ON POSITIVE SOLUTIONS OF THE
HEAT EQUATION

MASASUMI KATO[1]

To Professor Kiyoshi Noshiro on the occasion of his 60th birthday

1. Consider the positive and twice continuously differentiable solutions u of the heat equation

$$(1) \qquad \left(\Delta - \frac{\partial}{\partial t} \right) u = 0 , \qquad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

in an open t-strip $\Omega = R_n \times (0, T)$ for some $T > 0$, where R_n is the n-dimensional Euclidean space.

In this note, we prove a theorem of Fatou type on u and, as its application, the uniqueness theorem for the Cauchy problem of (1).

2. The following theorem corresponds to Fatou's theorem on harmonic functions.

THEOREM 1. *Let u be any positive solution of* (1) *in Ω. Then $\lim\limits_{t \to 0+} u(x, t)$ exists for almost every $x = (x_1, x_2, \ldots\ldots, x_n) \in R_n$.*

Proof. We begin with the Poisson-Stieltjes integral representation of $u(x, t)$ which is valid at least near the hyperplane $t = 0$. The representation is classical when $n = 1$ (cf. [5]).

Let t_0 be any fixed value such that $0 < t_0 < T$ and ε be any number such that $0 < \varepsilon < T - t_0$. Then we can represent $u(x, t + \varepsilon)$ as follows:

$$(2) \qquad u(x, t + \varepsilon) = \int_{R_n} k(y - x, t) u(y, \varepsilon) dy \qquad \text{in } \Omega_{\varepsilon} ,$$

where $k(y, t) = (4\pi t)^{-n/2} \exp (-|y|^2/4t)$, $|y|^2 = \sum_{i=1}^{n} y_i^2$

and $\Omega_{\varepsilon} = R_n \times (0, T - \varepsilon)$ (e.g. [2] p. 42–48).

203

By choosing $x = 0$, $t = t_0$ in (2), we see that the Borel measures

$$d\mu_\varepsilon = k(y, t_0)u(y, \varepsilon)dy$$

are uniformly bounded for sufficiently small $\varepsilon > 0$. Hence, by Frostman's selection theorem, there exists a sequence $\{\mu_{\varepsilon_j}\}_{j=1}^\infty$ of Borel measures converging to some Borel measure μ as $\varepsilon_1 > \varepsilon_2 > \ldots \ldots > \varepsilon_j > \ldots \ldots \to 0$. This means that $\lim \mu_{\varepsilon_j}(X) = \mu(X)$ for each Borel set X with the boundary of μ-measure zero. Hence, for any point (x, t) in $\Omega_0 = R_n \times (0, t_0)$, we have

$$u(x, t + \varepsilon_j) = \int_{R_n} k(y - x, t)u(y, \varepsilon_j)dy$$

$$= \int_{R_n} \frac{k(y - x, t)}{k(y, t_0)}d\mu_{\varepsilon_j} \to \int_{R_n} \frac{k(y - x, t)}{k(y, t_0)}d\mu \qquad \text{as } j \to \infty \ ,$$

where the passage to the limit is justified by noting the choice of μ_{ε_j} and by the obvious estimate

$$k(y - x, t)/k(y, t_0) = O(e^{-\text{const.} |y|^2}) \text{ as } |y| \to \infty \ .$$

By setting $d\sigma = d\mu/k(y, t_0)$, we obtain the desired representation

$$(3) \qquad\qquad u(x, t) = \int_{R_n} k(y - x, t)d\sigma \qquad \text{in } \Omega_0 \ ,$$

where σ is obviously finite for the bounded Borel sets in R_n.

Now, we consider the Lebesgue decomposition of $d\sigma$:

$$d\sigma = \varphi(y)dy + ds \ ,$$

where the density $\varphi \geqq 0$ is locally summable on R_n and $s \geqq 0$ is singular. By the strong version of Lebesgue's theorem and by the fact that the *symmetric derivative* $D_{\text{sym}}s(x)$ of s vanishes at almost every $x \in R_n$, we have

$$(4) \qquad\qquad a^{-n}\int_{|y-x|\leq a}\{|\varphi(y) - \varphi(x)|dy + ds(y)\} \to 0 \qquad \text{as } a \to 0$$

for almost every $x \in R_n$.

We may assume that (4) holds for $x = 0$. Then, for any $\varepsilon > 0$, there exists $a_0 > 0$ such that the left hand side of (4) is less than ε whenever $0 < a \leq 2a_0$. Moreover, for any t such that $0 < t < \text{Min}(a_0^2, t_0)$ we choose a positive integer N such that

$$2^{N-1}b \leqq a_0 < 2^N b \ , \quad b = t^{1/2} \ .$$

By (3), we see

$$|u(0, t) - \varphi(0)| \leqq$$

$$\left(\int_{|y|<b} + \sum_{j=1}^{N} \int_{2^{j-1}b \leqq |y| < 2^{j}b} + \int_{|y|>a_0} \right) k(y, t)\{|\varphi(y) - \varphi(0)| dy + ds(y)\} \ .$$

Estimating each integral, we see easily that

$$|u(0, t) - \varphi(0)| \leqq \text{Const. } \varepsilon + \text{Const. } \varepsilon \sum_{j=1}^{\infty} 2^{-nj}$$

$$+ \varphi(0) \int_{|y|>a_0} k(y, t) dy + \mu(R_n) t_0^{n/2} e^{a_0^2/4t_0} t^{-n/2} e^{-a_0^2/4t}$$

$$\leqq \text{Const. } \varepsilon \text{ as } t \to 0+ \ ,$$

which proves our assertion.

Remark. P.C. Rosenbloom ([3], p. 191–200) remarked without proof the validity of Fatou's theorem under a somewhat strong growth condition about $u(x, t)$.

3. Here, with the aid of Theorem 1, we prove the uniqueness theorem for positive solutions under some weak conditions (cf. [5] and [3]). For the purpose, we prepare the following lemma.

LEMMA. *If $\sigma \geqq 0$ is a Borel measure on R_n and if the upper symmetric derivative $\bar{D}_{\text{sym}}\sigma(x)$ is finite at each point $x \in R_n$, then σ is absolutely continuous with respect to the n–dimensional Lebesgue measure.*

Although we could deduce its proof from Ward's decomposition theorem ([4], p. 151–152), we state here a direct proof along the way suggested by Prof. S. Ito.

Proof. Assume that there exists a compact set K such that $\sigma(K) > 0$ and such that the Lebesgue measure $|K|$ of K equals zero. Then, for a sufficiently large $M > 0$, there exists a compact subset K_0 of K such that $\sigma(K_0) > 0$ and $\sigma(S)/r^n \leqq M$ whenever S is a closed sphere with center in K_0 and of radius r less than M^{-1}. On the other hand, on account of $|K_0| = 0$, there exists a sequence of open cubes $\{I_j\}_{j=1}^{\infty}$ such that

(a)   the diameter a_j of I_j is less than M^{-1} ,

(b) $\bigcup_{j=1}^{\infty} I_j \supset K_0$ and

(c) $\sum_{j=1}^{\infty} |I_j| < (n^{n/2}M)^{-1}\sigma(K_0)$.

By (b) and (c), we have, for at least one I_{j_0} ,

$$|I_{j_0}| < (n^{n/2}M)^{-1}\sigma(K_0 \cap I_{j_0}) \ .$$

Considering a closed sphere S_0 with center in $K_0 \cap I_{j_0} \neq \phi$ and of radius a_{j_0}, we have a contradiction.

Now we can prove the following.

THEOREM 2. *If* $\lim\sup\limits_{t \to 0+} u(x, t) < +\infty$ *at each point* $x \in R_n$ *and if* $\lim\limits_{t \to 0+} u(x, t)$ $=0$ *for almost every* $x \in R_n$, *then* $u = 0$ *in* Ω.

Proof. Assume that, for some point, e.g., $x = 0$, we have $\bar{D}_{\text{sym}}\sigma(0) = +\infty$. Then, there exists a sequence of radii $\{r_j\}_{j=1}^{\infty}$ converging to zero such that

$$\lim \sigma(S_j)/r_j^n = +\infty ,$$

where S_j denotes the closed sphere with center $x = 0$ and of radius r_j. Thus, for a sufficiently large j, we have by (3)

$$u(0, r_j^2) \geqq \int_{|y| \leqq r_j} k(y, r_j^2) d\sigma \geqq \text{Const. } \sigma(S_j)/r_j^n ,$$

which, by letting $j \to +\infty$, leads us to a contradiction.

Hence, by the above lemma, σ is absolutely continuous, that is, $s = 0$. On the other hand, by our assumption and Theorem 1, we see that $\varphi(x) = 0$ for almost every $x \in R_n$. Thus, we have $\sigma = 0$, that is, $u(x, t) = 0$ in Ω_0. Since t_0 is arbitrary in $(0, T)$, we conclude that $u(x, t) = 0$ in Ω.

4. By replacing $k(y, t)$ by the fundamental solution given in [1], we can replace Laplacian Δ in our theorems by an elliptic differential operator A of the following form:

$$A = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i} + c(x) ,$$

where the matrix $(a_{ij}(x))$ is symmetric and strictly positive definite for any $x \in R_n$ and

$$D^2 a_{ij}(x), \ D a_{ij}(x), \ D b_i(x), \ c(x) \text{ and } \det (a_{ij}(x))^{-1}$$

are bounded and Hölder continuous on R_n.[2]

REFERENCES

[1] F.G. Dressel: The fundamental solution of the parabolic equation. Duke Math. J., **7** (1940) 186–203, **13**(1946) 61–70.
[2] A. Friedman: Partial differential equations of parabolic type. Prentice-Hall (1964).
[3] P.C. Rosenbloom: Linear equations of parabolic type with constant coefficients. Contribu-

---

[2] For this remark the author owes to Prof. S. Ito.

tions to the theory of partial differential equations. Ann. Math. Studies. no. 33 (Princeton Univ. Press, 1954), p. 191–200.

[4] S. Saks: Theory of the integral. Warsaw. (1937).

[5] D.V. Widder: Positive temperatures on an infinite rod. Tran. Amer. Math. Soc., **55**, 85–95 (1944).

Suzuka College of Technology