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ON A VARIANT OF A QUESTION PROPOSED BY K. MAHLER CONCERNING LIOUVILLE NUMBERS

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Abstract

In this note we shall prove the existence of an uncountable subset of Liouville numbers (which we call the set of *ultra-Liouville numbers*) for which there exist uncountably many transcendental analytic functions mapping the subset into itself.

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1. Introduction

A real number ξ is called a *Liouville number*, if there exists a rational sequence $(p_k/q_k)_{k\geq 1}$, with $q_k > 1$, such that

$$0 < \left|\xi - \frac{p_k}{q_k}\right| < q_k^{-k} \quad \text{for } k = 1, 2, \ldots.$$

The set of the Liouville numbers is denoted by $\mathbb{L}$.

The name arises because Liouville [4] showed in 1844 that all Liouville numbers are transcendental, thus establishing the first explicit examples of transcendental numbers. The number $\ell := \sum_{n\geq 1} 10^{-n!}$, the so-called *Liouville constant*, is a standard example of a Liouville number. In 1962, Erdős [3] proved that every real number can be written as the sum and (if it is nonzero) the product of two Liouville numbers, as a consequence of the fact that $\mathbb{L}$ is a rather large set in a topological sense: it is a dense G_δ set.

In his pioneering book, Maillet [6, Ch. III] discusses some arithmetic properties of Liouville numbers. One of them is that, given a rational function f, with rational coefficients, if ξ is a Liouville number, then so is $f(\xi)$. We observe that the converse of this result is not valid in general; for example, taking $f(x) = x^2$, then $\zeta := \sqrt{(3+\ell)/4}$ is not a Liouville number [1, Theorem 7.4], but $f(\zeta)$ is. Also the rational coefficients cannot be taken to be algebraic (with at least one of them nonrational). For instance, $\ell\sqrt{3/2}$ is not a Liouville number, see [6, Theorem I_3]. In fact, $\ell\sqrt{3/2}$ is a U_2-number (for the definition of a U_2-number and this result, see [2]).

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An *algebraic function* is a function $f(x)$ which satisfies $P(x, f(x)) = 0$, where $P(x, y)$ is a polynomial with complex coefficients. For instance, functions that can be constructed using only a finite number of elementary operations are examples of algebraic functions. A function that is not algebraic is, by definition, a *transcendental function*. Common examples are the trigonometric functions, the exponential function, and their inverses.

In 1984, in one of his last papers, Mahler [5] posed several questions for which, according to him, 'perhaps further research might lead to interesting results'. His first question is related to Liouville numbers. In particular, this question asks the following:

QUESTION. Are there transcendental entire functions $f(z)$ such that if ξ is any Liouville number, then so is $f(\xi)$?

He said that: 'The difficulty of this problem lies of course in the fact that the set of all Liouville numbers is non-enumerable.'

The study of similar problems has attracted the attention of several mathematicians. Let A and B be subsets of $\mathbb{C}$ with $A \subset B$ and let $\Sigma_A(B)$ be the set of all transcendental analytic functions $f : B \to B$ such that $f(A) \subseteq A$. In 1886, Weierstrass proved that the set $\Sigma_{\mathbb{Q}}(\mathbb{R})$ has the power of continuum. Moreover, he asserted that $\Sigma_{\overline{\mathbb{Q}}}(\mathbb{C}) \neq \emptyset$. In 1896, Stäckel [7] confirmed the Weierstrass assertion by proving that for each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$, there is a transcendental entire function f such that $f(\Sigma) \subseteq T$. In particular, if A is a countable dense subset of $\mathbb{C}$, then $\Sigma_A(\mathbb{C})$ is uncountable. Consult the very extensive annotated bibliography of [8] for additional references and history. Note that the Mahler question can be rephrased as: is $\Sigma_{\mathbb{L}}(\mathbb{C}) \neq \emptyset$?

Set, inductively, $\exp^{[n]}(x) = \exp(\exp^{[n-1]}(x))$ and $\exp^{[0]}(x) = x$. Now let us define the following class of numbers:

DEFINITION. A real number ξ is called an *ultra-Liouville number* if, for every positive integer k, there exist infinitely many rational numbers p/q, with $q > 1$, such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{\exp^{[k]}(q)}.$$

The set of the ultra-Liouville numbers will be denoted by $\mathbb{L}_{\text{ultra}}$.

It follows from the definition that $\mathbb{L}_{\text{ultra}} \subseteq \mathbb{L}$ is also a dense G_δ set (in particular, it is uncountable) which means that $\mathbb{L}_{\text{ultra}}$ is a large set in a topological sense. In particular, every real number can be written as the sum and (if it is nonzero) the product of two ultra-Liouville numbers, as in [3]. However, from a metrical point of view, both sets $\mathbb{L}$ and $\mathbb{L}_{\text{ultra}}$ are very small: they have Hausdorff dimension zero.

The aim of this paper is to investigate a problem related to Mahler's question concerning $\mathbb{L}_{\text{ultra}}$. More precisely, our main result is the following

THEOREM 1.1. *The set $\Sigma_{\mathbb{L}_{\text{ultra}}}(\mathbb{C})$ is uncountable.*

In order to prove that, we shall prove a stronger result about the behaviour of some functions in $\Sigma_{\mathbb{Q}}(\mathbb{C})$. For a rational number z, we denote its denominator by $\text{den}(z)$. We prove the following result.

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THEOREM 1.2. *There exist uncountably many functions* $f \in \Sigma_{\mathbb{Q}}(\mathbb{C})$ *with* $\frac{1}{2} < f'(x) < \frac{3}{2}$, *for all* $x \in \mathbb{R}$, *such that*

$$\mathrm{den}(f(p/q)) < q^{8q^2}, \qquad (*)$$

for all $p/q \in \mathbb{Q}$, *with* $q > 1$. *In particular,* $\mathrm{den}(f(p/q)) < e^{e^q} - 1$, *if* $q \geq 7$.

2. The proofs

2.1. Proof that Theorem 1.2 implies Theorem 1.1. Given an ultra-Liouville number ξ and a positive integer k, there exist infinitely many rational numbers p/q with $q \geq 7$ and such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{\exp^{[k+2]}(q)}.$$

Let f be a function as in Theorem 1.2. By the mean value theorem, we obtain

$$\left|f(\xi) - f\left(\frac{p}{q}\right)\right| \leq \frac{3}{2}\left|\xi - \frac{p}{q}\right| < \frac{3}{2\exp^{[k+2]}(q)}.$$

We know that $f(p/q) = a/b$, with $b < e^{e^q} - 1$. Then $\frac{3}{2}\exp^{[k]}(b) < \exp^{[k+2]}(q)$ and hence

$$\left|f(\xi) - \frac{a}{b}\right| = \left|f(\xi) - f\left(\frac{p}{q}\right)\right| < \frac{1}{\exp^{[k]}(b)}.$$

This implies that $f(\xi)$ is an ultra-Liouville number as desired. □

2.2. Proof of Theorem 1.2. Before starting the proof, we shall state three useful facts (which can be easily proved)

- For any distinct $y, b \in [-1, 1]$, we have $|\sin(y - b)| > |y - b|/3$.
  (Indeed, the function $\sin(x)/x$ is decreasing for $x \in (0, \pi]$, and $\sin(2)/2 > \frac{1}{3}$.)
- For any distinct $x, y \in \mathbb{Q} \cap [0, \frac{1}{2}]$, with $\mathrm{den}(x), \mathrm{den}(y) \leq n$, we have

$$|\cos(2\pi x) - \cos(2\pi y)| \geq \frac{4}{n^3}.$$

  (Indeed, we first assume $0 \leq x < y \leq \frac{1}{4}$, so we have two cases: if $x = 0$ then $\cos(2\pi x) - \cos(2\pi y) = 1 - \cos(2\pi y) = 2\sin^2(\pi y) \geq 8/n^2 \geq 16/n^3$, since $\mathrm{den}(y) \geq 2$; and if $0 < x < y$ then $x \geq 1/n$ and, by the mean value theorem, $|\cos(2\pi x) - \cos(2\pi y)| \geq 2\pi\sin(2\pi x)(2\pi y - 2\pi x) \geq 8\pi x(y - x) \geq 8\pi(y - x)/n \geq 8\pi/n^3 > 16/n^3$. If $\frac{1}{4} < x, y < \frac{1}{2}$, replace x, y by $\frac{1}{2} - x$, $\frac{1}{2} - y$ and argue similarly.)
- For every $\epsilon \in (0, 2]$, any interval of length greater than ϵ contains at least two rational numbers with denominator less than or equal to $\lceil 2/\epsilon \rceil$.

Consider the following enumeration of $\mathbb{Q} \cap [0, \frac{1}{2}]$:

$$\{x_1, x_2, \ldots\} = \{\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{6}, \ldots\},$$

where we consider only irreducible fractions ordered in the following way: $x_1 = \frac{0}{1}$; for every $k \geq 1$, if $x_k = p/q$ with $2p < q - 2$ then $x_{k+1} = r/q$ where r is the minimum with $p < r \leq q/2$ and $\gcd(r, q) = 1$, and if $2p \geq q - 2$ then $x_{k+1} = 1/(q + 1)$. The set $A = \mathbb{Q} \cap [0, \frac{1}{2}]$ has the properties that $\cos(2\pi x) \neq \cos(2\pi y)$ for every $x \neq y$ in A, and that for every $z \in \mathbb{Q}$ there is (exactly one) $x \in A$ with $\cos(2\pi x) = \cos(2\pi z)$.

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One can see that $\mathrm{den}(x_n) \geq \sqrt{n}$, for all $n \geq 1$: indeed, the number of positive integers n for which the denominator of x_n is equal to k is at most k for every $k \geq 1$, so the maximum positive integer n for which the denominator of x_n is at most k is at most $1 + 2 + \cdots + k = k(k+1)/2 \leq k^2$.

Define $B_n = \{y_1, y_2, \ldots, y_n\}$ with $y_k := \cos(2\pi x_k)$ and define f by

$$f(x) = x + g(\cos(2\pi x)),$$

where $g(y) = \sum_{n=1}^{\infty} c_n g_n(y)$ and $g_n(y) = \prod_{b \in B_n} \sin(y - b)$. Note that $f(x+1) = f(x) + 1$ and so it is enough to consider $\mathbb{Q} \cap [0, 1)$ in order to characterise f on $\mathbb{Q}$. Notice also that, in order to show that $f(x) \in \mathbb{Q}$ for every $x \in \mathbb{Q}$, it is enough to prove this for $x \in A$. Indeed, given $z \in \mathbb{Q}$, take $x \in A$ with $\cos(2\pi x) = \cos(2\pi z)$. Then we have $f(z) - z = g(\cos(2\pi z)) = g(\cos(2\pi x)) = f(x) - x$, and so, if $f(x) \in \mathbb{Q}$, then $f(z) = f(x) + z - x \in \mathbb{Q}$; in particular, if $z \in \mathbb{Z}$ then $f(z) = z$, since $f(0) = 0$.

Now we shall choose the constants c_n inductively so that f will satisfy the desired conditions in Theorem 2. The first requirements are $c_n = 0$ for $1 \leq n \leq 5$ and $|c_n| < 1/n^n$ for every positive integer n. On the other hand, for all y belonging to the open ball $B(0, R)$, one has that

$$|g_n(y)| < \prod_{b \in B_n} e^{|y-b|} \leq e^{n(R+1)},$$

where we use the fact that $b \in [-1, 1]$. Thus, since $|c_n| < 1/n^n$, we get $|c_n g_n(y)| \leq (e^{R+1}/n)^n$ from which g (and so f) is an entire function, since the series $g(y) = \sum_{n=1}^{\infty} c_n g_n(y)$, which defines g, converges uniformly in any of these balls. Moreover, for $x \in \mathbb{R}$, we have $|g_n'(x)| \leq n$, and so $f'(x) = 1 - 2\pi \sin(2\pi x) \sum_{n=1}^{\infty} c_n g_n'(\cos(2\pi x)) \in (\frac{1}{2}, \frac{3}{2})$, since $\sum_{n=6}^{\infty} n/n^n < \frac{1}{4}\pi$.

Suppose that $c_1, \ldots, c_{n-1}$ have been chosen such that $f(x_1), \ldots, f(x_n)$ have the desired property (notice that the choice of $c_1, \ldots, c_{n-1}$ determines the values of $f(x_1), \ldots, f(x_n)$, independently of the values of $c_k, k \geq n$; in particular, since $c_k = 0$ for $1 \leq k \leq 5$, we have $f(x_n) = x_n$ for $1 \leq n \leq 6$). Now we shall choose c_n for which $f(x_{n+1})$ satisfies the requirements.

Let $t \leq n$ be positive integers with $n \geq 5$. Then $\mathrm{den}(x_{n+1}), \mathrm{den}(x_t) \leq n$ (indeed, $\mathrm{den}(x_6) = 5$ and $\mathrm{den}(x_{n+1}) - \mathrm{den}(x_n) \leq 1$, for all $n \geq 1$). Since $\cos(2\pi x_{n+1}) \neq \cos(2\pi x_t)$, we have $|y_{n+1} - y_t| \geq 4/n^3$. Therefore

$$|\sin(y_{n+1} - y_t)| > \frac{|y_{n+1} - y_t|}{3} > \frac{4}{3n^3} > \frac{1}{n^3}$$

yielding $|g_n(y_{n+1})| > n^{-3n}$. Thus $c_n g_n(y_{n+1})$ runs through an interval of length larger than $2/n^{4n}$. Now we may choose c_n (in at least two ways) such that $g(y_{n+1})$ is a rational number with denominator at most n^{4n}.

Given $z \in \mathbb{Q}$, let $q = \mathrm{den}(z)$. If $q = 1$ then $z \in \mathbb{Z}$ and so $f(z) = z$ and thus $\mathrm{den}(f(z)) = 1 \leq q^{8q^2}$. Otherwise $q > 1$, and there is a positive integer k with $\cos(2\pi x_k) = \cos(2\pi z)$, so x_k and z have the same denominator; indeed, in this case, we have $z - x_k \in \mathbb{Z}$ or $z + x_k \in \mathbb{Z}$. Thus $\mathrm{den}(f(z) - z) = \mathrm{den}(g(\cos(2\pi z))) = \mathrm{den}(g(\cos(2\pi x_k))) = \mathrm{den}(g(y_k)) \leq (k-1)^{4(k-1)} < k^{4(k-1)}$. Since $q = \mathrm{den}(z) = \mathrm{den}(x_k) \geq \sqrt{k}$, we get $\mathrm{den}(f(z) - z) \leq$

$k^{4(k-1)} \le (q^2)^{4(q^2-1)} = q^{8(q^2-1)}$. Then

$$\mathrm{den}(f(z)) \le \mathrm{den}(z)\,\mathrm{den}(f(z)-z) = q\,\mathrm{den}(f(z)-z) \le q \cdot q^{8(q^2-1)} \le q^{8q^2}$$

as desired.

The proof that we can choose f to be transcendental follows because there is a binary tree of different possibilities for f. (If we have chosen $c_1, c_2, \ldots, c_{n-1}$, different choices of c_n give different values of $f(y_{n+1})$, which does not depend on the values of c_k for $k > n$, and so different functions f.) Thus, we have constructed uncountably many possible functions, and the algebraic entire functions taking $\mathbb{Q}$ into itself must be polynomials belonging to $\mathbb{Q}[z]$, which is a countable subset.

In fact, we can prove that all functions constructed above are transcendental, unless $c_n = 0$, for all $n \in \mathbb{N}$: if such a function f is not transcendental, then f would be polynomial, since it is an entire function. However, the property $f(x+1) = f(x) + 1$ would imply $f(x) = x + c$, for some $c > 0$. Then $g(\sin(2\pi x))$ is a constant, but this leads to a contradiction, since $g(y_1) = 0$ and $g(y_{k+1}) = c_k \prod_{b \in B_k} \sin(y_{k+1} - b) \ne 0$, where k is minimal such that $c_k \ne 0$. □

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