a = \sum_{x\in Q} a_x,$$

where $a_x \in A_x$. Note that the decomposition $a = \sum_{x\in Q} a_x$ is unique, as independence of the family $(A_x)_{x \in P}$ follows (as shown in 2.3) from 2.3 (iv) and injectivity in 2.3 (v), which we have already established. We shall prove in this case the slightly sharper fact that

$$I = \sum_{x \in R} A_x$$

where R is the ideal of Q generated by

$$\text{support}(a) = \{x \in Q; a_x \neq 0\}.$$

We shall argue by induction on card(support(a)). We must show that $A_y \subseteq I$ for all y in R. Note that 2.1 (i) follows from 2.3 (iv) and injectivity in 2.3 (v), which we have already established. Recall that, as shown above, $x < y$ implies $A_x A_y \neq 0$. It follows that it is sufficient to show that $A_y \subseteq I$ for all y maximal in R, i.e., for all y maximal in support(a). Fix y maximal in support(a). In order to show that $A_y \subseteq I$, replacing a by aa_y^*, and a_x by $a_x a_y^*$ for each $x \in Q$, we may suppose that $a_y \geqq 0$, and that $a_x = 0$ unless $x \leqq y$. (By 2.1 (i), injectivity in 2.3 (v), and simplicity of A_x for $x \in Q$, we have $A_x A_y = 0$ if x and y are not comparable.) Since $0 \leqq a_y \neq 0$ in A_y, there exists $g \in \mathbf{Z}^{(P)}$ such that

$$(g, g) \in S_y \quad \text{and} \quad (g, g)_y a_y (g, g)_y \neq 0.$$

Since $(g, g)_y$ is a minimal projection in the elementary C^*-algebra A_y, we may replace a by a scalar multiple of $(g, g)_y a (g, g)_y$ and suppose that $a_y = (g, g)_y$, and, further, that

$$(g, g)_y a_x (g, g)_y = a_x \quad \text{for each } x \in Q$$

(recall that either $a_x = 0$ or $x \leqq y$). If support(a) $= \{y\}$, then, as A_y is simple, $I = A_y$. If support(a) $\neq \{y\}$ then, as now support(a) $\leqq y$, we may choose $z \in$ support(a) maximal such that $z < y$. As before, there exists $h \in \mathbf{Z}^{(P)}$ such that $(h, h) \in S_z$, $(h, h)_z \leqq (g, g)_y$, and $\|(h, h)_z a_z\|$ is arbitrarily small. By the inductive hypothesis, since support($(h, h)_z a$) is contained in support(a) and omits y, the closed ideal of $\sum_{x \in Q} A_x$ generated by $(h, h)_z a$ is equal to $\sum_{x \in R_1} A_x$ for some ideal R_1 of Q. Since y is the only element of support(a) greater than z, and $a_y = (g, g)_y$, we have

$$((h, h)_z a)_z = (h, h)_z (a_y + a_z) = (h, h)_z + (h, h)_z a_z.$$

Choosing h so that

$$\|(h, h)_z a_z\| < 1,$$

we have

$$((h, h)_z a)_z \neq 0.$$

By independence (see above), it follows that $z \in R_1$. It follows that A_z is contained in the closed ideal of $\sum_{x \in Q} A_x$ generated by $(h, h)_z a$, and hence in that generated by a, namely, I. In particular, a_z and

$$\sum_{x \in Q, x \neq z} a_x = a - a_z$$

belong to I. Since $a_z \neq 0$, by the inductive hypothesis the closed ideal of $\sum_{x \in Q} A_x$ generated by $\sum_{x \in Q, x \neq z} a_x$ contains A_y. Hence I, the closed ideal of $\sum_{x \in Q} A_x$ generated by a_z and $a - a_z$, contains A_y.

Denote by A_P the C^*-algebra generated by $\sum_{x \in P} A_x$. We have shown that 2.3 (iv) and 2.3 (v) hold, and it follows by 2.3 that A_P is the internal direct sum of the ordered family of simple sub-C^*-algebras $(A_x)_{x \in P}$. As we have remarked, A_x is elementary of order

$$c_x = \aleph_0 \operatorname{card}\{y \in P; y \geqq x\}.$$

Finally, let P_1 and P_2 be ordered sets, and let $\phi: P_1 \to P_2$ be an injection which is an order isomorphism onto its image. If $g \in \mathbf{Z}^{(P_1)}$, define $\phi(g) \in \mathbf{Z}^{(P_2)}$ by

$$\phi(g)\,|\phi(P_1) = g\phi^{-1},$$

$$\phi(g)\,|P_2 \backslash \phi(P_1) = 0.$$

This map ϕ from $\mathbf{Z}^{(P_1)}$ into $\mathbf{Z}^{(P_2)}$ clearly has the property that if $x \in P_1$ and $(g, h) \in S_x$, then

$$(\phi(g), \phi(h)) \in S_{\phi(x)}.$$

Hence there is an isomorphism ϕ from A_x onto a hereditary sub-C^*-algebra of $A_{\phi(x)}$ such that for each pair $(g, h) \in S_x$,

$$\phi((g, h)_x) = (\phi(g), \phi(h))_{\phi(x)}.$$

Extension of ϕ to $\sum_{x \in P_1} A_x$ by linearity is easily seen to be an injective *-algebra morphism. Since each finite sum $\sum_{x \in Q} A_x$ is a C^*-algebra, by 1.8.3 of [7] ϕ is isometric on finite sums and hence extends to an isometric morphism from A_{P_1} into A_{P_2}. This correspondence from $\phi: P_1 \to P_2$ to $\phi: A_{P_1} \to A_{P_2}$ clearly takes the identity map into the identity map, and compositions into compositions. In other words, it is a functor.

2.7. *Definition*. Let P be an ordered set, and let d be a function from P to $\mathbf{Z}^+ \cup \{+\infty\}$ with the following two properties:

(i) If $d(x) = 0$ then $d(y) > 0$ for some $y > x$.

(ii) If $d(x) < +\infty$ then $d(y) < +\infty$ for all $y > x$ and $d(y) = 0$ for all except finitely many $y > x$.

We shall say that d is a *positive defector on P*.

Set

$$D(P, d) = \{g \in \mathbf{Z}^{(P)\mathrm{lex}}; 0 \leqq g \leqq d\}$$

where $g \leqq d$ means that if $g(x) > d(x)$ then $g(y) < d(y)$ for some $y > x$.

For each finite subset Q of P such that $d(Q) \subseteq \mathbf{Z}^+$, denote by d_Q the function on P defined by

$$d_Q(x) = \begin{cases} d(x) \text{ if } x \geqq y \text{ for some } y \in Q, \\ 0 \text{ otherwise.} \end{cases}$$

Then by (ii), $d_Q(P) \subseteq \mathbf{Z}^+$ and $d_Q \in \mathbf{Z}^{(P)}$. It follows that

$$d_Q \in (\mathbf{Z}^{(P)\text{lex}})^+.$$

It is immediate from the definition of d_Q that $d_Q \leqq d$. Hence

$$d_Q \in D(P, d).$$

Furthermore, $g \in D(P, d)$ if and only if for some such $Q \subseteq P$, $0 \leqq g \leqq d_Q$. (Given $g \in D(P, d)$, take Q to be the set $\{x \in P; g(x) \neq 0, d(x) < +\infty\}$.)

It follows that $D(P, d)$ is an interval in $\mathbf{Z}^{(P)\text{lex}}$. (If $Q_1 \subseteq Q_2$ are subsets as above then it is immediate that $d_{Q_1} \leqq d_{Q_2}$; this shows that $D(P, d)$ is upward directed. $D(P, d)$ generates the group $\mathbf{Z}^{(P)}$ by (i).)

2.8. THEOREM. *Let A be a C^*-algebra, let P be an ordered set, and suppose that A is the internal lexicographic direct sum of the ordered family of elementary sub-C^*-algebras $(A_x)_{x \in P}$. It follows that A is approximately finite-dimensional, and that the ordered group $K_0(A)$ is isomorphic to the lexicographic direct sum $\mathbf{Z}^{(P)\text{lex}}$. In fact, $K_0(A)$ is equal to the internal lexicographic direct sum of the ordered family of sub ordered groups $(K_0(A_x))_{x \in P}$. With respect to this identification of $K_0(A)$ as $\mathbf{Z}^{(P)\text{lex}}$, the dimension range of A is equal to the interval $D(P, d)$ in $(\mathbf{Z}^{(P)\text{lex}})^+$ for some positive defector d.*

Proof. To prove that A is approximately finite-dimensional it is sufficient to consider the case that P is finite. In this case the assertion was proved in 5.2 of [**10**].

Again, to prove that $K_0(A)$ is the internal lexicographic direct sum of the ordered family of sub ordered groups $(K_0(A_x))_{x \in P}$, it is sufficient to consider the case that P is finite, and in this case the assertion was proved in 5.2 of [**10**].

To compute the dimension range, $D(A)$, of A, let us first prove that every element of $D(A)$ is majorized by an element of $D(A)$ which, as an element of $\mathbf{Z}^{(P)}$, has only positive coordinates. Clearly, to prove this it is sufficient to consider the case that P is finite. If no subalgebra A_x with x maximal in P has a unit, then for each x maximal in P there exist elements of $D(A)$ with arbitrarily large coordinate at x, and it follows that $D(A)$ is all of $(\mathbf{Z}^{(P)\text{lex}})^+$; in particular, in this case, the assertion to be verified holds. Suppose that, for some y maximal in P, A_y has a unit, e. Denote by P' the set of $x \in P$ such that

$$(1 - e)A_x \neq 0.$$

Since y is maximal in P,

$$eA_x \subseteq A_x \quad \text{for every } x \in P,$$

and it follows that

$$(1 - e)A_x(1 - e) \subseteq A_x \quad \text{for every } x \in P'.$$

Hence $(1 - e)A(1 - e)$ is the internal lexicographic direct sum of the ordered family of elementary sub-C^*-algebras

$$((1 - e)A_x(1 - e))_{x \in P}.$$

(Recall that by 1.6 of **[17]**, the lattice of closed ideals of a hereditary sub-C^*-algebra is naturally identified with that of the closed ideal it generates.) Since P' is a proper subset of P, we may assume, inductively, that the assertion to be proved holds for $(1 - e)A(1 - e)$. Since $D(A)$ is upward directed, if $g \in D(A)$ then there exists $h \in D(A)$ with $h \geqq g$ and $h \geqq [e]$. Then

$$h - [e] \in D((1 - e)A(1 - e)),$$

so there exists $k \in D((1 - e)A(1 - e))$ such that $k \geqq h - [e]$ and k has only positive coordinates in $\mathbf{Z}^{(P')\text{lex}}$. Hence

$$g \leqq h \leqq k + [e]$$

in $D(A)$, and $k + [e]$ has only positive coordinates in $\mathbf{Z}^{(P)}$.

Define a positive defector d on P as follows. Fix $x \in P$, and denote the ideal $\{y \in P; y \ngeqq x\}$ of P by R_x. If the image of $D(A)$ in the quotient $\mathbf{Z}^{(P \setminus R_x)\text{lex}}$ of $\mathbf{Z}^{(P)\text{lex}}$ does not have a largest element, set $d(x) = +\infty$. If this image of $D(A)$ has a largest element, say d_x, set $d(x) = d_x(x)$. It is immediate that d defined in this way satisfies 2.7 (i) and 2.7 (ii). Furthermore, by the result of the preceding paragraph, we have

$$d(P) \subseteq \mathbf{Z}^+ \cup \{+\infty\}.$$

Therefore d is a positive defector on P.

Let us show that $D(A) = D(P, d)$. Fix $g \in (\mathbf{Z}^{(P)\text{lex}})^+$. It is immediate from the definition of $D(P, d)$ in 2.7 that $g \in D(P, d)$ if and only if $g \leqq d$ in any quotient $\mathbf{Z}^{(P \setminus R_x)\text{lex}}$ with $d(x) < +\infty$, where as above

$$R_x = \{y \in P; y \ngeqq x\}.$$

By the definition of d in the preceding paragraph, $d(x) < +\infty$ if and only if the image of $D(A)$ in the quotient $\mathbf{Z}^{(P \setminus R_x)\text{lex}}$ has a largest element. It follows that $g \in D(P, d)$ if and only if $g \in D(A)$ in any quotient $\mathbf{Z}^{(P \setminus R_x)\text{lex}}$ in which $D(A)$ has a largest element. Clearly, if $g \in D(A)$ then $g \in D(A)$ in any quotient of $\mathbf{Z}^{(P)\text{lex}}$. It remains to show that if $g \in D(A)$ in any quotient $\mathbf{Z}^{(P \setminus R_x)\text{lex}}$ in which $D(A)$ has a largest element, then $g \in D(A)$. Denote by Q the set

$$\{x \in P;\ d(x) < +\infty \text{ and } g(x) \neq 0\},$$

and set

$$\bigcap\nolimits_{x \in Q} R_x = R.$$

Since $D(A)$ has a largest element in the quotient $\mathbf{Z}^{(P \setminus R_x)\text{lex}}$ for each $x \in Q$, and $g \in D(A)$ in each such quotient, it follows, as Q is finite and $D(A)$ is upward directed, that $D(A)$ has an element h such that $h \geqq g$ in each such quotient $\mathbf{Z}^{(P \setminus R_x)\text{lex}}$, and hence, by the definition of lexicographical direct sum order, such that $h \geqq g$ in $\mathbf{Z}^{(P \setminus R)\text{lex}}$. By 3.1 below, if $d(x) = +\infty$ then

$$1_x + D(A) \subseteq D(A),$$

where 1_x denotes the element of $\mathbf{Z}^{(P)}$ equal to 1 at x and 0 elsewhere. Hence, changing h at each $x \in R$ such that $h(x) \ngeqq g(x)$, that is, at finitely many $x \in P$ such that $d(x) = +\infty$, we have $h \geqq g$ in $\mathbf{Z}^{(P)\text{lex}}$, whence $g \in D(A)$.

2.9. THEOREM. *There exists a functor $(P, d) \mapsto A(P, d)$ from the category of ordered sets with positive defectors, and arrows $(P, d) \to (P', d')$ consisting of maps $\phi: P \to P'$ such that ϕ is an order isomorphism onto its image and the function $\phi(d)$ on P' which is equal to $d\phi^{-1}$ on $\phi(P)$ and to 0 on $P' \setminus \phi(P)$ satisfies $\phi(d) \leqq d'$, in the sense that if $\phi(d)(x) > d'(x)$ then $\phi(d)(y) < d'(y)$ for some $y > x$, to the category of C^*-algebras and injective morphisms, such that $A(P, d)$ is a lexicographic direct sum of elementary C^*-algebras of order*

$$c_x = \aleph_0 \operatorname{card}\{y \in P;\ y \geqq x\},\ x \text{ not maximal in } P,$$

$$d(x),\ x \text{ maximal in } P,$$

and, moreover, $D(A(P, d)) = D(P, d)$.

Proof. We shall construct $A(P, d)$ as a hereditary sub-C^*-algebra of the C^*-algebra A_P constructed in 2.6. If $d(P) = \{+\infty\}$, take $A(P, d) = A_P$. Suppose that $d(P) \subseteq \mathbf{Z}^+$. For each $x \in P$ and each $n \in \mathbf{Z}^+ \setminus \{0\}$, denote by n_x the function on P equal to n at x and 0 elsewhere, so that

$$n_x \in \mathbf{Z}^{(P)} \subseteq l^2(\mathbf{Z}^{(P)}),$$

and note that, in the notation of the construction of 2.6, $(n_x, n_x) \in S_x$, so that $(n_x, n_x)_x$ is a projection in A_x. (Here A_x is as in 2.6.) For each $x \in P$ consider the projection

$$e_x = \sum\nolimits_{0 < n \leqq d(x)} (n_x, n_x)_x \in A_x,$$

and note that if $x \neq y$ then $e_x e_y = 0$. For each finite subset $Q \subseteq P$ consider the projection

$$e_Q = \sum\nolimits_{x \in Q} e_x \in \sum\nolimits_{x \in Q} A_x.$$

Denote by $A(P, d)$ the closure of the union of the upward directed collection of hereditary sub-C^*-algebras $\{e_Q A_P e_Q;\ Q \subseteq P,\ Q \text{ finite}\}$.

If, now, d is an arbitrary positive defector on P, then the set

$$R = \{x \in P;\ d(x) = +\infty\}$$

is an ideal of the ordered set P, and the restriction of d to $P\backslash R$ is a positive defector with values in $\mathbf{Z}^+$. Denote by $A(P, d)$ the inverse image in A_P under the quotient map

$$A_P \rightarrow A_{P\backslash R}$$

of the hereditary sub-C^*-algebra $A(P\backslash R, d|(P\backslash R))$ defined as in the preceding paragraph.

This construction is easily seen to be functorial in the specified sense.

Let us show that $A(P, d)$ is a lexicographic direct sum of elementary C^*-algebras. We shall show that $A(P, d)$ is the internal lexicographic direct sum of a family of hereditary sub-C^*-algebras of the family $(A_x)_{x \in P}$. In the case $d(P) \subseteq \mathbf{Z}^+$, this follows from the algebraic identity, for each finite $Q \subseteq P$,

$$\begin{aligned}&(\textstyle\sum_{y \in Q} e_y)(\sum_{x \in P} A_x)(\sum_{z \in Q} e_z)\\ &= \textstyle\sum_{x \in P}(\sum_{y \in Q, y \geqq x} e_y) A_x (\sum_{z \in Q, z \geqq x} e_z).\end{aligned}$$

Thus, if, in this case, for each $x \in P$ we denote by C_x the closure of the union of the collection of hereditary sub-C^*-algebras

$$\{e_Q A_x e_Q;\ Q \subseteq P,\ Q \text{ finite},\ Q \geqq x\}$$

of A_x, then $A(P, d)$ is the internal lexicographic direct sum of the family $(C_x)_{x \in P}$. (Independence and properties 2.1 (i) and 2.1 (iii) are verified immediately, as is the first part of property 2.1 (ii); the second part of 2.1 (ii) follows from the same property of the family $(A_x)_{x \in P}$ by 1.6 of [**17**].) It follows that the assertion holds in the case that d is an arbitrary positive defector: use that the quotient map

$$A_P \rightarrow A_{P\backslash R}$$

with R as defined above is injective on $(\sum_{x \in P\backslash R} A_x)^-$ (by 2.2) to define a hereditary sub-C^*-algebra C_x of A_x for each $x \in P\backslash R$ such that 2.1 (i) and 2.1 (ii) are verified for C_x and C_y with x and y in $P\backslash R$, and take $C_x = A_x$ for each $x \in R$. (A second application of 1.6 of [**17**] verifies the second part of 2.1 (ii) for C_x and C_y with $x \in R$ and $y \in P\backslash R$.)

If $x \in R$, then $C_x = A_x$ and so C_x has order

$$c_x = \aleph_0 \operatorname{card}\{y \in P;\ y \geqq x\}.$$

If $x \in P\backslash R$ then C_x contains the nonzero hereditary subalgebra $e_y A_x e_y$ for any $y \geqq x$ in P. Since A_x has order c_x and $e_y A_x e_y$ is infinite if $y > x$, it follows that if $x \in P\backslash R$ and x is not maximal in P then C_x has order c_x

also. If x is maximal in P it is clear that C_x has order $d(x)$.

Finally, let us show that

$$D(A(P, d)) = D(P, d).$$

By 2.8, $D(A(P, d)) = D(P, d')$ for some positive defector d' on P. Consider first the case $d(P) = \{+\infty\}$. Then for each x maximal in P, as C_x has order $d(x) = +\infty$, it follows that $d'(x) = +\infty$. Hence $d'(P) = \{+\infty\}$. (If $d'(x) < +\infty$, then by 2.7 (i) and 2.7 (ii), $d'(y) < +\infty$ for some $y \geqq x$ with y maximal in P.) Thus, in this case, $d' = d$. Now consider the general case. If $d(x) < +\infty$ then, by 2.7 (ii),

$$d|(P \backslash R_x) \in \mathbf{Z}^{(P \backslash R_x)},$$

where

$$R_x = \{y \in P; y \ngeqq x\},$$

and by construction $d|(P\backslash R_x)$ is the largest element of the image of $D(A(P, d))$ in $\mathbf{Z}^{(P\backslash R_x)\text{lex}}$. If $d(x) = +\infty$, then, as $A(P, d)$ is the preimage in A_P of $A(P\backslash R, d|(P\backslash R))$ where

$$R = \{x \in P; d(x) = +\infty\},$$

so that $D(A(P, d))$ is the preimage in $D(A_P)$ of a subset of $\mathbf{Z}^{(P\backslash R)\text{lex}}$, and since it was shown above that

$$D(A_P) = D(P, +\infty) = (\mathbf{Z}^{(P)\text{lex}})^+,$$

we have

$$1_x + D(A(P, d)) \subseteq D(A(P, d))$$

where 1_x is the element of $\mathbf{Z}^{(P)}$ equal to 1 at x and 0 elsewhere. Thus, we have shown that the image of $D(A(P, d))$ in $\mathbf{Z}^{(P\backslash R_x)\text{lex}}$ has a largest element if and only if $d(x) < +\infty$, and in this case the largest element is equal to $d|(P\backslash R_x)$. On the other hand, it is immediate from the definition of $D(P, d')$ in 2.7 that the image of $D(P, d')$ in $\mathbf{Z}^{(P\backslash R_x)\text{lex}}$ has a largest element if and only if $d'(x) < +\infty$, and in this case the largest element is equal to $d'|(P\backslash R_x)$. Since

$$D(A(P, d)) = D(P, d'),$$

it follows from the preceding two sentences that $d' = d$, and so

$$D(A(P, d)) = D(P, d).$$

2.10. *Remark*. In case P is finite, a C^*-algebra was constructed in Theorem 2 of [**2**], corresponding to a given positive defector d on P, which is a lexicographic direct sum of separable elementary C^*-algebras indexed by P, and which can be shown to have dimension range equal to the interval $D(P, d)$ in $\mathbf{Z}^{(P)\text{lex}}$. (By 4.3 of [**9**], it is therefore isomorphic to the algebra $A(P, d)$ of 2.9.) This construction is not obviously functorial in the sense of 2.6 and 2.9 above, and does not obviously extend to the case that P is infinite. On the other hand, it is clear that the construction in 2.6 and 2.9 above is a modification of the construction in [**2**].

2.11. *Remark.* The functorial nature of the constructions in 2.6 and 2.9 implies that any automorphism of the ordered set P leads naturally to an automorphism of the C^*-algebra A_P, and that any automorphism of the ordered set P leaving invariant the positive defector d leads naturally to an automorphism of the C^*-algebra $A(P, d)$.

It would appear to be interesting to consider the C^*-algebra crossed product of A_P by the group of automorphisms of P, considered as a discrete group. For instance, if $P = \mathbf{Z}$, this crossed product is stably isomorphic to the C^*-algebra O_∞ considered by Cuntz in [**6**]. If P is countable, if no proper nonempty ideal of P is invariant under all automorphisms, and if no nonempty ideal of P is fixed pointwise by any nontrivial automorphism of P, then by [**11**] the crossed product C^*-algebra is simple.

3. Description of the intervals.

3.1. LEMMA. *Let P be an ordered set and let D be an interval in the lexicographic direct sum $\mathbf{Z}^{(P)\mathrm{lex}}$. For each $x \in P$, the following two properties are equivalent.*

(i) *The image of D in the quotient $\mathbf{Z}^{(\{y \in P; y \geqq x\})\mathrm{lex}}$ does not have a largest element.*

(ii) $D + 1_x \subseteq D$.

Proof. (ii) $\Rightarrow$ (i) is immediate. (Recall that 1_x denotes the function on P equal to 1 at x and 0 elsewhere.)

(i) $\Rightarrow$ (ii). We use an idea from the proof of Proposition 7.5 of [**14**].

Fix $g \in D$, and denote by I the ideal of $\mathbf{Z}^{(P)\mathrm{lex}}$ generated by $\{h \in D; g + h \in D\}$. It follows that $g + I$ is the largest element of the image of D in the quotient by I: if $k \in D$ then with $f \in D$ such that $k \leqq f$ and $g \leqq f$, we have $f - g \in I$, and so

$$k + I \leqq f + I = g + I.$$

Hence if (i) holds, the image of I in the quotient

$$\mathbf{Z}^{(\{y \in P; y \geqq x\})}$$

is not zero. This says that $1_x \in I$; this in turn is equivalent to (ii).

3.2. THEOREM. *Let P be an ordered set and let D be an interval in $\mathbf{Z}^{(P)\mathrm{lex}}$. Denote by $P \backslash U$ the set of elements $x \in P$ such that the image of D in the quotient*

$$\mathbf{Z}^{(\{y \in P; y \geqq x\})\mathrm{lex}}$$

has a largest element. It follows that U is an ideal of P and D is determined by U together with the image of D in the quotient $\mathbf{Z}^{(P \setminus U)\mathrm{lex}}$.

Proof. It is the same to say that D is equal to the preimage in $(\mathbf{Z}^{(P)\mathrm{lex}})^+$ of the image of D in the quotient $\mathbf{Z}^{(P \setminus U)\mathrm{lex}}$. To see that these statements are the same, note that this preimage is also an interval in $\mathbf{Z}^{(P)\mathrm{lex}}$, has the same image as D in $\mathbf{Z}^{(P \setminus U)\mathrm{lex}}$, and has the same set of elements $x \in P$ such that the image in

$$\mathbf{Z}^{(\{y \in P; y \geqq x\})\mathrm{lex}}$$

has a largest element (this uses only the trivial part of 3.1).

That D is equal to the preimage in $(\mathbf{Z}^{(P)\mathrm{lex}})^+$ of the image of D in the quotient $\mathbf{Z}^{(P \setminus U)\mathrm{lex}}$ follows from 3.1. (Let $g \in (\mathbf{Z}^{(P)\mathrm{lex}})^+$ be such that $g = h + k$ with $h \in D$ and $k \in \mathbf{Z}^{(U)}$. Write $k = k^+ - k^-$ where k^+, k^- belong to $\mathbf{Z}^{(U)}$ and have positive coordinates. By 3.1, $h + k^+ \in D$. Since

$$0 \leqq g = h + k^+ - k^- \leqq h + k^+$$

and D is hereditary, $g \in D$.)

3.3. LEMMA. *Let P be an ordered set, and let D be an interval in* $\mathbf{Z}^{(P)\mathrm{lex}}$. *D is determined by the family of its images in prime quotients of* $\mathbf{Z}^{(P)\mathrm{lex}}$.

Proof. It is sufficient to show that if g is an element of $\mathbf{Z}^{(P)}$ such that g belongs to D modulo each prime ideal of $\mathbf{Z}^{(P)\mathrm{lex}}$, then g belongs to D.

Suppose that $g \in \mathbf{Z}^{(P)}$ and $g \notin D$. By Zorn's lemma, choose an ideal I of $\mathbf{Z}^{(P)}$ such that $g \notin D$ modulo I, and such that I is maximal with this property. Then I is prime. Otherwise, passing to the quotient by I, we would have two ideals J_1 and J_2 with $J_1 \cap J_2 = 0$, but neither $J_1 = 0$ nor $J_2 = 0$. Hence, by maximality of I, $g \in D$ modulo J_1 and $g \in D$ modulo J_2. Thus, as D is upward directed, there exists $h \in D$ such that

$$0 \leqq g \leqq h \text{ modulo } J_i,\ i = 1, 2.$$

Since $J_i = \mathbf{Z}^{(R_i)}$ where R_1, R_2 are ideals of P with $R_1 \cap R_2 = \emptyset$, it follows that $0 \leqq g \leqq h$. Since D is hereditary, $g \in D$. This contradicts the choice of I such that $g \notin D$ modulo I. (Note that I is now assumed to be 0.) This proves the implication of the preceding paragraph (or, rather, its contrapositive).

3.4. THEOREM. *Let P be an ordered set, and let D be an interval in* $\mathbf{Z}^{(P)\mathrm{lex}}$. *Let* $g \in (\mathbf{Z}^{(P)\mathrm{lex}})^+$. *It follows that g belongs to D if and only if, in each prime quotient of* $\mathbf{Z}^{(P)\mathrm{lex}}$ *in which the image of D has a least upper bound, the image of g is majorized by this least upper bound, and majorized strictly if this least upper bound does not belong to the image of D.*

Proof. Necessity of the condition is clear.

By 3.2, to prove the theorem we may pass to the quotient, $\mathbf{Z}^{(P\setminus U)\mathrm{lex}}$. We may therefore suppose that for each $x \in P$ the image of D in the quotient

$$\mathbf{Z}^{(\{y\in P; y\geqq x\})\mathrm{lex}}$$

has a largest element. It follows that there is a function d from P into $\mathbf{Z}$ such that for each $x \in P$ the restriction of d to $\{y \in P;\ y \geqq x\}$ belongs to

$$\mathbf{Z}^{(\{y\in P; y\geqq x\})\mathrm{lex}}$$

and is the largest element of the image of D in this quotient. (For each $x \in P$ take $d(x)$ to be the coordinate at x of the largest element of the image of D in the corresponding quotient.)

Let us show that if $d \in \mathbf{Z}^{(P)\mathrm{lex}}$, i.e., if $d(x) = 0$ for all except finitely many $x \in P$, then d is the least upper bound of D in $\mathbf{Z}^{(P)\mathrm{lex}}$. Suppose that $d \in \mathbf{Z}^{(P)\mathrm{lex}}$. Then $d \geqq D$ (if $g \in D$ and $d(x) < g(x)$, then by definition of d, $d(y) > g(y)$ for some $y > x$). Furthermore, if $h \in \mathbf{Z}^{(P)\mathrm{lex}}$ and $h \geqq D$, then $h \geqq d$ (if $h(x) < d(x)$ then, as the images $\dot{D}$, $\dot{h}$, and $\dot{d}$ in the quotient

$$\mathbf{Z}^{(\{y\in P; y\geqq x\})\mathrm{lex}}$$

satisfy $\dot{d} \in \dot{D} \leqq \dot{h}$, there exists $y > x$ in P with $h(y) > d(y)$). This shows that d is the least upper bound of D.

Now let us prove sufficiency of the condition. Suppose that the condition holds, and assume that $g \notin D$. Set

$$R = \{x \in P;\ \exists y \geqq x \text{ with } d(y) > g(y)\}.$$

R is an ideal of P. Let $x \notin R$. Then $d(x) \leqq g(x)$. With $\dot{D}$ and $\dot{g}$ the images of D and g in the prime quotient $\mathbf{Z}^{(\{y\in P; y\geqq x\})}$,

$$\dot{d} = d|\{y \in P;\ y \geqq x\}$$

is the largest element of $\dot{D}$. Hence by the condition on g, $\dot{g} \leqq \dot{d}$. Thus, if $d(x) < g(x)$ then there exists $y > x$ in P such that $d(y) > g(y)$. This conclusion contradicts $x \notin R$. Therefore, if $x \notin R$, $d(x) = g(x)$.

Next, let us show that $g \notin D$ holds also after passing to the quotient $\mathbf{Z}^{(P\setminus R)\mathrm{lex}}$ of $\mathbf{Z}^{(P)\mathrm{lex}}$. Denote by S the largest ideal of P on which g is zero. Then

$$P\setminus S = \{y \in P;\ y \geqq x \text{ for some } x \text{ with } g(x) \neq 0\}.$$

Since $g(x) \neq 0$ for only finitely many $x \in P$, and since D is upward directed, it follows from the definition of d that there exists $h \in D$ with

$$h|(P\setminus S) = d|(P\setminus S).$$

Furthermore, we may suppose that h majorizes any given element of D. Recalling that $g|(P\backslash R) = d|(P\backslash R)$, we have

$$h - g = k_1 + k_2 + k_3$$

where

$$k_1 = \sum_{x \in S \cap (P\backslash R)} h(x)1_x,$$
$$k_2 = \sum_{x \in S \cap R} h(x)1_x,$$
$$k_3 = \sum_{x \in (P\backslash S) \cap R} (d(x) - g(x))1_x.$$

(Here $1_x \in \mathbf{Z}^{(P)}$ is the function equal to 1 at x and 0 elsewhere.) Note that $k_1, k_2, k_3 \in \mathbf{Z}^{(P)\mathrm{lex}}$, and $k_3 \geqq 0$. Denote by T the largest ideal of P on which both g and h are zero. We have $T \subseteq S$. As above, with now T in place of S, choose $h' \in D$ with

$$h'|(P\backslash T) = d|(P\backslash T).$$

We may choose h' so that $h' \geqq h$.

Let us show that

$$h' - h + k_2 + k_3 \geqq 0.$$

Denote the element $h' - h + k_2 + k_3$ by k. Suppose that $k(x) < 0$. We must find $y > x$ with $k(y) > 0$. Since $T \subseteq S$,

$$(h' - h)(P\backslash S) = (d - d)(P\backslash S) = 0.$$

Also, $k_2(P\backslash S) = 0$. Therefore, if $x \in P\backslash S$ then $k_3(x) < 0$; in particular, $x \in R$. By definition of R there exists $y \geqq x$ in R with $d(y) > g(y)$. Since also $y \in P\backslash S$, we have

$$k(y) = d(y) - g(y) > 0.$$

If, on the other hand, $x \in S$, then either

$$(h' - h)(x) < 0 \quad \text{or} \quad (h' - h)(x) \geqq 0.$$

Assume first that $(h' - h)(x) \geqq 0$. This combined with $k(x) < 0$ gives $k_2(x) < 0$. We shall show that whenever $k_2(x) \neq 0$, i.e., $x \in S \cap R$ and $h(x) \neq 0$, then $k(y) > 0$ for some $y \geqq x$. By the definition of R, there exists $y \geqq x$ in R with $d(y) > g(y)$. If $y \in P\backslash S$, then $k_3(y) < 0$, and as shown above there exists $y' > y \geqq x$ such that $k(y') > 0$. If $y \in S$, then $g(y) = 0$, so $d(y) > 0$. As $h(x) \neq 0$, we have $x \in P\backslash T$ and therefore $y \in P\backslash T$, so $h'(y) = d(y)$. Thus, if $y \in S$, $h'(y) > 0$, and hence

$$k(y) = (h' - h)(y) + h(y) = h'(y) > 0.$$

Now assume that $(h' - h)(x) < 0$. Then, as

$$h' - h \geqq 0 \quad \text{and} \quad (h' - h)(P\backslash S) = 0,$$

there exists $y_1 > x$ in S such that $(h' - h)(y_1) > 0$. If $k(y_1) > 0$, we may take $y = y_1$. If $k(y_1) \leqq 0$, then necessarily $k_2(y_1) < 0$. By what precedes, this implies that $k(y) > 0$ for some $y \geqq y_1$.

It follows that $k_1 \ngeqq 0$. Otherwise,

$$h' - g = h' - h + h - g = h' - h + k_1 + k_2 + k_3 \geqq 0,$$

so $g \in D$; this contradicts our assumption $g \notin D$. Since k_1 is the image of $h - g$ in the quotient $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$, we deduce that the image of $h - g$ in $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$ is not positive. Since, as remarked above, h may be chosen greater than or equal to any given element of D, it follows that the image of g in $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$ does not belong to the image of D. This is the conclusion announced above.

On the other hand, as shown earlier, the image of g in the quotient $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$ is equal to the image of d, i.e., to $d|\,(P \setminus R)$. Hence, as shown above, the least upper bound of the image of D in $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$ exists and is equal to the image of d, and to the image of g. Furthermore, the same argument shows that this holds also in any prime quotient of $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$. Hence by the condition of the theorem, the image of g in any prime quotient of $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$ belongs to the image of D. Hence by 3.3, or rather the first sentence of the proof of 3.3 (applied to the images of g and D in $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$), the image of g in $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$ belongs to the image of D. This is in direct contradiction with the conclusion of the preceding paragraph. Since that conclusion is based on the assumption $g \notin D$, it follows that $g \in D$.

3.5. COROLLARY. *Let P be an ordered set, and let D be an interval in $\mathbf{Z}^{(P)\mathrm{lex}}$. Define a function $d:P \to \mathbf{Z} \cup \{+\infty\}$ as follows. Fix $x \in P$. If the image of D in the quotient*

$$\mathbf{Z}^{(\{y \in P; y \geqq x\})\mathrm{lex}}$$

has a largest element, set the coordinate of this element at x equal to $d(x)$. Otherwise, set $+\infty = d(x)$. We shall refer to d as the defector of D.

D is determined by its defector, together with the set of prime ideals of $\mathbf{Z}^{(P)\mathrm{lex}}$, not of the form $\mathbf{Z}^{(\{y \in P; y \ngeqq x\})}$, modulo which D has a largest element.

Proof. This is immediate from the following modified form of 3.4 which is also established by the proof of 3.4.

Let $g \in (\mathbf{Z}^{(P)\mathrm{lex}})^+$. It follows that g belongs to D if and only if for each prime ideal R of P such that $d|\,(P \setminus R) \in \mathbf{Z}^{(P \setminus R)}$, the images $\dot{g}$, $\dot{D}$, and $\dot{d}$ of g, D, and d in the prime quotient $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$ satisfy the relations $\dot{g} \leqq \dot{d}$ and, moreover, $\dot{g} < \dot{d}$ if $\dot{d} \notin \dot{D}$.

(In fact, this is just a restatement of 3.4. The condition $d \in \mathbf{Z}^{(P)}$ is equivalent to the condition that D have a least upper bound in $\mathbf{Z}^{(P)\mathrm{lex}}$. (This also holds of course in any quotient $\mathbf{Z}^{(P \setminus R)}$.) To see this, note that one implication is immediate. Let us prove the other, i.e., that if D has a least upper bound b in $\mathbf{Z}^{(P)\mathrm{lex}}$, then $d \in \mathbf{Z}^{(P)}$. We prove $b = d$. First, let us show that $b \geqq d$, in the sense that if $b(x) < d(x)$ then $b(y) > d(y)$ for some $y > x$. If $d(x) < +\infty$ this is clear from the definition of d. If $d(x) = +\infty$, then by 3.1, $1_x + D \subseteq D$, so $b - 1_x$ is an upper bound of D, an absurdity. (So always $d(x) < +\infty$.) Hence, if $b \neq d$, then $b(y) > d(y)$ for any y. If y is minimal in P then $b - 1_y \geqq d \geqq D$, and if $x < y$ then $b - 1_x \geqq d \geqq D$. In either case we have an absurdity, so $b = d$.)

3.6. Some obvious questions concerning defectors as defined in 3.5 are the following.

What functions are defectors? Of course, a positive defector in the sense of 2.7 is a defector. (A positive defector d is the defector of the interval $D(P, d)$.)

Exactly when is a defector equivalent to a positive defector? If d is a defector, and if the set

$$D(P, d) = \{g \in \mathbf{Z}^{(P)\mathrm{lex}};\ 0 \leqq g \leqq d\}$$

is an interval (i.e., if this set is upward directed), then is d equivalent to a positive defector? (By 2.7 this condition is necessary for d to be equivalent to a positive defector.)

4. Classification of the intervals.

4.1. Let P be an ordered set. By 5.6 of [**9**], two intervals in $\mathbf{Z}^{(P)\mathrm{lex}}$ are isomorphic as abstract abelian local semigroups if and only if they are conjugate by an automorphism of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$. We shall express this by saying that the two intervals belong to the same orbit of the action of this automorphism group (on the set of intervals).

Any automorphism of the ordered set P gives rise to an automorphism of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$ by permuting the standard basis $(1_x)_{x \in P}$. Accordingly, in classifying the intervals in $\mathbf{Z}^{(P)\mathrm{lex}}$ we shall consider only automorphisms of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$ which induce the identity automorphism of the ordered set P (considered as a sub ordered set of the prime ideal spectrum of $\mathbf{Z}^{(P)\mathrm{lex}}$). In other words, we shall consider only the group of ideal-preserving automorphisms of $\mathbf{Z}^{(P)\mathrm{lex}}$, and we are interested in the orbit of an interval under this group.

Let d be a function from P to $\mathbf{Z} \cup \{+\infty\}$ with the property 2.7 (ii): If $d(x) < +\infty$ then $d(y) < +\infty$ for all $y > x$ and $d(y) = 0$ for all except finitely many $y > x$. Any defector as defined in 3.5 has this property. Let α be an ideal-preserving automorphism of $\mathbf{Z}^{(P)\mathrm{lex}}$. Note that the matrix entry $\alpha_{x,y}$ (i.e., the coordinate at x of $\alpha 1_y$) can be nonzero only if $y > x$. Hence we may define a function αd from P to $\mathbf{Z} \cup \{+\infty\}$ as follows:

$$\alpha d(x) = \begin{cases} \sum_{y > x} \alpha_{x,y} d(y) & \text{if } d(x) < +\infty, \\ +\infty & \text{if } d(x) = +\infty. \end{cases}$$

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If d is the defector of the interval D, it follows easily that αd is the defector of the interval αd. We shall say that αd and d are equivalent defectors.

4.2. *Remark*. The class of lexicographic direct sums of ordered families of elementary C^*-algebras is not closed under passing to hereditary sub-C^*-algebras. This is seen by using defectors as follows.

Consider the ordered set $P = -\mathbf{N}$, the positive defector $d = (\cdots, 0, 0, 1)$, and the interval $D = [0, d[$. By 2.9, there is a lexicographic direct sum of elementary C^*-algebras $A = A(P, d)$ with dimension range $D(P, d)$. Clearly $d \in D(P, d)$, so

$$D = [0, d[\subseteq D(P, d).$$

It follows that there is a hereditary sub-C^*-algebra B of A with dimension range D. Suppose that B is a lexicographic direct sum of elementary C^*-algebras. By 2.8, the dimension range of B, i.e., D, is isomorphic to $D(P, d')$ for some positive defector d'. It is easy to see that the defector of $D = [0, d[$ is equal to d. It is not difficult to check (using 3.1) that the defector of the interval $D(P, d')$ is d'. (This holds for any positive defector in place of d'; see the proof of 4.7.) Hence, d' is equivalent to d, and, in particular, $d' \in \mathbf{Z}^{(P)}$. By definition of $D(P, d')$ it follows that

$$D(P, d') = [0, d'].$$

Clearly, however, the intervals $[0, d[$ and $[0, d']$ are not isomorphic. (One has a largest element, and one not.) This contradiction shows that the hereditary sub-C^*-algebra B of $A(P, d)$ is not a lexicographic direct sum of elementary C^*-algebras.

4.3. THEOREM. *Let P be an ordered set, and let D be an interval in $\mathbf{Z}^{(P)\mathrm{lex}}$. The orbit of D (under the group of ideal-preserving automorphisms of $\mathbf{Z}^{(P)\mathrm{lex}}$) is determined by the orbit of the defector of D, together with the set of prime ideals of $\mathbf{Z}^{(P)\mathrm{lex}}$, not of the form*

$$\mathbf{Z}^{(\{y \in P; y \not\geqq x\})},$$

modulo which D has a largest element.

Proof. This is immediate from 3.5.

4.4. LEMMA. *Let P be an ordered set satisfying the decreasing chain condition. Let α be an endomorphism of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$ taking each ideal into itself and inducing the identity in each simple subquotient. It follows that α is an automorphism of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$ taking each ideal onto itself.*

Proof. The decreasing chain condition, restated, says that every subset of P has a minimal element. It follows that P has a composition series with singleton relative differences, i.e., a well-ordered family of ideals

$(P_\gamma)_{0 \leqq \gamma \leqq \beta}$, where β is some ordinal number, $P_0 = \emptyset$, $P_\beta = P$, $P_\gamma \subseteq P_{\gamma+1}$, $P_\gamma = \bigcup_{\delta<\gamma} P_\delta$ if γ is a limit ordinal and

$$\mathrm{card}(P_{\gamma+1} \setminus P_\gamma) = 1.$$

By transfinite induction, it suffices to establish the conclusion assuming that it holds with P_γ in place of P for all $\gamma < \beta$. Hence we may assume that $\beta = \gamma + 1$, and write $P = P_\gamma \cup \{x\}$.

Thus, by inductive assumption the restriction of α to the ideal $\mathbf{Z}^{(P_\gamma)\mathrm{lex}}$ is an ideal-preserving isomorphism of the ordered group $\mathbf{Z}^{(P_\gamma)\mathrm{lex}}$. Since

$$\mathbf{Z}^{(P)} = \mathbf{Z}^{(P_\gamma)} + \mathbf{Z}1_x \quad \text{and} \quad 1_x - \alpha 1_x \in \mathbf{Z}^{(P_\gamma)},$$

it follows immediately that α is injective and surjective, i.e., α is an automorphism of the group $\mathbf{Z}^{(P)}$.

It remains to show that α^{-1} is an endomorphism of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$, taking each ideal into itself. Suppose that

$$g \in (\mathbf{Z}^{(P)\mathrm{lex}})^+.$$

Then $g = g_1 + g_2$ where

$$g_1 \in (\mathbf{Z}^{(P_\gamma)\mathrm{lex}})^+ \quad \text{and}$$

$$g_2 = a_x 1_x + \textstyle\sum_{y<x} a_y 1_y$$

with $a_x > 0$ (this implies $g_2 \geqq 0$). To show $\alpha^{-1}g \geqq 0$ it is sufficient to show that $\alpha^{-1}g_2 \geqq 0$, since $\alpha^{-1}g_1 \geqq 0$ by the inductive assumption. By the hypotheses of the lemma,

$$1_x - \alpha 1_x \in \mathbf{Z}^{(P_\gamma)}, \quad \alpha 1_x \geqq 0,$$

and $\alpha 1_x$ belongs to the ideal generated by 1_x. Therefore,

$$\alpha 1_x = 1_x + \textstyle\sum_{y<x} b_y 1_y.$$

Hence

$$\alpha^{-1} 1_x = 1_x - \textstyle\sum_{y<x} b_y \alpha^{-1} 1_y,$$

and by the inductive assumption $\alpha^{-1}1_y$ belongs to the ideal generated by 1_y for each $y < x$, so

$$\alpha^{-1} 1_x = 1_x + \textstyle\sum_{y<x} c_y 1_y,$$

and, furthermore,

$$\alpha^{-1} g_2 = a_x 1_x + \textstyle\sum_{y<x} d_y 1_y.$$

This shows that α^{-1} is positive. At the same time we see that $\alpha^{-1}1_x$ belongs to the ideal generated by 1_x. Since, as pointed out, by assumption $\alpha^{-1}1_y$ belongs to the ideal generated by 1_y for every $y \in P_\gamma$, and since every ideal of $\mathbf{Z}^{(P)\mathrm{lex}}$ is generated (as an ideal, and in fact as a subgroup) by the set $\{1_y; y \in R\}$ for some ideal R of P, it follows that α^{-1} takes every ideal of $\mathbf{Z}^{(P)\text{lex}}$ into itself.

4.5. *Remark*. The decreasing chain condition on P in 4.4 may not be dropped. (Consider the case $P = -\mathbf{N}$, and define α by

$$\alpha 1_{-n} = 1_{-n} + 1_{-n-1}, \quad n = 0, 1, 2, \ldots .$$

α is not surjective as the sum of the coordinates of αg is even for any $g \in \mathbf{Z}^P$.)

4.6. COROLLARY. *Let P be an ordered set satisfying the decreasing chain condition. Let d and d' be functions from P to $\mathbf{Z} \cup \{+\infty\}$ such that $d(x) < +\infty$ if and only if $d'(x) < +\infty$. Suppose that if $d(x) < +\infty$ then $d(y) < +\infty$ for all $y > x$, and $d(y) = 0$ for all except finitely many $y > x$. Suppose that, furthermore, if $x < +\infty$ then*

$$d'(x) = d(x) + \sum\nolimits_{y>x} a_{x,y} d(y),$$

where $a_{x,y} \in \mathbf{Z}$; this sum, by the preceding hypothesis, is a finite one. Suppose that for each $y \in P$, $a_{x,y} = 0$ for all except finitely many x.

It follows that d is a defector if and only if d' is a defector. In case d and d' are defectors, they are equivalent. (*Of course, if d and d' are equivalent defectors, the hypotheses hold.*)

Proof. This is immediate from 4.1 and 4.4.

4.7. LEMMA. *Let P be an ordered set and let D be an interval in $\mathbf{Z}^{(P)\text{lex}}$. Suppose that the defector d of D is positive, and that for any prime ideal R of P not of the form $\{y \in P; y \ngeqq x\}$, the restriction $d|(P\backslash R)$ does not belong to $\mathbf{Z}^{(P\backslash R)}$. It follows that D is equal to the interval $D(P, d)$.*

Proof. As pointed out before, the defector of the interval $D(P, d)$ is d. Let us prove this. We must show that, for each $x \in P$, $d(x) < +\infty$ if and only if the image of $D(P, d)$ in the quotient

$$\mathbf{Z}^{(\{y \in P; y \geqq x\})\text{lex}}$$

has a largest element, and that in this case the coordinate of this element at x is equal to $d(x)$. If $d(x) < +\infty$, then by 2.7 (ii), the restriction

$$d|\{y \in P; y \geqq x\}$$

belongs to $\mathbf{Z}^{(\{y \in P; y \geqq x\})}$. By definition of $D(P, d)$, the function on P equal to d on $\{y \in P; y \geqq x\}$ and equal to 0 elsewhere belongs to $D(P, d)$, so $d|\{y \in P; y \geqq x\}$ belongs to the image of D in

$$\mathbf{Z}^{(\{y \in P; y \geqq x\})\text{lex}}.$$

The element $d|\{y \in P; y \geqq x\}$ is then clearly the largest element of this image of D, and of course has coordinate $d(x)$ at x. If $d(x) = +\infty$, then by definition of $D(P, d)$,

$$1_x + D(P, d) \subseteq D(P, d).$$

Hence by 3.1, the image of $D(P, d)$ in the quotient

$$\mathbf{Z}^{(\{y \in P; y \geqq x\})\mathrm{lex}}$$

does not have a largest element.

Thus, the intervals D and $D(P, d)$ have the same defector d. To show they are equal, by 3.5 it suffices to show that neither interval has a largest element modulo any prime ideal of $\mathbf{Z}^{(P)\mathrm{lex}}$ not of the form

$$\mathbf{Z}^{(\{y \in P; y \ngeqq x\})}.$$

If such a prime ideal is given, necessarily of the form $\mathbf{Z}^{(R)}$ for some ideal R of P, by hypothesis

$$d|(P \setminus R) \notin \mathbf{Z}^{(P \setminus R)}.$$

Suppose that the image of D in the quotient $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$ has a largest element g. Then for any $x \in P \setminus R$, the image of g in the quotient

$$\mathbf{Z}^{(\{y \in P; y \geqq x\})\mathrm{lex}}$$

must be the largest element of the image of D in this quotient, so

$$g = d|(P \setminus R),$$

a contradiction. The same argument with $D(P, d)$ in place of D shows that also the image of $D(P, d)$ in the quotient $\mathbf{Z}^{(P \setminus R)\mathrm{lex}}$ cannot have a largest element. Therefore, by 3.5, $D = D(P, d)$.

4.8. THEOREM. *Let P be an ordered set satisfying the decreasing chain condition. It follows that every interval in $\mathbf{Z}^{(P)\mathrm{lex}}$ is isomorphic to $D(P, d)$ for some positive defector d.*

Proof. By 4.7, it is sufficient to show that any interval in $\mathbf{Z}^{(P)\mathrm{lex}}$ is isomorphic to an interval with positive defector. (By 4.7, as P satisfies the decreasing chain condition $D(P, d)$ is the only interval with positive defector d.) Let D be an interval in $\mathbf{Z}^{(P)\mathrm{lex}}$.

By 3.2, we may suppose that the image of D in any quotient

$$\mathbf{Z}^{(\{y \in P; y \geqq x\})\mathrm{lex}}$$

has a largest element. Choose an increasing sequence $(g_0, g_1, g_2, \dots)$ in D with $g_0 = 0$ such that

$$\bigcup [0, g_n] = D.$$

For each $x \in P$, the sequence of integers $(g_n(x))$ is eventually constant (and equal to $d(x)$ where d is the defector of D).

For each $n = 0, 1, 2, \dots$ define subsets X_n, Y_n of P as follows:

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$$X_n = \{x \in P; (g_{n+1} - g_n)(x) < 0\},$$
$$Y_n = \{x \in P; (g_{n+1} - g_n)(x) > 0\}.$$

For each $x \in X_n$, choose $w(x) \in Y_n$ with $w(x) > x$ (recall that $g_{n+1} - g_n \geqq 0$). This defines a map

$$w:X = \bigcup X_n \to Y = \bigcup Y_n \subseteq P.$$

Note that $w^{-1}(y)$ is finite for each $y \in Y$. Indeed, for each $y \in Y$, as $(g_n(y))$ converges, y can belong to Y_n for only finitely many n. As, for each n, $w^{-1}(Y_n) \subseteq X_n$ and X_n is finite, it follows that $w^{-1}(y)$ is finite.

For each $x \in X$, as $(g_n(x))$ converges,

$$\inf(g_{n+1} - g_n)(x) > -\infty.$$

Set

$$\inf(g_{n+1} - g_n)(x) = h(x), \quad x \in X.$$

Define an endomorphism α of the group $\mathbf{Z}^{(P)}$ as follows:

$$\alpha 1_z = 1_z + \textstyle\sum_{x \in w^{-1}(z)} |h(x)| 1_x.$$

(Recall that $w^{-1}(z)$ is finite for each $z \in P$.) As $x < w(x)$ for each $x \in X$, it follows that α is an endomorphism of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$ taking each ideal into itself. Clearly α induces the identity in each simple subquotient of $\mathbf{Z}^{(P)\mathrm{lex}}$. Hence by 4.4, α is an automorphism of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$, taking each ideal onto itself.

If $x \in X$ then

$$\alpha(g_{n+1} - g_n)(x) = (g_{n+1} - g_n)(x) + |h(x)|(g_{n+1} - g_n)(w(x)).$$

If $x \notin X$ then

$$\alpha(g_{n+1} - g_n)(x) = (g_{n+1} - g_n)(x).$$

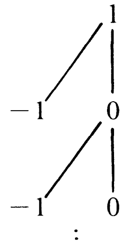
Hence, by the definition of X, w, and h, for any $x \in P$,

$$\alpha(g_{n+1} - g_n)(x) \geqq 0.$$

Therefore the defector of αD is positive.

4.9. *Remark*. 4.8 implies in particular that if P satisfies the decreasing chain condition then every defector is equivalent to a positive defector.

This does not hold if P is an arbitrary ordered set. To see this consider the defector



where the numbers indicate the values of the function at the corresponding points of P, and the order among points of P decreases downwards along the connecting lines. It is not difficult to see that this function is indeed the defector of an interval (in fact, by 3.5 (cf. proof of 4.7), the defector of a unique interval). (Consider the finite sums of the element

$$\begin{array}{ccc} & & 1 \\ -1 & - & 1 \end{array}$$

and its translates downwards.) It is obvious that this defector is not equivalent to a positive defector. (It does not even satisfy the equations of 4.6 with a positive defector d' and $\alpha_{x,y} \in \mathbf{Z}$ as in 4.6, since the value 1 at the top would have to be added at infinitely many positions.)

We note that this gives a second example showing that the class of lexicographic direct sums of elementary C^*-algebras is not closed under passing to hereditary sub-C^*-algebras. (Cf. 4.2.) (Use 2.6 and 2.8.)

If a hereditary sub-C^*-algebra of a separable lexicographic direct sum of elementary C^*-algebras has a unit, it must again belong to this class. To see this, note that any defector which is equal to 0 at all except finitely many points is equivalent to a positive defector (see 4.6 and 4.10). Note also that if D is an interval in $\mathbf{Z}^{(P)\mathrm{lex}}$ such that D has a largest element, and the defector d of D is positive, then $d \in \mathbf{Z}^{(P)}$ and

$$D = [0, d] = D(P, d)$$

(cf. 4.2). Then use 2.9 and the isomorphism theorem 4.3 of [**9**].

Another condition which is sufficient for a defector to be equivalent to a positive defector is given in Theorem 4.11 which follows.

4.10. LEMMA. *Let P be an ordered set, and let $(P_i)_{i \in I}$ be a partition of P into subsets. For each $i \in I$, let α_i be an ideal-preserving automorphism of the ordered group $\mathbf{Z}^{(P_i)\mathrm{lex}}$. Define an endomorphism α of the group $\mathbf{Z}^{(P)}$ by*

$$\alpha 1_x = \alpha_i 1_x, \quad x \in P_i.$$

It follows that α is an ideal-preserving automorphism of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$. We shall say that α is of block form.

Proof. It is immediate that α is an automorphism of the group $\mathbf{Z}^{(P)}$.

It follows from the definition of $\mathbf{Z}^{(P)\mathrm{lex}}$ that α is an endomorphism of the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$. Similarly, this holds for α^{-1}. It follows from the fact that each ideal of $\mathbf{Z}^{(P)\mathrm{lex}}$ is generated (as an ideal, and in fact as a subgroup) by a subset of $\{1_x; x \in P\}$ that α takes each ideal of $\mathbf{Z}^{(P)\mathrm{lex}}$ into itself. Similarly, this holds for α^{-1}. This shows that α is an ideal-preserving automorphism of $\mathbf{Z}^{(P)\mathrm{lex}}$.

4.11. THEOREM. *Let P be an ordered set, and let d be a defector on P. Denote by P' the subset $\{x \in P; d(x) \neq 0\}$, with the relative order. Suppose* *that every* $x \in P'$ *covers only finitely many* $y \in P'$ *with* $d(y) < 0$. *It follows that d is equivalent to a positive defector.*

Proof. By the definition of a defector, for each $x \in P'$ the set $\{y \in P'; y \geqq x\}$ is finite. For each $x \in P'$, denote by length(x) the length of the set $\{y \in P'; y \geqq x\}$, i.e., the length of the longest chain in this set.

Partition P into the sets $P \setminus P'$ and

$$P'_1 \cup P'_2, \quad P'_3 \cup P'_4, \ldots,$$

where P'_n denotes the set

$$\{x \in P'; \text{length}(x) = n\}.$$

Define an ideal-preserving automorphism α of $\mathbf{Z}^{(P)\text{lex}}$ of block form as follows. On $\mathbf{Z}^{(P \setminus P')}$, α is the identity. On $\mathbf{Z}^{(P'_{2n})}$, α is the identity. For each $x \in P'_{2n-1}$,

$$\alpha 1_x = 1_x + \sum\nolimits_{y<x, y \in P'_{2n}, d(y)<0} \alpha_{y,x} 1_y$$

where $\alpha_{y,x} \in \mathbf{Z}$ is such that

$$\alpha_{y,x} d(x) + d(y) \geqq 0.$$

By hypothesis, the sum is a finite one. Defined in this way, α is an ideal-preserving automorphism of each of the ordered groups

$$\mathbf{Z}^{(P \setminus P')\text{lex}} \quad \text{and}$$

$$\mathbf{Z}^{(P'_{2n-1} \cup P'_{2n})\text{lex}}, \quad n = 1, 2, \ldots,$$

and therefore by 4.10 is an ideal-preserving automorphism of $\mathbf{Z}^{(P)\text{lex}}$. Furthermore,

$$\alpha d(\bigcup P'_{2n}) \subseteq \mathbf{Z}^+,$$

$$\alpha d | \bigcup P'_{2n-1} = d | \bigcup P'_{2n-1}.$$

Note that $d(P'_1) \subseteq \mathbf{Z}^+$ (in fact, $d(P'_1) \subseteq \mathbf{Z}^+ \setminus \{0\}$, as each $x \in P'_1$ is maximal in P' and therefore also in P). Hence, partitioning P into the sets $(P \setminus P') \cup P'_1$ and

$$P'_2 \cup P'_3, \quad P'_4 \cup P'_5, \ldots,$$

we may in a similar way, using 4.10, define an ideal-preserving automorphism β of $\mathbf{Z}^{(P)\text{lex}}$ of block form (with respect to this second partition), such that

$$\beta\alpha d | \bigcup P'_{2n} = \alpha d | \bigcup P'_{2n},$$

$$\beta\alpha d(\bigcup P'_{2n+1}) \subseteq \mathbf{Z}^+.$$

Here we use that for each $x \in P'_{2n}$, $\alpha d(y) < 0$ for only finitely many $y \in P_{2n+1}$ with $y < x$ (recall that for such y, $\alpha d(y) = d(y)$). Since

$$\alpha d(\bigcup P'_{2n}) \subseteq \mathbf{Z}^+,$$

it follows that

$$\beta\alpha d(P) \subseteq \mathbf{Z}^+,$$

i.e., $\beta\alpha d$ is a positive defector.

4.12. *Definition*. Let P be an ordered set, and let d be a defector on P. Let R be a prime ideal of P, not of the form $\{y \in P; y \not\geqq x\}$, such that $d(P\backslash R) \subseteq \mathbf{Z}$, and such that $d(x)$ is nonzero for infinitely many $x \in P\backslash R$. Define a generalized integer

$$n = 2^{n_1}3^{n_2}5^{n_3}7^{n_4}\ldots \quad (n_i = 0, 1, 2, \ldots, +\infty)$$

as follows: m divides n if and only if m divides $d(x)$ for all except finitely many $x \in P\backslash R$. We shall refer to n as the *ultimate divisor of d at the prime ideal R*. Clearly, any defector equivalent to d will have the same ultimate divisor at R. (If d' is equivalent to d and m divides $d(x)$ for all except finitely many $x \in P\backslash R$, then for all except finitely many $x \in P\backslash R$,

$$d'(x) \equiv d(x) \pmod m$$

(see (4.6)), and hence for all except finitely many $x \in P\backslash R$, m divides $d'(x)$.)

In the case that P is the totally ordered set $-\mathbf{N}$, the only prime ideal of P not of the form $\{y \in P; y \not\geqq x\}$ is the empty set. Thus, a finite-valued defector with infinite support has just one ultimate divisor. This, together with the additional information consisting of the orbit of the defector on each quotient of P of the form $\{y \in \mathbf{N}; y \geqq x\}$, is, as we shall now show, a complete invariant.

4.13. THEOREM. *Let d be a defector on the ordered set* $-\mathbf{N}$. *If* $d(x) \in \mathbf{Z}\backslash\{0\}$ *for only finitely many* $x \in -\mathbf{N}$, *then there is a unique positive defector d' in the orbit of d such that, for each* $x \in -\mathbf{N}$, $d'(x)$ *either is equal to* $+\infty$ *or is strictly less than the greatest common divisor of the set*

$$\{d(y); y > x \quad \textit{and} \quad d(y) \in \mathbf{Z}\}.$$

We shall call this the normalized form of d.

If $d(x) \in \mathbf{Z}\backslash\{0\}$ *for infinitely many* $x \in -\mathbf{N}$ (*in this case* $d(-\mathbf{N}) \subseteq \mathbf{Z}$), *then the orbit of d is determined by the ultimate divisor of d* (*at the prime ideal* $\emptyset$), *together with the normalized form of the top part of d, an invariant defined as follows. The top part of d is the finite sequence*

$$(d(-1), \ldots, d(x))$$

where $x \in -\mathbf{N}$ *is such that the greatest common divisor of the set*

$$\{d(y); y > x\}$$

is equal to the greatest common divisor of all of $d(-\mathbf{N})$, *and, moreover, x is maximal such* (*x defined in this way is clearly invariant under equivalence, so the normalized form of the top part of d is an invariant*).

Proof. The first statement follows immediately from 3.2, 4.6, and 4.10.

Before proving the second statement, by 4.11 we may suppose that d is a positive defector.

By 4.6 and 4.10, we may normalize the top part of d by acting on d with an (ideal-preserving) automorphism of $\mathbf{Z}^{(-\mathbf{N})\mathrm{lex}}$. Since the greatest common divisor of the top part of d divides all of d, we may divide by this and suppose it is equal to 1. We may then act on d by an automorphism (see 4.6 and 4.10) so that the first coefficient of d after the top part of d is equal to 1.

In this way (by now neglecting the top part of d) the problem is reduced to the case that $d(0) = 1$. In this case we must show that the orbit of d is determined by the ultimate divisor of d. Choose a sequence $0 = x_0 > x_1 > x_2 > \ldots$ in $-\mathbf{N}$ such that for each $k = 0, 1, 2, \ldots$, the greatest common divisor of the set

$$d(\{x_k - 1, \ldots, x_{k+1} + 1\})$$

is equal to the greatest common divisor of the set

$$d(\{x_k - 1, x_k - 2, \ldots\}),$$

say n_{k+1}. Then by an application of 4.10 with the blocks

$$\{0\}, \{1, \ldots, x_1\}, \{x_1 + 1, \ldots, x_2\}, \ldots,$$

we may act on d by an automorphism of $\mathbf{Z}^{(-\mathbf{N})\mathrm{lex}}$ of block form in such a way as to replace $d(x_k)$ by n_k for each $k = 1, 2, \ldots$, keeping the other coefficients of d fixed. Hence by another application of 2.10, with the blocks

$$\{0, \ldots, x_1 - 1\}, \{x_1, \ldots, x_2 - 1\}, \ldots,$$

we may act on d by an automorphism of $\mathbf{Z}^{(-\mathbf{N})\mathrm{lex}}$ of block form to obtain the equivalent defector

$$(1, 0, \ldots, 0, n_1, 0, \ldots, 0, n_2, 0, \ldots).$$

Note that n_k divides n_{k+1} for each $k = 1, 2, \ldots$, so the ultimate divisor of d, say n, is just the least common multiple of $\{n_1, n_2, \ldots\}$, among generalized integers.

We must show that if $(n_1', n_2', \ldots)$ is a sequence of nonzero positive integers with n_k' dividing n_{k+1}' for each k, such that also the least common multiple of $\{n_1', n_2', \ldots\}$ is n, then d is equivalent to the defector

$$(1, 0, \ldots, 0, n_1', 0, \ldots, 0, n_2', 0, \ldots),$$

where the blocks of zeros are of arbitrary length.

Using 4.6 and 4.10, by an automorphism of block form we may transform any coinfinite subsequence of the nonzero coordinates of the defector

$$d = (1, 0, \ldots, 0, n_1, 0, \ldots, 0, n_2, 0, \ldots)$$

to zero, and similarly for the defector

$$d' = (1, 0, \ldots, 0, n'_1, 0, \ldots, 0, n'_2, 0, \ldots).$$

Note that we are not keeping track of how long the blocks of zeros are. Therefore we may replace $(n_1, n_2, \ldots)$ and $(n'_1, n'_2, \ldots)$ by subsequences. We now do so, choosing subsequences such that, after changing notation, we have the divisibility relations

$$n_1 | n'_1 | n_2 | n'_2 | \ldots,$$

and, furthermore, the position of n'_k in d' lies strictly between the positions of n_k and n_{k+1} in d, for each $k = 1, 2, \ldots$. Then d and d' are both subsequences of the intertwined defector

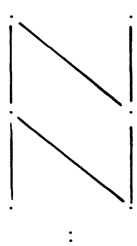
$$(1, 0, \ldots, 0, n_1, 0, \ldots, 0, n'_1, 0, \ldots, 0, n_2, 0, \ldots, 0, n'_2, 0, \ldots).$$

As the nonzero coefficients of this defector divide one another successively, we see as before that d and d' are both equivalent to this defector, and are therefore equivalent to each other. This shows that the orbit of d (with $d(0) = 1$) depends only on the ultimate divisor n.

4.14. *Remarks*. In 4.13 the case $d(0) = 1$ is particularly simple; the normalized form of the top part of d is just the one-term sequence (1), so all information about the orbit of d is contained in the ultimate divisor of d.

The analysis in 4.13 is easily extended to any totally ordered set P. While P may have more than one prime ideal not of the form $\{y \in P; y \ngeq x\}$, a defector can have an ultimate divisor at no more than one such prime ideal; this, together with the normalized top part defined in a similar way, is again a complete invariant. Of course, if P does not have a largest element, any defector is identically infinite.

Let us consider an example similar to that of 4.13 but in which P is not totally ordered. For P consider the union of two copies of $-\mathbf{N}$ with intertwining relations as follows:



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There are two prime ideals of P not of the form $\{y \in P; y \geqq x\}$, namely, $\emptyset$ and the right hand column, R. Let D be an interval in $\mathbf{Z}^{(P)\text{lex}}$, with defector d. If d is infinite somewhere, the classification is reduced by 3.2 to the case that either the left or the right column (or both) is cut off at a finite stage. This is the case that the ordered set is either $-\mathbf{N}$ or a finite perturbation of $-\mathbf{N}$ (or is finite), and is dealt with as in 4.13. Assume that d takes on only finite values. As in the totally ordered case (using 4.6, 4.10, and 4.11), we may suppose that d is positive with normalized top part. If d has infinite support on the left side then the ultimate divisors of d at $\emptyset$ and at R are both defined. The second of these divides the first, but otherwise they are arbitrary. Together with the normalized top part of d, they determine the orbit of d under equivalence. (This is proved much as in 4.13.) If d has finite support on the left side and infinite support on the right side, then only the ultimate divisor of d at $\emptyset$ is defined, but this together with the normalized top part of d determines the orbit of d. If d has finite support on all of P then d can be normalized, and its orbit is determined by the normalized form.

Finally, by 3.5, D is determined by d and the additional information whether D or the image of D in $\mathbf{Z}^{(P\backslash R)\text{lex}}$ has a largest element. Such a largest element can exist if, and only if, the support of d is finite in the first case, and the support of d in $P\backslash R$ (the left column) is finite in the second case.

5. The postliminary case.

5.1. THEOREM. *Let G be a dimension group. The following two statements are equivalent.*

(i) *Each simple subquotient of G is isomorphic to $\mathbf{Z}$, and a subset of the prime ideal spectrum* Spec G *is closed if it contains the closure of each of its points* (*that is, the Jacobson topology in* Spec G *is determined by the inclusion order relation*).

(ii) *G is isomorphic to the lexicographic direct sum $\mathbf{Z}^{(P)\text{lex}}$ where the ordered index set P satisfies the decreasing chain condition.*

Proof. (ii) $\Rightarrow$ (i). This follows immediately from the description of the ideals of a lexicographic direct sum of ordered groups $\mathbf{Z}^{(P)\text{lex}}$ which was given in the Introduction.

(i) $\Rightarrow$ (ii). First let us show that if the second condition of (i) holds then Spec G, ordered by inclusion, satisfies the decreasing chain condition. Let $t_1 \geqq t_2 \geqq \ldots$ be a decreasing chain of prime ideals of G. Then the ideal t of G with $t^+ = \bigcap t_n^+$ is prime, and by definition the closures of the sets $\{t\}$ and $\{t_1, t_2, \ldots\}$ in Spec G in the Jacobson topology are the same. By hypothesis, any union of closed subsets of Spec G is closed, and, in particular, the closure of the set $\{t_1, t_2, \ldots\}$ is equal to the union of the closures of the sets $\{t_1\}, \{t_2\}, \ldots$. It follows that for some $n = 1, 2, \ldots, t$ is in the closure of the set $\{t_n\}$, i.e., $t \geqq t_n$. Hence

$$t_n = t_{n+1} = \ldots (=t).$$

This shows that the chain $t_1 \geqq t_2 \geqq \ldots$ is finite.

Now suppose that (i) holds. We shall prove that G is isomorphic to $\mathbf{Z}^{(P)\text{lex}}$ where $P = \text{Spec } G$. By the preceding paragraph, this proves (ii).

For each $t \in P = \text{Spec } G$, choose a sub ordered group G_t of G isomorphic to $\mathbf{Z}$ as follows. Consider the ideals of G with spectra the open sets

$$\{s \in P;\, s \leqq t\} \quad \text{and} \quad \{s \in P;\, s < t\}.$$

The quotient of the first ideal by the second has spectrum $\{t\}$, so by hypothesis is isomorphic to $\mathbf{Z}$. Choose G_t to be a lifting of this quotient, i.e., to be the subgroup generated by a positive element of the first ideal mapping onto the positive generator of the quotient.

Let us now show that the family of subgroups $(G_t)_{t \in P}$ is independent with sum G, and that the resulting direct sum decomposition $G = \sum G_t$ defines an isomorphism of G with the ordered group $\mathbf{Z}^{(P)\text{lex}}$.

For this purpose we define a composition series $(I_\rho)_{0 \leqq \rho \leqq \alpha}$ for G as follows: $I_0 = 0$, and if $I_\rho \neq G$, $I_{\rho+1}/I_\rho$ is the sum of the minimal nonzero ideals of G/I_ρ. That G (or G/I_ρ) has a minimal nonzero ideal if it is not already zero is seen as follows. P has a minimal element by the chain condition. Any minimal point of P is in the complement of the closure of every other point of P, and hence (as by (i), any intersection of open sets is open) is isolated. This shows that P has an isolated point; this constitutes the spectrum of a minimal nonzero ideal of G. (The same argument is valid with a nonempty closed subspace of P in place of P, and shows that the quotient G/I_ρ has a minimal nonzero ideal if it is not zero.)

Fix $t \in P$, and denote by ρ the unique ordinal such that

$$t \in \text{Spec } I_{\rho+1} \quad \text{and} \quad t \notin \text{Spec } I_\rho.$$

Then $\{s \in P;\, s \leqq t\}$ is contained in Spec $I_{\rho+1}$, but not in Spec I_ρ. Furthermore, Spec $I_{\rho+1}/I_\rho$ is discrete, so $\{s \in P;\, s < t\}$ is contained in Spec I_ρ. By definition, the sum of G_t and the ideal of G with spectrum $\{s \in P;\, s < t\}$ is the ideal with spectrum $\{s \in P;\, s \leqq t\}$. Hence G_t is contained in $I_{\rho+1}$ and the image of G_t in $I_{\rho+1}/I_\rho$ is the minimal nonzero ideal with spectrum $\{t\}$.

Since each $I_{\rho+1}/I_\rho$ is the sum of its minimal nonzero ideals, and by distributivity these are independent, it follows from the preceding paragraph that the family $(G_t)_{t \in P}$ is independent and $\sum G_t = G$.

By (i), the open subsets of P ($=$Spec G) are the same as the ideals of P in the inclusion ordering. Let us show that the ideal of $G = \sum_{t \in P} G_t$ which corresponds to the ideal S of P is just $\sum_{t \in S} G_t$. It is enough to consider the case that S is singly generated, i.e., for some $s \in P$,

$$S = \{t \in P;\, t \leqq s\}.$$

Moreover, if ρ is such that

$$s \in \operatorname{Spec} I_{\rho+1} \quad \text{and} \quad s \notin \operatorname{Spec} I_\rho,$$

we may assume inductively that the statement holds for any ideal of P contained in Spec I_ρ. As above, we have

$$\{t \in P;\ t < s\} \subseteq \operatorname{Spec} I_\rho,$$

so the ideal of G with spectrum $\{t \in P;\ t < s\}$ is by the inductive assumption equal to $\sum_{t<s} G_t$. Hence by definition of G_s, the ideal of G with spectrum $\{t \in P;\ t \leqq s\}$ is

$$G_s + \sum_{t<s} G_t = \sum_{t \leqq s} G_t.$$

Now let us show that the order in $G = \sum G_t$ is the lexicographic order. This means that if $g = \sum g_t$ with $g_t \in G_t$ then g is positive in G if and only if g_t is positive in G_t for each t maximal in P such that $g_t \neq 0$. If $g = \sum g_t$ is an element of G, denote by S the ideal of P consisting of those elements strictly less than some $t \in P$ with $g_t \neq 0$, and denote by I the ideal $\sum_{t \in S} G_t$ of G. If $g \geqq 0$, then $g + I \geqq 0$, and since $g_t + I$ is zero unless t is minimal in $P \backslash S$, in which case the ordered group G_t maps isomorphically onto a minimal nonzero ideal of G/I, we see that for each such t, $g_t + I \geqq 0$ and hence $g_t \geqq 0$; in particular, $g_t \geqq 0$ for each t maximal with $g_t \neq 0$.

Suppose, conversely, that $g_t \geqq 0$ for each t maximal with $g_t \neq 0$, and let us prove that $g \geqq 0$. The hypothesis may be restated as follows:

$$g - \sum_{t \in S} g_t \geqq 0.$$

(Here S is as defined above.) The conclusion may be strengthened (in fact, just restated) as follows:

$$g - h \geqq 0 \quad \text{for any } h \in I.$$

(Here $I = \sum_{t \in S} G_t$, as above.) Replacing g by $g - \sum_{t \in S} g_t$ does not change this statement of the conclusion, and it transforms the hypothesis into the more convenient form $g \geqq 0$ (the original form of the conclusion).

Suppose, then, that

$$g = \sum g_t \geqq 0,$$

and that

$$h = \sum_{t \in S} h_t \in I$$

(with S and I as above), and let us show that $g - h \geqq 0$. Since $I = I^+ - I^+$, we may suppose that $h \geqq 0$. Then by what we proved above, h is also positive in the lexicographic order. Denote by T the ideal of P consisting of those elements strictly less than some $t \in P$ with $h_t \neq 0$, and denote by J the ideal of $\sum_{t \in T} G_t$ of G; we have $J \subseteq I \subseteq I_\rho$. It is sufficient to show that $g - h + J \geqq 0$ in G/J. For then, using this with $2h$ in place of h, we have

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$$g - 2h + k \geqq 0 \quad \text{for some } k \in J.$$

Since then both h and k belong to I_ρ, and $h - k$ is positive in the lexicographic order (as h is, and $k \in J$), by the inductive assumption it follows that $h - k \geqq 0$. Adding this to the inequality $g - 2h + k \geqq 0$ gives $g - h \geqq 0$.

To show that $g - h \geqq 0$, then, we may suppose that $J = 0$. In this case each h_t is a positive multiple of an atom of I^+ (i.e., of a minimal nonzero element of I^+) and so h is a finite sum of atoms of I^+, say

$$h = h_1 + \ldots + h_n.$$

Now consider the ideals of G containing g. Since every ideal of G is the sum of certain subgroups G_t (namely, those subgroups G_t for which t is in the spectrum of the ideal), and since the family (G_t) is independent, any ideal containing $g = \sum g_t$ must contain G_t for all t with $g_t \neq 0$. Hence any ideal of G containing g contains

$$I = \textstyle\sum_{t \in S} G_t.$$

On the other hand, the set of all positive elements of G majorized by some positive multiple of g is the positive part of an ideal of G. This ideal contains g since g is positive, and therefore it contains I. In particular, as $h_1 \in I^+$, we have $h_1 \leqq mg$ for some $m = 1, 2, \ldots$. Hence by Riesz decomposition,

$$h_1 = g_1 + \ldots + g_m \quad \text{with } 0 \leqq g_i \leqq g.$$

Since I is an ideal of G, each g_i belongs to I. Since h_1 is an atom of I^+, h_1 is equal to some g_i; in particular, $g - h_1 \geqq 0$. Replacing g by $g - h_1$ we obtain, in the same way,

$$g - h_1 - h_2 \geqq 0.$$

Repeating this yields after n steps the desired inequality

$$g - h = g - h_1 - \ldots - h_n \geqq 0.$$

5.2. *Remark*. The decreasing chain condition on the ordered set P may be reformulated as follows: the complement of any prime ideal of P has a smallest element. In this way one sees that the decreasing chain condition on P is equivalent to the condition on the ordered group $\mathbf{Z}^{(P)\mathrm{lex}}$ that each prime quotient have an ideal isomorphic to $\mathbf{Z}$. One also sees directly the fact established in the course of the proof of 5.1 that in this case the prime ideal spectrum of $\mathbf{Z}^{(P)\mathrm{lex}}$ is equal to P.

5.3. *Remark*. In 5.1 the proof of (i) $\Rightarrow$ (ii) does not use that G is unperforated. Therefore this property is a consequence of (i) in an ordered abelian group with the Riesz decomposition property.

5.4. *Remark*. The lexicographic direct sum $\mathbf{Z}^{(P)\mathrm{lex}}$ cannot in general be characterized by its simple subquotients (all equal to $\mathbf{Z}$) and its ideal lattice (the ideal lattice of P). (5.1 implies that it can when P satisfies the decreasing chain condition.) A counterexample is the case $P = \mathbf{Z}$; see 4.8 of [**10**].

5.5. COROLLARY. *Let A be a separable C^*-algebra. The following three statements are equivalent.*

(i) *Each simple subquotient of A is elementary, and a subset of the primitive ideal spectrum* Prim A *is closed if it contains the closure of each of its points.*

(ii) *A is approximately finite-dimensional and the dimension group of A is isomorphic to the lexicographic direct sum $\mathbf{Z}^{(P)\mathrm{lex}}$ where P satisfies the decreasing chain condition.*

(iii) *A is a lexicographic direct sum of a family of elementary C^*-algebras $(B_x)_{x \in P}$ where P satisfies the decreasing chain condition.*

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds and let us show first that Prim A satisfies the decreasing chain condition. Let $t_1 \geqq t_2 \geqq \dots$ be a decreasing chain in Prim A. Then in particular each t_n is a prime ideal, so the intersection $t = \bigcap t_n$ is also a prime closed two-sided ideal of A. Let us pass to the quotient and change notation so that $t = 0$, Prim $t = \emptyset$. Suppose that no t_n is equal to 0. Then, as A is prime, each Prim t_n is dense in Prim A, whence by 3.4.13 of [7], $\bigcap$Prim t_n is dense in A. But by (i), $\bigcap$Prim t_n is open, so

$$\bigcap \mathrm{Prim}\ t_n = \mathrm{Prim}\ t = \emptyset.$$

This contradiction shows that some t_n is equal to 0, i.e., the decreasing chain $t_1 \geqq t_2 \geqq \dots$ is finite.

As in the proof of (i) $\Rightarrow$ (ii) of 5.1, it follows by further use of (i) that every nonempty closed subspace of Prim A has an isolated point and hence that A has a composition series with elementary subquotients. By 7 of [**5**] combined with transfinite induction, it follows that A is approximately finite-dimensional.

By 5.1 of [**10**] the lattice of closed ideals of A is isomorphic to the lattice of ideals of the dimension group $K_0(A)$. Since, by 3.9.1 of [7], every prime closed ideal of A is primitive, it follows that Prim A is homeomorphic to Spec $K_0(A)$. Hence $K_0(A)$ satisfies 5.1 (i), and therefore 5.1 (ii), from which condition (ii) of the corollary follows.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. By 4.8, the dimension range of A, an interval in $\mathbf{Z}^{(P)\mathrm{lex}}$, is isomorphic to $D(P, d)$ for some positive defector d. By 2.10, there is a separable C^*-algebra $A(P, d)$ which is a lexicographic direct sum of a family of elementary C^*-algebras $(B_x)_{x \in P}$, and which is therefore by 2.8 approximately finite-dimensional, such that the dimension range of A is isomorphic to $D(P, d)$. By 4.3 of [**9**], A is isomorphic to $A(P, d)$.

(iii) $\Rightarrow$ (i). Suppose that (iii) holds. By 2.2, the lattice of closed ideals of A is naturally isomorphic to the lattice of ideals of P. In particular, it follows that every simple subquotient of A is elementary. By hypothesis, the complement of any prime ideal of P has a smallest element (see 5.2). Thus, the prime ideal spectrum of P is equal to P, and the open sets are the ideals of P. Since Prim A is homeomorphic to the prime ideal spectrum of P, i.e., to P with open sets the ideals of P, (i) follows.

5.6. *Remark.* While the implication (iii) $\Rightarrow$ (i) in 5.5 does not need the assumption that A is separable, and the implications (i) $\Rightarrow$ (ii) and also (ii) $\Rightarrow$ (i) are easily proved without this assumption (using 5.1), it is not clear how to prove the implication (ii) $\Rightarrow$ (iii) without assuming that A is separable.

5.7. *Remark.* The classification of separable postliminary lexicographic direct sums of elementary C^*-algebras follows, in principle, from 4.6 and 4.8. Thus, only the defector need be considered, and equivalence of defectors may be described in an explicit way.

Of course, one might hope for a simple parametrization of the equivalence classes, which is after all possible in the case of the ordered set $-\mathbf{N}$ (see 4.13), in which case the dimension group is not postliminary.

If a defector can be put in normalized form then this is a canonical label for the equivalence class. As follows from 3.2, 4.6, and 4.10 (cf. 4.13), this is possible for a defector with only finitely many nonzero finite coordinates (see also [**1**], [**2**]). As shown recently by Jensen in [**16**], a defector can be put in normalized form if, more generally, every point at which it is finite and nonzero majorizes only finitely many other such points.

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