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A CHARACTERIZATION OF $PSL(2, 31)$ AND ITS GEOMETRY

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1. Introduction. The aim of this paper is the characterization of $PSL(2, 31)$ in terms of its action on a certain polygonal graph. A *polygonal graph* is a pair $(\mathscr{H}, \mathscr{E})$ consisting of a graph $\mathscr{H}$ which is regular, connected and has girth m for some $m \geqq 3$, and a set $\mathscr{E}$ of m-gons (circuits of length m) of $\mathscr{H}$ such that every 2-claw (i.e. path of length 2) of $\mathscr{H}$ is contained in a unique element of $\mathscr{E}$. (See Section 2 for the definitions of the terms used here.) If $\mathscr{E}$ is the set of all m-gons of H, so that there is in $\mathscr{H}$ a unique m-gon on every one of its 2-claws, then we write $\mathscr{H}$ for $(\mathscr{H}, \mathscr{E})$ and call $\mathscr{H}$ a *strict polygonal graph.* If we wish to emphasize the integer m, then we call $(\mathscr{H}, \mathscr{E})$ an *m-gon-graph* (respectively, a *strict m-gon-graph*). For convenience, a strict 5-gon-graph will be called a *pentagraph.*

If $(\mathscr{H}, \mathscr{E})$ is a polygonal graph on a set Ω and $G \leqq \mathrm{Aut}\ (\mathscr{H})$, then we shall denote by $[H]$ the following hypothesis:

Suppose that for any 2-claw $(x: y, z)$, $x, y, z \in \Omega$, every involution in G_{xyz} fixes the m-gon in $\mathscr{E}$ on $(x: y, z)$, but no other m-gon on $(x: y, z)$, where in the case that G_{xyz} has no involutions we interpret this to mean that G_{xyz} fixes the m-gon in $\mathscr{E}$ on $(x: y, z)$, and no other m-gon on $(x: y, z)$.

Note that $[H]$ is automatically satisfied if $\mathscr{H}$ is a strict m-gon-graph.

THEOREM 1.1. (Theorem 2 of [**4**]) *Let $(\mathscr{H}, \mathscr{E})$ be a polygonal graph of valency $k \geqq 3$ on a set Ω, with girth $m \geqq 5$, where k and m are odd. Suppose that $G \leqq \mathrm{Aut}\ (\mathscr{H})$ is a group of automorphisms of $\mathscr{H}$ transitive on Ω with G_x 3-transitive on $\Delta(x)$ (the vertices of $\mathscr{H}$ adjacent to x). Assume hypothesis $[H]$ and that $\mathscr{H}$ contains no strict m-gon-graph of valency 3 as a subgraph.*

Then $k = 5$ and $G_x \cong A_5$.

Now in the case that $m = 5$, there is a unique pentagraph of valency 3, which we shall call the dodecahedral graph. This graph and the Petersen graph are discussed in Section 2, as well as the definitions and notation to be used. In Section 3 we will discuss the pentagraph of valency 5, to be denoted by $\mathscr{H}_{31}$, on which the group $PSL(2, 31)$ acts.

In Section 4 we will investigate in detail pentagraphs of valency 5, finally proving the following.

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THEOREM 1.2. *Let $\mathscr{H}$ be a pentagraph of valency* 5 *and* $G \leqq \mathrm{Aut}\,(\mathscr{H})$ *with* $G_x^{\Delta(x)} \cong A_5$ *for all vertices* x *of* $\mathscr{H}$. *Then either* (i) $\mathscr{H}$ *contains dodecahedral subgraphs, or* (ii) $G \cong PSL(2, 31)$ *and* $\mathscr{H} \cong \mathscr{H}_{31}$.

Theorem 1.1 together with Theorem 1.2 now give the following characterization of $PSL(2, 31)$.

THEOREM 1.3. *Let* $(\mathscr{H}, \mathscr{E})$ *be a* 5*-gon-graph on a set* Ω *with the valency of* $\mathscr{H}$ *odd. Let* $G \leqq \mathrm{Aut}\,(\mathscr{H})$ *be such that* G_x *is* 3*-transitive on* $\Delta(x)$ *for all* $x \in \Omega$, *and assume hypothesis* [H]. *Then the following are equivalent:*

(i) $G \cong PSL(2, 31)$ *and* $\mathscr{H} \cong \mathscr{H}_{31}$.

(ii) $\mathscr{H}$ *contains no dodecahedral subgraph.*

2. Notation and preliminaries. All groups and graphs will be finite and all graphs will be undirected, with no loops or multiple edges. If $\mathscr{H}$ is such a graph with vertex set Ω, then for $x, y \in \Omega$, we write $x \sim y$ to mean x is adjacent to y. As mentioned before, $\Delta(x)$ denotes the set of vertices of Ω adjacent to $x \in \Omega$. If Γ is a subset of Ω, the *induced subgraph* of $\mathscr{H}$ on Γ is the maximal subgraph of $\mathscr{H}$ with vertices the set Γ. The *valency* of x is $|\Delta(x)|$ and $\mathscr{H}$ is called *regular* if the valency of each vertex is the same. A *path of length* n is a sequence $(x_0, x_1, \ldots, x_n)$ of $n + 1$ vertices in Ω with $x_i \sim x_{i+1}$, $i = 0, \ldots, n-1$ and $x_i \neq x_{i+2}$, $i = 0, \ldots, n-2$. This path is called a *circuit* if it is a closed path (i.e. $x_0 = x_n$ and $x_1 \neq x_{n-1}$), and the *girth* of $\mathscr{H}$ is the length of the smallest circuit of $\mathscr{H}$. A *pentagon* is a circuit consisting of 5 distinct vertices. The *distance* from x to y is the length of the shortest path from x to y (if one exists). We say $\mathscr{H}$ is *connected* if there is a path from x to y for all $x, y \in \Omega$. By a 2*-claw* $(x{:}y, z)$ we mean a path (y, x, z) of length 2, so that $y \neq z$ and $y, z \in \Delta(x)$. The automorphism group of $\mathscr{H}$ will be denoted by $\mathrm{Aut}\,(\mathscr{H})$.

If G is a group acting on a set Ω, we denote by x^g the image of $x \in \Omega$ by the element $g \in G$. For $W = \{x, y, z, \ldots\}$ a subset of Ω, $G_{xyz\ldots} = G_{[W]}$ will denote the pointwise stabilizer, and G_W the setwise stabilizer, of W. G^Ω will denote the group of permutations of Ω induced by the action of G, so that $G^\Omega \cong G/G_{[\Omega]}$. For $g \in G$, $\Omega(g)$ is the subset of Ω fixed (pointwise) by g.

If G is transitive on Ω and $\Gamma \neq \{x\}$ is an orbit for G_x, $x \in \Omega$, then the *graph* $\mathscr{H}$ *defined with respect to the suborbit* Γ is the graph whose vertex set is Ω and edge set the set of pairs $\{(x^g, y^g) : g \in G\}$, for some $y \in \Gamma$. Then $\mathscr{H}$ is a regular graph of valency $|\Gamma|$ and is undirected if and only if there exists a $g \in G$ with $x^g \in \Gamma$ and $g^2 \in G_x$. Also $\mathscr{H}$ is connected if and only if $G = \langle G_x, h \rangle$ for any $h \in G$ such that $x^h \in \Gamma$.

We shall use Z_n, D_n to denote respectively the cyclic group of order n and the dihedral group of order n. Σ_n and A_n denote the symmetric and alternating groups of degree n, respectively.

The Petersen graph. Consider $G = \Sigma_5$ acting on the set Ω of two-element subsets of $\{1, 2, 3, 4, 5\}$, so $|\Omega| = 10$. Let $x = \{1, 2\}$ and $u = \{3, 4\}$. Define a graph $\mathscr{H}$ on Ω whose edges are $\{(x^g, u^g): g \in G\}$. Alternatively, and equivalently, join pair $\{i, j\}$ to pair $\{i', j'\}$ if and only if $\{i, j\} \cap \{i', j'\} = \emptyset$. Then $\mathscr{H}$ is an undirected, connected, regular graph of valency 3 isomorphic with the Petersen graph. Note that the girth of $\mathscr{H}$ is 5 and that there are exactly two pentagons on each 2-claw.

Direct construction can be used to show the following.

LEMMA 2.1. *The Petersen graph is the unique (up to isomorphism) undirected, connected graph of valency* 3 *and girth* 5 *having exactly two pentagons on each* 2-*claw.*

Now it is not difficult to show that $G = \mathrm{Aut}\,(\mathscr{H})$, $G_x \cong D_{12}$ and $G_x^{\Delta(x)} \cong \Sigma_3$. Also G has a subgroup $H \cong A_5$ and H is also transitive on Ω. Furthermore, $H_x \cong \Sigma_3$ is faithful on $\Delta(x)$. Note also that G and H are the only subgroups of $\mathrm{Aut}\,(\mathscr{H})$ whose one point stabilizers are 3-transitive on the points adjacent to the given point.

The dodecahedral graph. Consider the group $H = A_5$ acting on the set of ordered pairs $\Omega = \{(i, j): i \neq j,\ i, j \in \{1, 2, 3, 4, 5\}\}$. Then $|\Omega| = 20$. Let $x = (1, 2)$ and $u = (3, 4)$, and define a graph $\mathscr{H}$ on Ω whose edges are $\{(x^g, u^g): g \in H\}$. Then $\mathscr{H}$ is an undirected, connected, regular graph of valency 3 isomorphic with the graph of vertices and edges of a regular dodecahedron, and we shall thus call it the dodecahedral graph. Note that $\mathscr{H}$ has girth 5 and on each 2-claw there is an unique pentagon.

Once again, direct construction can be used to show the following.

LEMMA 2.2. *The dodecahedral graph is the unique (up to isomorphism) undirected, connected graph of valency* 3 *and girth* 5 *having a unique pentagon on each* 2-*claw (i.e. it is the unique pentagraph of valency* 3).

Let s be the following permutation on Ω: $(i, j)^s = (j, i)$. Clearly s is an involution and preserves adjacency in $\mathscr{H}$, so $s \in \mathrm{Aut}\,(\mathscr{H})$. Further, s centralizes H. Now it is not difficult to show that $G = H \times \langle s \rangle = \mathrm{Aut}\,(\mathscr{H})$. Furthermore, $G_x \cong \Sigma_3$ is faithful and 3-transitive on $\Delta(x)$, and in fact G is the only subgroup of $\mathrm{Aut}\,(\mathscr{H})$ whose one point stabilizer is 3-transitive on the points adjacent to the given point.

3. The pentagraph $\mathscr{H}_{31}$. The group $G = PSL(2, 31)$ has a subgroup $H \cong A_5$ of index 248. In the action of G on the set Ω of right cosets of H, there is a unique suborbit for H of length 5 so that the graph $\mathscr{H}_{31}$ constructed with respect to this suborbit has valency 5. Further, since H is maximal in G, this graph is connected. As $G_x \cong A_5$ is 3-transitive on $\Delta(x)$ for any $x \in \Omega$, $\mathscr{H}_{31}$ has no triangles.

Now all elements of order 3 are conjugate in G and for any $x \in \Omega$, an element of order 3 in G_x fixes exactly two points in $\Delta(x)$. An easy counting argument shows that an element of order 3 in G fixes exactly 5 points in Ω, so that these fixed points must lie in a pentagon of $\mathscr{H}_{31}$. So now if $x \in \Omega$, then each 2-

element subset of $\Delta(x)$ (of which there are 10) determines a pentagon, viz. the fixed points of the subgroup of order 3 in G_x which fixes the 2-element subset. Now each of these 10 pentagons determines two points at distance 2 from x. That these 10 pentagons determine 20 distinct points at distance 2 from x can be seen by observing that two elements of order 3 in A_5 (in its action on 5 points) whose fixed points are disjoint must generate A_5; but G_x fixes no vertex of Ω other than x. Thus the girth of $\mathscr{H}_{31}$ is 5.

We now have the following.

LEMMA 3.1. *$\mathscr{H}_{31}$ is a pentagraph of valency 5 on 248 vertices, whose pentagons are the fixed points in Ω of elements of order 3 in G.*

Proof. All that remains to be shown is the uniqueness of pentagons on 2-claws.

Let $x \in \Omega$ and let $\Delta(x) = \{y_1, y_2, y_3, y_4, y_5\}$. Let $\Pi = (x, y_1, z_1, z_2, y_2)$ be the pentagon in $\mathscr{E}$ on $(x:y_1, y_2)$ fixed by $G_{xy_1y_2} = \langle w \rangle$, say, where w has order 3. We claim that Π is the unique pentagon on the 2-claw $(x: y_1, y_2)$. If there is another pentagon on this 2-claw, then, since w fixes exactly two points of $\Delta(y_1)$ (viz. x and z_1), there are in fact three more pentagons on $(x: y_1, y_2)$ other than Π, permuted transitively by w. Furthermore if (x, y_1, z_3, z_4, y_2) is one of these, then $\{z_1, z_2\}$ and $\{z_3, z_4\}$ are disjoint since the girth of $\mathscr{H}_{31}$ is 5.

Since G_x is 3-transitive on $\Delta(x)$, there are thus 4 pentagons on each 2-claw $(x: y_i, y_j)$, $i \neq j$, so that by considering the 4 pentagons on each of the 2-claws $(x: y_1, y_i)$, $i = 2, 3, 4, 5$, it can be seen that G has rank 3 and degree 26 on Ω. No such rank 3 group exists ([**2**]), and in any event $|\Omega| = 248$. So Π is the unique pentagon on $(x: y_1, y_2)$ and $\mathscr{H}_{31}$ is a pentagraph.

LEMMA 3.2. *$\mathscr{H}_{31}$ does not contain a dodecahedral subgraph.*

Proof. Let Γ be a dodecahedral graph which is a subgraph of $\mathscr{H}$. For $y_1, y_2, y_3 \in \Delta(x)$, the subgraph of $\mathscr{H}_{31}$ on the set $\{x, y_1, y_2, y_3\}$ is called a 3-claw, denoted by $(x: y_1, y_2, y_3)$. Observe that since G_x is 3-transitive on $\Delta(x)$ and G is transitive on Ω, G is transitive on the 3-claws of $\mathscr{H}_{31}$. Observe also that Γ is uniquely determined by any one of its 3-claws. So if C_1 and C_2 are any two 3-claws of Γ there is an element $f \in G$ such that $C_1{}^f = C_2$, whence $\Gamma^f = \Gamma$.

Thus G_Γ is transitive on Γ and for $x \in \Gamma$, $(G_\Gamma)_x$ is 3-transitive on $\Delta(x) \cap \Gamma$. Hence $G_\Gamma{}^\Gamma \cong A_5 \times Z_2$. But a Sylow 2-subgroup of $PSL(2, 31)$ does not contain a section isomorphic with $Z_2 \times Z_2 \times Z_2$ (which is isomorphic with a Sylow 2-subgroup of $G_\Gamma{}^\Gamma$), and so we have a contradiction.

THEOREM. 3.1. Aut $(\mathscr{H}_{31}) = G = PSL(2, 31)$.

Proof. Let $F = $ Aut $(\mathscr{H}_{31})$. Since $G \leqq F$, we have that for $x \in \Omega$, $F_x{}^{\Delta(x)}$ is isomorphic with a subgroup of Σ_5 containing A_5. So suppose $F_x{}^{\Delta(x)} \cong \Sigma_5$.

Let $s \in F_x{}^{\Delta(x)}$ be an involution fixing 3 points of $\Delta(x)$. It is fairly easy to show that the connected component of $\mathscr{H}_{31}$ on $\Omega(s)$ which contains x is itself a pentagraph of valency 3, whence by Lemma 2.2 it is a dodecahedral graph. This contradicts Lemma 3.2.

Hence $F_x^{\Delta(x)} \cong A_5$, but as shall be shown in Lemma 4.2, F_x is faithful on $\Delta(x)$. Thus $F_x \cong A_5$, whence $G = F$.

Remark. Although there are two conjugacy classes of subgroups of $PSL(2, 31)$ isomorphic with A_5, these are all conjugate in $PGL(2, 31)$, so that the graph $\mathscr{H}_{31}$ is independent of the choice of the subgroup $H \cong A_5$.

4. Pentagraphs of valency 5. We first prove the following general result about m-gon-graphs with m odd.

LEMMA 4.1. *Let* $(\mathscr{H}, \mathscr{E})$ *be an m-gon-graph with m odd, and* $G \leqq \mathrm{Aut}\ (\mathscr{H})$ *such that for any vertex x of* $\mathscr{H}$, G_x *is transitive on* $\Delta(x)$. *Then G is transitive on the vertices of* $\mathscr{H}$.

Proof. Given any two points x, y on an m-gon Π with y at distance 2 from x there is a unique point z on Π with $(z: x, y)$ a 2-claw of Π. Since G_z is transitive on $\Delta(z)$, we can move x to y, fixing z. In the same way we can move around Π two "steps" at a time, and so since m is odd, G is transitive on Π. Hence, by connectivity of $\mathscr{H}$, G is transitive on the vertices of $\mathscr{H}$.

LEMMA 4.2. *Let* $(\mathscr{H}, \mathscr{E})$ *be an m-gon-graph with* $G \leqq \mathrm{Aut}\ (\mathscr{H})$ *and suppose that for every 2-claw* $(x: y, z)$, $|G_{xyz}|$ *is independent of the 2-claw* $(x: y, z)$, *and* G_{xyz} *fixes the m-gon in* $\mathscr{E}$ *on* $(x: y, z)$. *Then* G_x *is faithful on* $\Delta(x)$.

Proof. Suppose $g \in G_x$ fixes $\Delta(x)$ pointwise. For $i \geqq 1$, let $\Delta_i(x) = \{y \in \Omega: y \text{ is at distance } i \text{ from } x\}$, and $\Delta_0(x) = \{x\}$. Assume g fixes $\Delta_i(x)$ pointwise for all $i < n$, $n \geqq 2$. We show g fixes $\Delta_n(x)$, whence by induction g fixes all the vertices of $\mathscr{H}$ (since $\mathscr{H}$ is connected), which implies that $g = 1$.

Let $y \in \Delta_n(x)$. Take $u \in \Delta_{n-1}(x)$, $v \in \Delta_{n-2}(x)$ with $v \sim u \sim y$. Let $\Pi = (y, u, v, w, \ldots)$ be the m-gon in $\mathscr{E}$ on $(u: v, y)$. Since Π is fixed by G_{uvy}, we have $G_{uvy} \leqq G_{uvw}$ and so by the hypotheses $G_{uvy} = G_{uvw}$. Now by the inductive hypothesis, $g \in G_{uvw}$ because $w \in \Delta_i(x)$ for some $i \leqq n - 1$. So g fixes Π, whence g fixes y.

Since y was arbitrary in $\Delta_n(x)$, this completes the proof.

LEMMA. 4.3. *Let* $(\mathscr{H}, \mathscr{E})$ *be a 5-gon-graph of valency* 5, *and* $G \leqq \mathrm{Aut}\ (\mathscr{H})$, *and suppose that for any 2-claw* $(x: y, z)$, G_{xyz} *fixes the pentagon in* $\mathscr{E}$ *on* $(x: y, z)$. *For all vertices x*,

(i) $G_x^{\Delta(x)}$ *is 3-transitive on* $\Delta(x)$ *if and only if* $G_x \cong A_5$, *and*
(ii) *if* $G_x^{\Delta(x)} \cong A_5$, *then* $\mathscr{H}$ *is a pentagraph.*

Proof. (i) Clearly $G_x \cong A_5$ implies G_x is 3-transitive on $\Delta(x)$.

Conversely, suppose $G_x^{\Delta(x)}$ is 3-transitive on $\Delta(x)$. Then by Lemmas 4.1 and 4.2, G_x is faithful on $\Delta(x)$, so that $G_x \cong A_5$ or Σ_5. So suppose $G_x \cong \Sigma_5$. Then involutions in G_x fix either one or three points of $\Delta(x)$, and there is an involution

$t \in G_x$ such that t fixes the three points $u, v, w \in \Delta(x)$ and interchanges the remaining two points $y, z \in \Delta(x)$. We now have two possible cases, viz., that t fixes more than one pentagon on $(x: u, v)$, or t fixes exactly one pentagon on $(x: u, v)$. Since the girth of $\mathscr{H}$ is 5, we can see quite easily in the former case that t fixes exactly two pentagons on $(x: u, v)$, so that if we let Γ be that connected component of the induced subgraph of $\mathscr{H}$ on $\Omega(t)$ which contains x, Γ is isomorphic to the Petersen graph (Lemma 2.1). In the latter case, Γ is a graph of valency 3 and girth 5 with a unique pentagon on each 2-claw, so that by Lemma 2.2, Γ is isomorphic to a dodecahedral graph.

Since $C_{G_x}(t)$ is isomorphic with a dihedral group of order 12, and $\langle t \rangle = G_{xuvw}$, $(C_G(t)^\Gamma)_x$ is isomorphic with Σ_3 and is 3-transitive on $\Omega(t) \cap \Delta(x)$. Similarly, $(C_G(t)^\Gamma)_{u'}$ is 3-transitive on $\Omega(t) \cap \Delta(u')$ for any $u' \in \Gamma$, so by Lemma 4.1, $C_G(t)^\Gamma$ is transitive on the vertices of Γ.

Now in the case of a Petersen graph, the stabilizer of a point in the full automorphism group is isomorphic with D_{12}, so that if Γ is a Petersen graph, $C_G(t)^\Gamma$ is isomorphic with A_5, while if Γ is a dodecahedral graph, $C_G(t)^\Gamma \cong A_5 \times Z_2$.

Let $\Pi = (x, y, y', z', z)$ be the pentagon in $\mathscr{E}$ on $(x: y, z)$, so that because G_{xyz} fixes Π, and t normalizes G_{xyz}, t interchanges y' and z'. Hence t normalizes $G_{y'z'} \cong \Sigma_4$. Note that $C_{G_x}(t)$ contains a subgroup H isomorphic with Σ_3 which fixes y and z, and also y' and z', and so $H \leqq G_{y'z'}$. Thus t does not act as an inner automorphism of $G_{y'z'}$. Suppose $G_{y'z'} \leqq C_G(t)$. Then $t \notin G_{y'z'}$ implies that

$$\Sigma_4 \cong G_{y'z'}{}^\Gamma \leqq C_G(t)^\Gamma.$$

But neither A_5 nor $A_5 \times Z_2$ contains a subgroup isomorphic with Σ_4. Thus $G_{y'z'} \nleqq C_G(t)$.

Thus t must be an outer automorphism of $G_{y'z'}$. No such automorphism, however, exists ([**3**], Satz II.5.5). This contradiction proves (i).

(ii) By Lemma 4.1, G is transitive on the vertices Ω of $\mathscr{H}$. We claim that G_x is transitive on $\Delta_2(x)$ for any $x \in \Omega$.

Clearly it suffices to show that if $y \in \Delta(x)$ and $z_1, z_2 \in \Delta(y)$, then there is a $g \in G_x$ with $z_1{}^g = z_2$. Since $G_y{}^{\Delta(y)} \cong A_5$ is (at least) 2-transitive on $\Delta(y)$, there is a $g \in G_y$ such that g fixes x and $z_1{}^g = z_2$, which is what we wanted to show.

Now in the same way as was done in the proof of Lemma 3.1, we can show the uniqueness of pentagons on 2-claws of $\mathscr{H}$. Thus $\mathscr{H}$ is a pentagraph.

COROLLARY 4.1. *Let* $(\mathscr{H}, \mathscr{E})$ *be a 5-gon-graph of valency 5 and* $G \leqq \operatorname{Aut}(\mathscr{H})$ *with* $G_x{}^{\Delta(x)}$ *3-transitive for all vertices x. Then for any 2-claw* $(x: y, z)$, G_{xyz} *fixes the pentagon in* $\mathscr{E}$ *on* $(x: y, z)$ *if and only if* $\mathscr{H}$ *is a pentagraph.*

Proof. Use Lemma 4.3.

Remark. Lemma 4.3 provides another proof of Theorem 3.1, namely, that $\operatorname{Aut}(\mathscr{H}_{31}) = PSL(2, 31)$.

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Proof of Theorem 1.2. From now on assume that $\mathscr{H}$ is a pentagraph of valency 5, and $G \leqq \mathrm{Aut}(\mathscr{H})$ with $G_x^{\Delta(x)} \cong A_5$ for all x in the set Ω of vertices of $\mathscr{H}$, so by Lemma 4.1, G is transitive on Ω. Pick a point $0 \in \Omega$ and let $\Delta(0) = \{1, 2, 3, 4, 5\}$. In its action on $\Delta(0)$, we represent $H = G_0$ by its usual representation as a permutation group of degree 5. So let $t = (1\,2)(3\,4)$, $z = (1\,2\,3)$ and $h = (1\,2)(4\,5)$, where $t, z, h \in H$. Then $G_{0,5} = K = \langle t, z\rangle \cong A_4$ and $H = \langle K, h\rangle \cong A_5$.

We first note the following.

LEMMA 4.4. *H is transitive on $\Delta_2(0)$; in fact there is a one to one correspondence between the points of $\Delta_2(0)$ and the set of ordered pairs of unequal points of $\Delta(0)$ preserving the action of H.*

Proof. Since there are no rectangles in $\mathscr{H}$, given any point $u \in \Delta_2(0)$ there is a unique point $i \in \Delta(0)$ with $(i\colon 0, u)$ a 2-claw. By uniqueness of pentagons on 2-claws, there is a $v \in \Delta_2(0)$ and a $j \neq i$, $j \in \Delta(0)$, with $(0, i, u, v, j)$ the unique pentagon on any of its 2-claws. Let u correspond to the ordered pair (i, j) and v to the pair (j, i). This defines a map from $\Delta_2(0)$ to the set of pairs (i, j), $i, j \in \Delta(0)$, $i \neq j$. Again by uniqueness of pentagons on 2-claws of the form $(0\colon i, j)$, we see that this map is in fact one to one, hence onto, and is thus a one to one correspondence.

Now if $g \in H$, then the pentagon on the 2-claw $(0\colon i, j)$ clearly gets mapped onto the pentagon on $(0\colon i^g, j^g)$ by g, so that the correspondence preserves the action of H. This proves the lemma.

From now on we will denote the point in $\Delta_2(0)$ which corresponds to the pair (i, j), $i, j \in \Delta(0)$, by the symbol ij, so that $(ij)^k = i^k j^k$ for $k \in H$.

LEMMA 4.5. *For $ij \in \Delta_2(0)$, $|\Delta(ij) \cap \Delta_2(0)| = 1$.*

Proof. We have $ji \in \Delta(ij) \cap \Delta_2(0)$. If $i'j' \in \Delta(ij) \cap \Delta_2(0)$ with $i'j' \neq ij$, then there will be two pentagons on the 2-claw $(i\colon 0, ij)$, viz. $(0, i, ij, ji, j)$ and $(0, i, ij, i'j', i')$, a contradiction.

LEMMA 4.6. *There is an involution $g \in G$ which interchanges the points 0 and 5 (and hence normalizes K).*

Proof. Since G is transitive on 3-claws of $\mathscr{H}$, there is a $g \in G$ with $(0\colon 5, 4, 3)^g = (5\colon 0, 54, 53)$ such that $0^g = 5$, $5^g = 0$, $4^g = 54$ and $3^g = 53$. But then it can readily be seen, by uniqueness of pentagons on 2-claws, that $54^g = 4$, $53^g = 3$ and g fixes 45 and 35.

Thus $g^2 \in H_{5,4,3} = \{1\}$, completing the proof of Lemma 4.6.

Since $\mathscr{H}$ is connected, $G = \langle H, g\rangle$, g as in Lemma 4.6. $L = \langle z\rangle$ fixes the pentagon $(0, 5, 54, 45, 4)$ pointwise, so we have

$$L = \langle z\rangle = G_{4,0,5} = G_{0,5,54} = (G_{4,0,5})^g = L^g,$$

so that $z^g = z$ or $z^g = z^{-1}$. Also

$$0^{gh} = 4,\ 0^{(gh)^2} = 45,\ 0^{(gh)^3} = 54,\ 0^{(gh)^4} = 5 \text{ and } 0^{(gh)^5} = 0,$$

so that $(gh)^5 \in H$ but $(gh)^i \notin H$ for $1 \leqq i \leqq 4$. Hence $(gh)^5 \in N_H(L) = L\langle h\rangle$. In fact, $(gh)^5 \in C_H(L) = \langle z\rangle$, for if $(gh)^5 = h$, zh or $z^{-1}h$, then $g^{(hg)^2} = 1$, z or z^{-1}; but $g^{(hg)^2}$ is an involution.

Suppose $g \in C_G(L)$. Then since h inverts z we have that $g^{(hg)^2}$ is an involution in $C_H(L) = L$, a contradiction. Thus $g \notin C_G(L)$. We have by Lemma 4.6 that $g \in N_G(K)$, and so g does not centralize K. If g acts as an inner automorphism of K, then since $Z(K) = 1$, there is an involution $k \in K$ such that $gk \in C_G(K)$. If $k = (1\ 2)(3\ 4)$, then $gk = (0\ 5)(4\ 53)\ldots$ which does not centralize z. If $k = (1\ 3)(2\ 4)$, then $gk = (0\ 5)(4\ 52\ldots)\ldots$ which also does not centralize z. Similarly, if $k = (1\ 4)(2\ 3)$, gk does not centralize z. Thus g is not inner on K.

Hence $G_{\{0,5\}} = \langle K, g\rangle \cong \Sigma_4$. Now as a product of disjoint cycles,

$$\begin{aligned} g &= (0\ 5)(4\ 54)(3\ 53)(45)(35)\ldots, \\ z &= (0)(5)(4)(54)(1\ 2\ 3)(51\ 52\ 53)\ldots, \text{ and} \\ t &= (0)(5)(1\ 2)(3\ 4)(53\ 54)(51\ 52)\ldots. \end{aligned}$$

So $z^g = (0)(5)(4)(54)(1^g\ 2^g\ 53)(51^g\ 52^g\ 3)\ldots$, and since $1^g, 2^g \in \Delta(0^g) = \Delta(5) = \{0, 51, 52, 53, 54\}$, and $g \notin C_G(z)$, we have that g contains the cycles $(1\ 52)$ and $(2\ 51)$. Hence $z^g = z^{-1}$ and $t^g = t$. (We can think of g as the transposition $(1\ 2)$ in its action on K (but not on the rest of H).)

Now for $\epsilon = 0$, 1 or -1, let $\mathbf{G}(\epsilon)$ be the abstract group defined in terms of generators and relations as follows:

$$\begin{aligned} \mathbf{G}(\epsilon) = \langle \mathbf{g}, \mathbf{h}, \mathbf{t}, \mathbf{z} : \mathbf{g}^2 = \mathbf{h}^2 = \mathbf{t}^2 = \mathbf{z}^3 = (\mathbf{gt})^2 = (\mathbf{gz})^2 = (\mathbf{hz})^2 \\ = (\mathbf{tz})^3 = (\mathbf{th})^3 = (\mathbf{gh})^5\mathbf{z}^{-\epsilon} = \mathbf{1}\rangle. \end{aligned}$$

Then, $\mathbf{K}(\epsilon) = \langle \mathbf{t}, \mathbf{z}\rangle \cong A_4$; since $|\mathbf{gtz}| = 4$ we have $\langle \mathbf{K}(\epsilon), \mathbf{g}\rangle \cong \Sigma_4$; and since $|\mathbf{htz}| = 5$ we have $\mathbf{H}(\epsilon) = \langle \mathbf{K}(\epsilon), \mathbf{h}\rangle = \langle \mathbf{h}, \mathbf{t}, \mathbf{z}\rangle \cong A_5$. Also

$$\begin{aligned} \mathbf{z}^3 = (\mathbf{gz})^2 = (\mathbf{hz})^2 = (\mathbf{tz})^3 = \mathbf{1} \Leftrightarrow (\mathbf{z}^{-1})^3 = (\mathbf{gz}^{-1})^2 = (\mathbf{hz}^{-1})^2 \\ = (\mathbf{tz}^{-1})^3 = \mathbf{1}, \end{aligned}$$

so that $\mathbf{G}(1) \cong \mathbf{G}(-1)$.

What we have shown in the previous paragraphs is that under the identification ϕ which maps $\mathbf{g} \to g$, $\mathbf{h} \to h$, $\mathbf{t} \to t$, and $\mathbf{z} \to z$, G is a quotient group of $\mathbf{G}(\epsilon)$, for $\epsilon = 0$ or 1. Further, $\mathbf{G}(\epsilon)$ (for the correct ϵ) acts on $\mathscr{H}$ via ϕ. Let $\mathbf{N}$ be the kernel of $\phi : \mathbf{G}(\epsilon) \to G$.

Consider first $\epsilon = 0$. $\mathbf{G}(0)$ contains the subgroup $\mathbf{R} = \langle \mathbf{g}, \mathbf{h}, \mathbf{t}\rangle$ which can be seen to be a homomorphic image of the finite reflection group

$$\mathbf{S} = \langle \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 : \mathbf{s}_1{}^2 = \mathbf{s}_2{}^2 = \mathbf{s}_3{}^2 = (\mathbf{s}_1\mathbf{s}_2)^5 = (\mathbf{s}_1\mathbf{s}_3)^2 = (\mathbf{s}_2\mathbf{s}_3)^3 = \mathbf{1}\rangle$$

which is the group of the dodecahedron (and hence of the dodecahedral graph) isomorphic with $A_5 \times Z_2$ (see Section 2). This reflection group has two non-trivial normal subgroups, viz. the center $\langle (\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1)^5 \rangle$ of order two whose image is $\mathbf{N}_1 = \langle (\mathbf{thg})^5 \rangle \leqq \mathbf{R}$, and $\langle \mathbf{s}_1\mathbf{s}_2, \mathbf{s}_2\mathbf{s}_3 \rangle \cong A_5$ whose image is $\mathbf{N}_2 = \langle \mathbf{gh}, \mathbf{ht} \rangle \leqq \mathbf{R}$. Under the induced action of $\mathbf{G}(0)$, $\mathbf{ht}$ fixes 0 while $\mathbf{gh}$ does not, so that $\phi(\mathbf{N}_2) \neq \{1\}$. Now it is easy to check that $0^{(thg)^3} = 43^g$, while $0^{(ght)^2} = 34$, so that if $(thg)^5 = 1$ in G then $43^g = 34$. But then by considering the image of the pentagon $(0, 3, 34, 43, 4)$ under g we will have that 53 is adjacent to 43, which contradicts Lemma 4.5 as 53 is adjacent to 35. Thus $\phi(\mathbf{N}_1) \neq \{1\}$. Hence $\mathbf{R} \cong \mathbf{S}$ and $\mathbf{R} \cap \mathbf{N} = \{\mathbf{1}\}$.

Let Λ be the orbit of $\mathbf{R}$ in Ω containing the point 0. It can be seen that, since $\mathbf{g} \in \mathbf{R}$, $\Lambda \cap \Delta(0) = \{3, 4, 5\}$ or $\Lambda \cap \Delta(0) = \Delta(0)$. But, since $\mathscr{H}$ contains no rectangles, any element of $\mathbf{R}$ taking 2 to 3 (or 1 to 3) must fix 0 and therefore be in $\mathbf{H} = \mathbf{H}(0)$. Now $\langle \mathbf{h}, \mathbf{t} \rangle \leqq \mathbf{H} \cap \mathbf{R} \leqq \mathbf{H}$ and since $\langle \mathbf{h}, \mathbf{t} \rangle$ is maximal in $\mathbf{H}$ and does not contain any element taking 2 to 3, or 1 to 3, if $\Lambda \cap \Delta(0) = \Delta(0)$ then $\mathbf{H} \cap \mathbf{R} = \mathbf{H} \cong A_5$. But then $\mathbf{H} = \mathbf{N}_2$. However $\mathbf{gh} \in \mathbf{N}_2$ does not fix 0, so this is not possible.

Thus $\Sigma_3 \cong \langle \mathbf{h}, \mathbf{t} \rangle = \mathbf{H} \cap \mathbf{R}$, $\Lambda \cap \Delta(0) = \{3, 4, 5\}$ and $|\Lambda| = |\mathbf{R}|/|\mathbf{R} \cap \mathbf{H}| = 20$. Hence we also see that for any $x \in \Lambda$, Λ contains exactly three points adjacent to x. Further as a product of disjoint cycles, $\mathbf{gh} = (0\ 4\ 45\ 54\ 5) \ldots$ so that Λ contains the vertices of the (unique) pentagon on the 2-claw $(0{:}\ 4, 5)$, and hence the vertices of the pentagon on any 2-claw $(x{:}\ y, z)$ for which $x, y, z \in \Lambda$. Thus any connected component of the subgraph of $\mathscr{H}$ induced by the vertices of Λ has valency 3 and has a unique pentagon on any of its 2-claws. Thus Λ induces a dodecahedral graph in $\mathscr{H}$ by Lemma 2.2.

We have thus proved that if $\epsilon = 0$, then $\mathscr{H}$ contains a dodecahedral subgraph.

Now consider $\epsilon = 1$. Using a coset enumeration program of R. Scott [**5**], the index of $\mathbf{H}(1) \cong A_5$ in $\mathbf{G}(1)$ is 248 (equal to the number of vertices of the graph $\mathscr{H}_{31}$ of Section 3). Thus $|\mathbf{G}(1)| = |PSL(2, 31)|$.

Consider the following matrices over $GF(31)$, which for convenience are also called $\mathbf{g}$, $\mathbf{h}$, $\mathbf{t}$, and $\mathbf{z}$.

$$\mathbf{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} 0 & 10 \\ 3 & 0 \end{pmatrix}$$

$$\mathbf{t} = \begin{pmatrix} -14 & 12 \\ 12 & 14 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 5 & 0 \\ 0 & -6 \end{pmatrix}.$$

It is easily checked that $\left(\text{modulo } \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle\right)$ these matrices satisfy the relations for $\mathbf{G}(1)$. Further, it can be checked that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = [\mathbf{z}^{\mathbf{tz}^{-1}(\mathbf{hg})^2\mathbf{tz}}]\mathbf{g}, \quad \text{modulo } \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle,$$

and so by ([**1**], Section 7.5), these matrices generate $PSL(2, 31)$ over the field of 31 elements. Hence $\mathbf{G}(1) \cong PSL(2, 31)$. Then clearly too, $\mathscr{H}$ is isomorphic with the graph $\mathscr{H}_{31}$ of Section 3.

We have thus proved Theorem 1.2.

Proof of Theorem 1.3. For (i) implies (ii), see Lemma 3.2. For (ii) implies (i), see Lemma 4.1, Theorem 1.1, Lemma 4.3 and Theorem 1.2.

Remark. We have that as an abstract group $PSL(2, 31)$ has generators and relations given by $\mathbf{G}(1)$. Another set of generators and relations for $PSL(2, 31)$ which can be derived from this is

$$PSL(2, 31) = \langle \mathbf{A}, \mathbf{B}, \mathbf{C}: \mathbf{A}^2 = \mathbf{B}^3 = (\mathbf{AB})^5 = \mathbf{C}^2 = (\mathbf{CB})^4 = (\mathbf{AC})^5\mathbf{BCBCB}^{-1} = \mathbf{1} \rangle,$$

where $\langle \mathbf{A}, \mathbf{B} \rangle \cong A_5$, $\langle \mathbf{B}, \mathbf{C} \rangle \cong \Sigma_4$, by letting $\mathbf{C} = \mathbf{g}$, $\mathbf{A} = \mathbf{h}$, and $\mathbf{B} = \mathbf{z}^{\mathbf{tzt}}$. Then $\mathbf{z} = (\mathbf{B}^{-1})^{\mathbf{CB}^{-1}}$ and $\mathbf{t} = \mathbf{A}^{\mathbf{BAB}}$. (As an element of $\langle \mathbf{A}, \mathbf{B} \rangle$ and $\langle \mathbf{B}, \mathbf{C} \rangle$, $\mathbf{B}$ acts like $(1\,3\,4)$.)

Finally, a remark about $\mathbf{G}(0)$. It is not known whether or not $\mathbf{G}(0)$ is finite. By letting $\mathbf{R}_1 = \mathbf{g}, \mathbf{R}_2 = \mathbf{gz}, \mathbf{R}_3 = \mathbf{h}$ and $\mathbf{R}_4 = \mathbf{t}$, we see that $\mathbf{G}(0)$ is a quotient of the infinite Coxeter group $\langle \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4: \mathbf{r}_i^2 = (\mathbf{r}_1\mathbf{r}_2)^3 = (\mathbf{r}_1\mathbf{r}_3)^5 = (\mathbf{r}_1\mathbf{r}_4)^2 = (\mathbf{r}_2\mathbf{r}_3)^{15} = (\mathbf{r}_2\mathbf{r}_4)^4 = (\mathbf{r}_3\mathbf{r}_4)^3 = \mathbf{1},\ i = 1, 2, 3, 4 \rangle$ under the obvious identification; however in $\mathbf{G}(0)$ we also have, for example, $(\mathbf{R}_2\mathbf{R}_3)^3 = (\mathbf{R}_1\mathbf{R}_3)^3$. If $\mathbf{G}(0)$ were finite, then it would lead to an example of a pentagraph of valency 5 with dodecahedral subgraphs. However we have shown the following.

COROLLARY 4.2. *With the hypotheses of Theorem* 1.2, *if* $\mathscr{H}$ *contains a dodecahedral subgraph, then* $G \cong \mathbf{G}(0)/\mathbf{N}$, *where* $\mathbf{N}$ *is a normal subgroup of* $\mathbf{G}(0)$ *intersecting* $\mathbf{R} = \langle \mathbf{g}, \mathbf{h}, \mathbf{t} \rangle$ *trivially.*

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