3
One-Dimensional Rectifiable Sets

Almost all of the theory of one-dimensional rectifiable sets in the plane was developed by Besicovitch in the three papers [62], [63] and [64]. Some generalizations are due to Morse and Randolph [352] and Moore [350]. For most of the proofs of the results in this chapter, see [158, 190]. Several ideas extend to higher dimensions, and in the next chapter we shall mainly discuss the new ideas needed. We consider here only subsets of the plane although essentially everything holds for one-dimensional rectifiable subsets of $\mathbb{R}^n$.

3.1 Definitions and Tangents

One-dimensional rectifiable sets generalize rectifiable curves and they have many properties that make this class flexible in a measure-theoretic sense and useful in many applications. In particular,

- Every rectifiable curve is a rectifiable set.
- Countable unions of rectifiable sets are rectifiable.
- Subsets of rectifiable sets are rectifiable.
- Sets of zero $\mathcal{H}^1$ measure are rectifiable.
- If for every $\varepsilon > 0$ there is a rectifiable set $F \subset E$ with $\mathcal{H}^1(E \setminus F) < \varepsilon$, then E is rectifiable.

These properties offer an immediate definition: E is 1-rectifiable if $\mathcal{H}^1$ almost all of it can be covered with rectifiable curves. We state it a bit differently:

Definition 3.1 A set $E \subset \mathbb{R}^2$ is 1-*rectifiable* if there are Lipschitz maps $f_i : \mathbb{R} \to \mathbb{R}^2, i = 1, 2, \ldots$ such that

$$\mathcal{H}^1\left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R})\right) = 0.$$

Next we shall see that rectifiable sets have, in an appropriate sense, the properties of rectifiable curves which we presented in Chapter 2. This is actually rather easy, but more essentially only rectifiable sets have these properties. That is, each of the properties allows a converse statement.

We shall begin with tangents. Consider the following example. Let q_i, $i = 1, 2, \ldots$ be all the points of the plane with rational coordinates and set $E = \bigcup_{i=1}^{\infty} S_i$ with $S_i = \{x \in \mathbb{R}^2 : |x - q_i| = 2^{-i}\}$. Then E is 1-rectifiable and $\mathcal{H}^1(E) < \infty$. But it is dense in $\mathbb{R}^2$. So how can it have any tangents? Obviously it cannot in the ordinary sense and we have to give a measure-theoretic definition of a tangent. For $a \in \mathbb{R}^2$, $s > 0$, $L \in G(2,1)$, define the cone (angular sector)

$$X(a, L, s) = \left\{x \in \mathbb{R}^2 : d(x, L + a) < s|x - a|\right\}.$$

Definition 3.2 A line $L \in G(2,1)$ is an *approximate tangent line* of a set $E \subset \mathbb{R}^2$ at a point $a \in \mathbb{R}^2$ if $\Theta^{*1}(E, a) > 0$ and for every $s > 0$,

$$\lim_{r \to 0} r^{-1} \mathcal{H}^1\left(E \cap B(a, r) \setminus X(a, L, s)\right) = 0.$$

It is convenient to define the approximate tangents as lines through the origin. Then the geometric approximate tangent at a is the translate by a.

Now we immediately have that the above union E of the circles S_i has an approximate tangent at almost all of its points: each S_i has an ordinary tangent at all of its points and at almost all points it is an approximate tangent of E by the density theorem, Theorem 1.1. Since rectifiable curves have tangents almost everywhere, essentially the same argument shows that every 1-rectifiable set with finite $\mathcal{H}^1$ measure has an approximate tangent at almost all of its points.

The converse is not very difficult either. I give the idea assuming that E has ordinary tangents almost everywhere. Then we can write E as a countable union of a set of measure zero and sets F for which there exist $s > 0$ and $L \in G(2,1)$ such that $F \setminus X(a, L, s) = \emptyset$ for $a \in F$. The set F is such that the tangents in its points are close to a fixed line L. This implies that the restriction of the projection P_L to F is one-to-one with a Lipschitz inverse. Hence $F = (P_L|F)^{-1}(P_L(F))$ is rectifiable.

The case of approximate tangents causes technical difficulties, but the main idea is the same. So we have

Theorem 3.3 *If E is $\mathcal{H}^1$ measurable and $\mathcal{H}^1(E) < \infty$, then E is 1-rectifiable if and only if it has an approximate tangent line at almost all of its points.*

Definition 3.4 A set $E \subset \mathbb{R}^2$ is *purely 1-unrectifiable* if $\mathcal{H}^1(E \cap F) = 0$ for every 1-rectifiable set $F \subset \mathbb{R}^2$ (or, equivalently, $\mathcal{H}^1(E \cap C) = 0$ for every rectifiable curve C).

The proof of the following proposition is an easy exercise.

Proposition 3.5 *If E is $\mathcal{H}^1$ measurable and $\mathcal{H}^1(E) < \infty$, then $E = R \cup P$ where R is 1-rectifiable and P is purely 1-unrectifiable.*

Example 3.6 A standard example of a purely 1-unrectifiable compact subset of the plane is the self-similar Cantor set C obtained by choosing the squares of side-length 1/4 in the corners of the unit square and continuing indefinitely. That is, $C = C_1 \times C_1$ where C_1 is a symmetric Cantor set on the line of Hausdorff dimension 1/2. Then $0 < \mathcal{H}^1(C) < \infty$. It is easy to check that C has no approximate tangents.

Obviously all properties of rectifiable sets correspond to properties of purely unrectifiable sets, and vice versa. In particular, E is purely unrectifiable if and only if it has no approximate tangent at almost all of its points. But looking at the above argument we can say more. We did not need large sectors $B(a,r) \setminus X(a,L,s)$ in the complement of F; arbitrarily narrow ones were enough. This leads to, although not quite trivially, [190], Theorem 3.29, or [321], Corollary 15.16,

Theorem 3.7 *If E is $\mathcal{H}^1$ measurable and purely 1-unrectifiable and $\mathcal{H}^1(E) < \infty$, then for every $L \in G(2,1)$ and every $s > 0$,*

$$\limsup_{r\to 0} r^{-1}\mathcal{H}^1(E \cap B(a,r) \cap X(a,L,s)) \gtrsim s$$

for $\mathcal{H}^1$ almost all $a \in E$.

3.2 Densities

Next we look at density properties of rectifiable sets. We again get easily from the area formula for Lipschitz maps that rectifiable sets have density 1 almost everywhere. Also the converse holds:

Theorem 3.8 *If E is $\mathcal{H}^1$ measurable and $\mathcal{H}^1(E) < \infty$, then E is 1-rectifiable if and only if $\Theta^1(E,x) = 1$ for $\mathcal{H}^1$ almost all $x \in E$.*

This is where Besicovitch used connectivity and Theorem 2.1 essentially. He did it in terms of circle pair regions

$$U(x,y) = U(x,|x-y|) \cap U(y,|x-y|),\ \ x,y \in \mathbb{R}^2.$$

To see how they appear, notice that if $C \subset \mathbb{R}^2$ is compact and $U(x,y) \cap C \neq \emptyset$ for $x,y \in C$, then C is connected. If not, C is the union of two disjoint compact sets C_1 and C_2. Let $x_i \in C_i$ with $|x_1 - x_2| = d(C_1, C_2)$. Then $C \cap U(x_1, x_2) = \emptyset$, which contradicts our assumption.

Besicovitch took the density 1 property as the starting point and defined a set to be regular if the conclusion of Theorem 3.8 holds. The term 'rectifiable' was introduced by Federer in [199], although his terminology too is somewhat different from mine, see [203, 3.2.14].

With some more work one can then prove, see [190, Lemma 3.22],

Lemma 3.9 *Let E be $\mathcal{H}^1$ measurable and $\mathcal{H}^1(E) < \infty$. Suppose that $\alpha > 0$ and $F \subset E$ is compact for which $\mathcal{H}^1(F) > 0$ and $\mathcal{H}^1(E \cap U(x,y)) > \alpha|x-y|$ for $x, y \in F$. Then there is a continuum C such that $0 < \mathcal{H}^1(C \cap E) \leq \mathcal{H}^1(C) < \infty$. In particular, E is not purely 1-unrectifiable.*

Theorem 3.8 follows from

Theorem 3.10 *If E is $\mathcal{H}^1$ measurable, $\mathcal{H}^1(E) < \infty$ and $\Theta^1_*(E,x) > 3/4$ for $\mathcal{H}^1$ almost all $x \in E$, then E is 1-rectifiable.*

Once we have Lemma 3.9 available, the proof of Theorem 3.10 is simple: if it is false, we can find, using also Theorem 1.2, a compact subset F of E with $\mathcal{H}^1(F) > 0$ and positive numbers α and r_0 such that

$$\mathcal{H}^1(E \cap U(x,r)) > \left(\tfrac{3}{4} + \alpha\right) 2r \text{ for } x \in F, 0 < r < r_0,$$

and

$$\mathcal{H}^1(E \cap B) \leq (1+\alpha)d(B) \text{ whenever } F \cap B \neq \emptyset \text{ and } d(B) < r_0.$$

If $d(F) < r_0/3$, then for $x, y \in F$ with $r = |x - y|$,

$$\begin{aligned}
&\mathcal{H}^1(E \cap U(x,y)) \\
&= \mathcal{H}^1(E \cap U(x,r)) + \mathcal{H}^1(E \cap U(y,r)) - \mathcal{H}^1(E \cap (U(x,r) \cup U(y,r))) \\
&> 2 \cdot \left(\tfrac{3}{4} + \alpha\right) 2r - (1+\alpha)3r = \alpha r = \alpha|x-y|.
\end{aligned}$$

This shows by Lemma 3.9 that E is not purely unrectifiable, which is enough.

The bound 3/4 is not sharp; Preiss and Tiser [384] improved it by small $\varepsilon > 0$. They did it in general metric spaces, and we shall return to this in Chapter 7. The best bound is not known but it is expected to be 1/2. This is the famous Besicovitch 1/2-problem:

Conjecture 3.11 *If E is $\mathcal{H}^1$ measurable, $\mathcal{H}^1(E) < \infty$ and E is purely 1-unrectifiable, then $\Theta^1_*(E,x) \leq 1/2$ for $\mathcal{H}^1$ almost all $x \in E$.*

Partial results, with some extra conditions on E, were obtained by Farag in [192] and [194].

Besicovitch used circle pairs to prove a stronger form of Theorem 3.8: rectifiability follows from the existence of density. So we have

Theorem 3.12 *Let E be $\mathcal{H}^1$ measurable with $\mathcal{H}^1(E) < \infty$. Then the following are equivalent:*

(1) *E is 1-rectifiable.*
(2) $\Theta^1(E, x) = 1$ *for $\mathcal{H}^1$ almost all $x \in E$.*
(3) $\Theta^1(E, x)$ *exists for $\mathcal{H}^1$ almost all $x \in E$.*

3.3 Projections

When discussing tangents, we saw that rectifiable sets have projections of positive length on tangent lines. A closer look at that argument shows that if E is 1-rectifiable with $\mathcal{H}^1(E) > 0$, then there is at most one line $L \in G(2, 1)$ such that $\mathcal{H}^1(P_L(E)) = 0$. We can characterize rectifiable sets via projections, but it is more natural to state this for purely unrectifiable sets:

Theorem 3.13 *If E is $\mathcal{H}^1$ measurable and $\mathcal{H}^1(E) < \infty$, then E is purely 1-unrectifiable if and only if $\mathcal{H}^1(P_L(E)) = 0$ for almost all $L \in G(2, 1)$.*

What is left to prove is that any purely 1-unrectifiable $\mathcal{H}^1$ measurable set E with $\mathcal{H}^1(E) < \infty$ projects to measure zero in almost all directions. This is the famous Besicovitch projection theorem. Here are some ideas for the proof.

The essence of the proof is to study how E behaves close to lines. Fix first $L \in G(2, 1)$. Let $\delta > 0$ and split E into three parts. The first set $E_1(\delta, L)$ consists of points $a \in E$ such that only a little measure is approaching a in the direction L:

$$\limsup_{s \to 0} \sup_{0<r<\delta} (sr)^{-1} \mathcal{H}^1(E \cap B(a, r) \cap X(a, L, s)) = 0.$$

The second set $E_2(\delta, L)$ consists of points $a \in E$ such that a lot of measure is approaching a in the direction L:

$$\limsup_{s \to 0} \sup_{0<r<\delta} (sr)^{-1} \mathcal{H}^1(E \cap B(a, r) \cap X(a, L, s)) = \infty.$$

It might be that nothing else is needed. Maybe almost all pairs (a, L) must satisfy one of these properties. But that is not known. What saves us is that it suffices to add another property. The third set $E_3(L)$ consists of points $a \in E$ such that $E \cap (L + a) \cap B(a, \delta) \neq \emptyset$ for all $\delta > 0$.

The last property is easy to deal with because it is rather simple to show that for any set $F \subset \mathbb{R}^2$ with $\mathcal{H}^1(F) < \infty$, $\text{card}(F \cap (L + x)) < \infty$ for almost all $x \in L^\perp$. Hence $\mathcal{H}^1(P_{L^\perp}(E_3(L))) = 0$.

The second case also is fairly easy. It is only slightly more complicated than the case where the cones $X(a, L, s)$ are replaced by cylinders $\{x \in B(a, r): d(x, L + a) < sr\}$. If many of these have big measure as compared to sr, then applying Vitali's covering theorem on $L^{\perp}$, we find that $\mathcal{H}^1(P_{L^{\perp}}(E_2(\delta, L))) = 0$.

We have already dealt with the first set: by Theorem 3.7 $\mathcal{H}^1(E_1(\delta, L)) = 0$. Then after some technicalities we are left to show that for almost all (a, L) if a is the only point of E on $E \cap B(a, \delta) \cap (L + a)$, then

$$\limsup_{s\to 0} \sup_{0<r<\delta} (sr)^{-1}\mathcal{H}^1(E \cap B(a, r) \cap X(a, L, s))$$

is either 0 or ∞.

For this we can use a general lemma due to Mickle and Radó [345]:

Lemma 3.14 *If ψ is an outer measure on $\mathbb{R}^n$ and A is Lebesgue measurable with $\psi(A) = 0$, then for $\mathcal{L}^n$ almost all $x \in A$, $\limsup_{r\to 0} r^{-n}\psi(B(x, r))$ is either 0 or ∞.*

The proof is a simple covering argument combined with the Lebesgue density theorem. This lemma is applied on $G(2, 1)$, identified with S^1, and we leave it to the reader to figure out how ψ is chosen, or see [321, Chapter 18]. Notice that we do not, and we cannot, assume that Borel sets are ψ measurable.

The proof sketched above is the original Besicovitch proof from [64], but modified and polished by several people. No essentially different proof is known. Tao [406] proved a quantitative multiscale version of Besicovitch's projection theorem, but he did this quantizing Besicovitch's proof. Davey and Taylor [132] proved a non-linear version of Tao's result.

As mentioned above, the third alternative $E_3(L)$ may not be needed at all; the following might be true:

Conjecture 3.15 *If E is $\mathcal{H}^1$ measurable, purely 1-unrectifiable and $\mathcal{H}^1(E) < \infty$, then for $\mathcal{H}^1$ almost all $a \in E$ almost all lines through a meet E only at a.*

Although this is not known, Csörnyei and Preiss [122] disproved a related conjecture which asked whether it is true that for every $\mathcal{H}^1$ measurable E with $\mathcal{H}^1(E) < \infty$ almost all lines through almost all points of E meet E in a finite set. Their set contains non-trivial rectifiable and purely unrectifiable parts.

The analogue of Crofton's formula (2.4) extends from rectifiable curves to 1-rectifiable sets with routine arguments.

3.4 Analyst's Travelling Salesman Problem

This topic is discussed in the books [67], [378] and [415].

An analogue of the classical travelling salesman problem for general compact planar sets is: when can all points of a compact set $F \subset \mathbb{R}^2$ be traversed via a rectifiable curve and how long must such a curve be? Jones proved in [263] the following theorem. Let $F \subset \mathbb{R}^2$ be compact. For any square Q set

$$\beta_F(Q) = \inf_{L \text{ a line}} \sup\{d(x, L)/d(Q) \colon x \in F \cap Q\},$$

meaning that $\beta_F(Q) = 0$ if $F \cap Q = \emptyset$. Let $\mathcal{D}$ be the family of all dyadic squares in the plane.

Theorem 3.16 *A compact set $F \subset \mathbb{R}^2$ is contained in a rectifiable curve if and only if the Jones square function*

$$\beta(F) := \sum_{Q \in \mathcal{D}} \beta_F(3Q)^2 d(Q) < \infty.$$

The length of the shortest such curve is comparable to $\beta(F) + d(F)$.

If $\beta(F) < \infty$, the key to the construction of the curve is the Pythagoras theorem. Suppose that a line segment I is a good approximation of F inside a rectangle of side lengths d and βd. To get good approximations in the next smaller scale, we might need to replace I by two line segments I_1 and I_2 with one common endpoint and the other endpoints common with those of I. Then the increase of length is roughly $d(\sqrt{1 + \beta^2} - 1) \sim \beta^2 d$. The complete construction is not easy, but it is easier than the proof of the converse statement.

Theorem 3.16 was extended to $\mathbb{R}^n$ by Okikiolu [370]. Schul generalized it in [394] to infinite-dimensional Hilbert spaces. This means that he showed that the constants are independent of n. A different proof of Theorem 3.16 and a nice treatment of this topic can be found in the book by Bishop and Peres [67]. They use arguments involving Crofton's formula.

Jones's theorem can easily be stated in integral form which leads to the following characterization of 1-rectifiable sets. Since sets of measure zero can affect greatly the β numbers, it is better to state it for purely unrectifiable sets. Let

$$\beta_E(x, r) = \inf_L \sup\{d(x, L)/r \colon x \in E \cap B(x, r)\}. \tag{3.1}$$

Theorem 3.17 *If $E \subset \mathbb{R}^2$ is $\mathcal{H}^1$ measurable and $\mathcal{H}^1(E) < \infty$, then E is purely 1-unrectifiable if and only if*

$$\int_{\mathbb{R}^2} \int_0^\infty \beta_F(x, r)^2 r^{-2}\, dr\, dx = \infty$$

for every Borel set $F \subset E$ with $\mathcal{H}^1(F) > 0$.

For a pointwise characterization in higher dimensions, see Theorem 4.20.

We shall briefly mention another related way to characterize rectifiability in terms of *Menger curvature* $c(x, y, z)$, which also measures approximation by lines. For three points x, y, z in the plane, it is defined as the reciprocal of the radius of the circle passing through these points. Thus $c(x, y, z) = 0$ if and only if the points are collinear. More quantitatively by elementary geometry,

$$c(x, y, z) = \frac{2d(x, L_{y,z})}{|x - y||x - z|} = \frac{4A(x, y, z)}{|x - y||x - z||y - z|}, \tag{3.2}$$

where $L_{y,z}$ is the line through y and z and $A(x, y, z)$ is the area of the triangle with vertices x, y and z.

Theorem 3.18 *If $E \subset \mathbb{R}^2$ is $\mathcal{H}^1$ measurable, $\mathcal{H}^1(E) < \infty$ and*

$$\int_E \int_E \int_E c(x, y, z)^2 \, d\mathcal{H}^1 x \, d\mathcal{H}^1 y \, d\mathcal{H}^1 z < \infty, \tag{3.3}$$

then E is 1-rectifiable

This was first proved by David (unpublished) and then by Léger in [288]. In [410], Tolsa gave a different proof based on the analytic capacity.

The main problem for proving Theorem 3.18 is that we do not have AD-regularity, or even positive lower density. With such assumptions the proof would be much easier. To get a vague idea, suppose that E is AD-1-regular, $x_i \in E, i = 1, 2, 3, |x_i - x_j| = 2r$, when $i \neq j$, and $\beta_E(x_i, r) \sim 1$. Then with some $c > 0$ and $A_i = E \cap B(x_i, r) \setminus B(x_i, cr)$,

$$\int_{A_1} \int_{A_2} \int_{A_3} c(x, y, z)^2 \, d\mathcal{H}^1 x \, d\mathcal{H}^1 y \, d\mathcal{H}^1 z \sim r.$$

Clearly this cannot happen too often if (3.3) holds.

Lerman and Whitehouse [290] and Meurer [344] proved higher-dimensional versions.

We shall return to this concept in several later chapters.

Here is another related square function: Let Γ be a Jordan curve in the plane and let Ω^+ and Ω^- be its interior and exterior domains. For $x \in \Gamma, r > 0$, let $I^+(x, r)$ and $I^-(x, r)$ be the longest arcs of $\partial B(x, r)$ contained in Ω^+ and Ω^-, respectively. Set

$$\varepsilon(x, r) = \max\{|\pi r - \mathcal{H}^1(I^+(x, r))|, |\pi r - \mathcal{H}^1(I^-(x, r))|\}/r$$

and define

$$\mathcal{E}(x)^2 = \int_0^1 \varepsilon(x, r)^2/r \, dr.$$

Jaye, Tolsa and Villa [260] proved the following theorem (and more) solving an old ε^2 conjecture of Carleson:

Theorem 3.19 *Let* Γ *be a Jordan curve. Then for* $\mathcal{H}^1$ *almost all* $x \in \Gamma$, Γ *has a tangent at* x *if and only if* $\mathcal{E}(x) < \infty$.

That $\mathcal{E}(x) < \infty$ at almost every tangential point x of Γ is classical, so these authors proved the converse. The proof uses methods developed in connection with Theorems 3.16 and 3.18, but a lot of new ideas and techniques are also required.