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Ricci Solitons and Geometry of Four-dimensional Non-reductive Homogeneous Spaces

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Abstract. We study the geometry of non-reductive four-dimensional homogeneous spaces. In particular, after describing their Levi-Civita connection and curvature properties, we classify homogeneous Ricci solitons on these spaces, proving the existence of shrinking, expanding, and steady examples. For all the non-trivial examples we find, the Ricci operator is diagonalizable.

1 Introduction

A (connected) pseudo-Riemannian manifold (M, g) is *homogeneous* provided there exists a group G of isometries acting transitively on it. Such manifold (M, g) can be then identified with $(G/H, g)$, where H is the isotropy group at a fixed point o of M and g is an invariant pseudo-Riemannian metric.

A homogeneous pseudo-Riemannian manifold (M, g) is said to be *reductive* if $M = G/H$ and the Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$. It is well known that when H is connected, this condition is equivalent to the algebraic condition $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. In the study of homogeneous pseudo-Riemannian manifolds, a fundamental difference arises between the Riemannian case and the non Riemannian one. In fact, while any homogeneous Riemannian manifold is reductive, there exist homogeneous pseudo-Riemannian manifolds that do not admit any reductive decomposition.

Although some differences naturally occur for curvature properties (see for example [6, 7] for the three-dimensional case), the study of reductive homogeneous pseudo-Riemannian manifolds parallels the Riemannian case in several ways, for instance, with regard to homogeneous structures [17] and algebraic characterizations of homogeneous geodesics [13]. On the other hand, to our knowledge, non-reductive examples have not been fully investigated, although it is likely that the most interesting differences between Riemannian and pseudo-Riemannian settings occur in such cases.

Two- and three-dimensional homogeneous pseudo-Riemannian manifolds are reductive [5, 14]. In the basic paper [14], the classification of four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifolds was obtained, showing the

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existence of both Lorentzian and neutral signature examples. Apart from the classification of Einstein metrics [14], we do not know of any further results concerning the geometric properties of these spaces. The aim of this paper is to provide a systematic study of the geometry of four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds, with particular regard to the existence of homogeneous Ricci solitons.

A *Ricci soliton* is a pseudo-Riemannian manifold (M, g) admitting a smooth vector field V such that

$$\mathcal{L}_V g + \varrho = \lambda g, \tag{1.1}$$

where $\mathcal{L}_V$ and ϱ respectively denote the Lie derivative in the direction of V and the Ricci tensor, and λ is a real number. A Ricci soliton is said to be *shrinking*, *steady*, or *expanding* according to whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Ricci solitons play an important role in understanding the singularities of the *Ricci flow*, of which they are the self-similar solutions. A survey and further references on the geometry of Ricci solitons may be found in [10].

First introduced and studied in the Riemannian case, Ricci solitons have been investigated in pseudo-Riemannian settings, with special attention to the Lorentzian case [3, 8, 11, 25]. The Ricci soliton equation also appears to be related to String Theory. Some physical aspects of the Ricci flow have been emphasized in [1, 15, 19]. Moreover, the interest of the Ricci soliton equation also relies on the fact that it is a special case of Einstein field equations.

If $M = G/H$ is a homogeneous space, a *homogeneous Ricci soliton* on M is a G-invariant metric g for which equation (1.1) holds. In particular, by an *invariant Ricci soliton* we mean a homogeneous one such that equation (1.1) is satisfied by an invariant vector field.

It is a natural question to determine which homogeneous manifolds G/H admit a G-invariant Ricci soliton [21]. Also with regard to this question, pseudo-Riemannian geometry allows more interesting behaviours with respect to Riemannian settings.

For example, there exist three-dimensional, Riemannian, homogeneous Ricci solitons (see, for example, [2, 21]), but there are no three-dimensional left-invariant Riemannian metrics, together with a left-invariant vector field V, such that equation (1.1) holds for a three-dimensional Lie group [12] (see also [18, 24]). On the other hand, there exist several non-trivial interesting examples of such left-invariant Lorentzian Ricci solitons in dimension three [3].

In this paper, we obtain the full classification of homogeneous Ricci solitons on four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds $M = G/H$, for solutions of (1.1) determined by vector fields $V \in \mathfrak{m}$. Non-trivial examples appear both in the Lorentzian case and for metrics of neutral signature $(2, 2)$.

We shall also investigate some curvature properties, which in the Riemannian case are related to natural reductivity and symmetry, proving that in pseudo-Riemannian settings these conditions can also be satisfied by non-reductive spaces. Finally, we classify invariant symplectic, complex, and Kähler structures on four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifolds.

The paper is organized as follows. The classification of four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds is reported in Section 2, with some needed corrections and integrated by the explicit list of the corresponding invariant pseudo-Riemannian metrics. In Section 3 we describe their Levi-Civita connection and curvature. The classification of Einstein-like examples and homogeneous Ricci solitons of these spaces will be given in Sections 4 and 5. Section 6 will be devoted to invariant symplectic and complex structures on four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifolds.

2 The Classification of four-dimensional Non-reductive Homogeneous Spaces

Let $M = G/H$ denote a homogeneous manifold, with H connected, $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of G and the isotropy subalgebra respectively, and $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ the factor space, which identifies with a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$. The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines the isotropy representation

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m}), \qquad \rho(x)(y) = [x, y]_{\mathfrak{m}} \quad \text{for all} \quad x \in \mathfrak{g}, y \in \mathfrak{m}.$$

Consider a basis $\{h_1, \dots, h_r, u_1, \dots, u_n\}$ of $\mathfrak{g}$, where $\{h_j\}$ and $\{u_i\}$ are bases of $\mathfrak{h}$ and $\mathfrak{m}$, respectively. Then any bilinear form on $\mathfrak{m}$ is determined by the matrix g of its components with respect to the basis $\{u_i\}$ and is invariant if and only if ${}^t\rho(x) \circ g + g \circ \rho(x) = 0$ for all $x \in \mathfrak{g}$. Invariant pseudo-Riemannian metrics on the homogeneous space $M = G/H$ are in a one-to-one correspondence with nondegenerate invariant symmetric bilinear forms g on $\mathfrak{m}$ [20]. Non-reductive homogeneous manifolds of dimension 4 were classified in [14] in terms of the corresponding non-reductive Lie algebras. We now report this classification and explicitly describe the corresponding pseudo-Riemannian metrics.

2.1 Lorentzian Case

(A1) $\mathfrak{g} = \mathfrak{a}_1$ is the decomposable 5-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{s}(2)$, where $\mathfrak{s}(2)$ is the 2-dimensional solvable algebra. There exists a basis $\{e_1, \dots, e_5\}$ of $\mathfrak{a}_1$ such that the non-zero products are

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_4, e_5] = e_4,$$

and the isotropy subalgebra is $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3 + e_4\}$. So, we can take

$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_5, u_4 = e_3 - e_4\}$$

and have the following isotropy representation for h_1:

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Consequently, with respect to $\{u_i\}$, invariant metrics g are of the form

$$g = \begin{pmatrix} a & 0 & -\frac{a}{2} & 0 \\ 0 & b & c & a \\ -\frac{a}{2} & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix} \tag{2.1}$$

and are nondegenerate whenever $a(a-4d) \neq 0$. Both Lorentzian and signature $(2,2)$ invariant metrics exist.

From the isotropy representation above it also easily follows that a vector field $V \in \mathfrak{m}$ is invariant if and only if $V \in \operatorname{Span}\{u_1 + 2u_3, u_4\}$.

(A2) $\mathfrak{g} = \mathfrak{a}_2$ is the one-parameter family of 5-dimensional Lie algebras $A_{5,30}$ of [23]. There exists a basis $\{e_1, \dots, e_5\}$ of $\mathfrak{a}_2$ such that the non-zero products are

$$\begin{array}{lll} [e_1, e_5] = (\alpha+1)e_1, & [e_2, e_4] = e_1, & [e_2, e_5] = \alpha e_2, \\ [e_3, e_4] = e_2, & [e_3, e_5] = (\alpha-1)e_3, & [e_4, e_5] = e_4, \end{array}$$

for any value of $\alpha \in \mathbb{R}$, and the isotropy is $\mathfrak{h} = \operatorname{Span}\{h_1 = e_4\}$. Hence, we take

$$\mathfrak{m} = \operatorname{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3, u_4 = e_5\}$$

and find the isotropy representation

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the invariant metrics are of the form

$$g = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \tag{2.2}$$

nondegenerate whenever $ad \neq 0$. Both Lorentzian and signature $(2,2)$ invariant metrics occur.

Moreover, it is easily seen that a vector field $V \in \mathfrak{m}$ is invariant if and only if $V \in \operatorname{Span}\{u_1, u_4\}$.

(A3) $\mathfrak{g} = \mathfrak{a}_3$ is one of the 5-dimensional Lie algebras $A_{5,37}$ or $A_{5,36}$ in [23]. There exists a basis $\{e_1, \dots, e_5\}$ of $\mathfrak{a}_3$ such that the non-zero products are

$$\begin{array}{lll} [e_1, e_4] = 2e_1, & [e_2, e_3] = e_1, & [e_2, e_4] = e_2, \\ [e_2, e_5] = -\varepsilon e_3, & [e_3, e_4] = e_3, & [e_3, e_5] = e_2, \end{array}$$

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with $\varepsilon = 1$ for $A_{5,37}$ and $\varepsilon = -1$ for $A_{5,36}$, and the isotropy is $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3\}$. Thus, we take

$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_4, u_4 = e_5\}$$

and obtain the isotropy representation

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

from which we deduce that invariant metrics are given by

$$g = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ a & 0 & c & d \end{pmatrix} \tag{2.3}$$

and are nondegenerate whenever $ab \neq 0$. Both Lorentzian and signature $(2,2)$ invariant metrics exist.

A vector field $V \in \mathfrak{m}$ is invariant if and only if $V \in \mathrm{Span}\{u_1, u_3\}$.

(A4) $\mathfrak{g} = \mathfrak{a}_4$ is the 6-dimensional Schroedinger Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}(3)$, where $\mathfrak{n}(3)$ is the 3-dimensional Heisenberg algebra. There exists a basis $\{e_1, \dots, e_6\}$ of $\mathfrak{a}_4$, where the non-zero products are

$$\begin{array}{llll} [e_1, e_2] = 2e_2, & [e_1, e_3] = -2e_3, & [e_2, e_3] = e_1, & [e_1, e_4] = e_4, \\ [e_1, e_5] = -e_5, & [e_2, e_5] = e_4, & [e_3, e_4] = e_5, & [e_4, e_5] = e_6, \end{array}$$

and the isotropy is $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3 + e_6, h_2 = e_5\}$. Therefore, we take

$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3 - e_6, u_4 = e_4\}$$

and have the following isotropy representation for h_1, h_2:

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We then find that the invariant metrics are of the form

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{2} \end{pmatrix}, \tag{2.4}$$

nondegenerate whenever $a \neq 0$. The eigenvalues of g are

$$a, \quad \tfrac{1}{2}a, \quad \tfrac{1}{2}(b + \sqrt{b^2 + 4a^2}), \quad \tfrac{1}{2}(b - \sqrt{b^2 + 4a^2}),$$

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and so, invariant metrics are necessarily Lorentzian.

Moreover, a vector field $V \in \mathfrak{m}$ is invariant if and only if $V \in \operatorname{Span}\{u_3\}$.

(A5) $\mathfrak{g} = \mathfrak{a}_5$ is the 7-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes A^1_{4,9}$, with $A^1_{4,9}$ as in [23]. It admits a basis $\{e_1,\dots,e_7\}$, such that the non-zero products are

$$\begin{array}{llll} [e_1,e_2]=2e_2, & [e_1,e_3]=-2e_3, & [e_1,e_5]=-e_5, & [e_1,e_6]=e_6, \\ [e_2,e_3]=e_1, & [e_2,e_5]=e_6, & [e_3,e_6]=e_5, & [e_4,e_7]=2e_4 \\ [e_5,e_6]=e_4, & [e_5,e_7]=e_5, & [e_6,e_7]=e_6. & \end{array}$$

The isotropy is $\mathfrak{h} = \operatorname{Span}\{h_1 = e_1 + e_7, h_2 = e_3 - e_4, h_3 = e_5\}$. So, we take

$$\mathfrak{m} = \operatorname{Span}\{u_1 = e_1 - e_7, u_2 = e_2, u_3 = e_3 + e_4, u_4 = e_6\}$$

and find the isotropy representation

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Hence, invariant metrics are of the form

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & \frac{a}{4} & 0 \\ 0 & \frac{a}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{8} \end{pmatrix} \tag{2.5}$$

and are nondegenerate whenever $a \neq 0$. The eigenvalues of g are

$$-\frac{1}{4}a, \quad \frac{1}{8}a, \quad \frac{1}{4}a, \quad a,$$

and so g is Lorentzian. Besides $V = 0$, no vector fields $V \in \mathfrak{m}$ are invariant.

2.2 Signature $(2,2)$ Case

Besides cases $A1, A2, A3$, which also admit invariant metrics of neutral signature $(2,2)$, the remaining possibilities are the following.

(B1) $\mathfrak{g} = \mathfrak{b}_1$ is the 5-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, admitting a basis $\{e_1,\dots,e_5\}$, where the non-zero products are

$$\begin{array}{llll} [e_1,e_2]=2e_2, & [e_1,e_3]=-2e_3, & [e_2,e_3]=e_1, & [e_1,e_4]=e_4, \\ [e_1,e_5]=-e_5, & [e_2,e_5]=e_4, & [e_3,e_4]=e_5, & \end{array}$$

and the isotropy is $\mathfrak{h} = \operatorname{Span}\{h_1 = e_3\}$. Thus, taking

$$\mathfrak{m} = \operatorname{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_4, u_4 = e_5\},$$

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we have the isotropy representation

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and the invariant metrics are of the form

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & c & a \\ a & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \tag{2.6}$$

nondegenerate whenever $a \neq 0$. The eigenvalues of g are the roots of

$$x^4 + (-b - d)x^3 + (bd - c^2 - 2a^2)x^2 + a^2(b + d)x + a^4 = 0,$$

and so g has signature $(2, 2)$, as $-(b + d)$ and $a^2(b + d)$ have opposite sign.

Starting from the above isotropy representation, we easily conclude that a vector field $V \in \mathfrak{m}$ is invariant if and only if $V \in \operatorname{Span}\{u_1, u_4\}$.

(B2) $\mathfrak{g} = \mathfrak{b}_2$ is the 6-dimensional Schroedinger Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{n}(3)$, but with isotropy $\mathfrak{h} = \operatorname{Span}\{h_1 = e_3 - e_6, h_2 = e_5\}$. Then taking

$$\mathfrak{m} = \operatorname{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3 + e_6, u_4 = e_4\},$$

we have the isotropy representation

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and the invariant metrics are of the form

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{2} \end{pmatrix}, \tag{2.7}$$

nondegenerate whenever $a \neq 0$. As the eigenvalues of G are

$$a, \quad -\frac{1}{2}a, \quad \frac{1}{2}(b + \sqrt{b^2 + 4a^2}), \quad \frac{1}{2}(b - \sqrt{b^2 + 4a^2}),$$

the metric has signature $(2, 2)$.

A vector field $V \in \mathfrak{m}$ is invariant if and only if $V \in \operatorname{Span}\{u_3\}$.

(B3) $\mathfrak{g} = \mathfrak{b}_3$ is the 6-dimensional Lie algebra $(\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2) \times \mathbb{R}$. It admits a basis $\{u_1, \dots, u_4, h_1 = u_5, h_2 = u_6\}$ such that $\mathfrak{h} = \mathrm{Span}\{h_1, h_2\}$, $\mathfrak{m} = \mathrm{Span}\{u_1, \dots, u_4\}$ and the non-zero products are

$$\begin{array}{lll} [h_1, u_2] = u_1, & [h_1, u_3] = -u_4, & [h_2, u_2] = -2h_2, \\ [h_2, u_3] = -u_2, & [h_2, u_4] = u_1, & [u_1, u_2] = -u_1, \\ [u_1, u_3] = u_4, & [u_2, u_3] = -2u_3, & [u_2, u_4] = -u_4. \end{array}$$

Thus, the isotropy representation is given by

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, the invariant metrics are of the form

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \tag{2.8}$$

nondegenerate whenever $a \neq 0$. The eigenvalues of g are

$$a, \quad -a, \quad \frac{1}{2}(b + \sqrt{b^2 + 4a^2}), \quad \frac{1}{2}(b - \sqrt{b^2 + 4a^2}),$$

and so g has signature $(2, 2)$.

Moreover, a vector field $V \in \mathfrak{m}$ is invariant if and only if $V \in \mathrm{Span}\{u_1\}$.

Remark 2.1 Lie algebra $\mathfrak{b}_3$ corresponds to [20, case $2.5^1.2$], as explained in [14, p. 302]). Here we reported this Lie algebra as it appears in [20], since the case listed in [14, Theorem 2.4] does not correspond to it. Some corrections were also needed and have been made for the Lie brackets of Lie algebra $\mathfrak{b}_1$.

3 Levi-Civita Connection and Curvature

In order to compute the Levi-Civita connection and the curvature of a non-reductive homogeneous space, consider again a basis $\{h_1, \dots, h_r, u_1, \dots, u_n\}$ of $\mathfrak{g}$, with $\{e_j\}$ and $\{u_i\}$ bases of $\mathfrak{h}$ and $\mathfrak{m}$ respectively. Following [20], an invariant nondegenerate symmetric bilinear form g on $\mathfrak{m}$ uniquely defines its invariant linear Levi-Civita connection, described in terms of the corresponding homomorphism of $\mathfrak{h}$-modules $\Lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda(x)(y_{\mathfrak{m}}) = [x, y]_{\mathfrak{m}}$ for all $x \in \mathfrak{h}$, $y \in \mathfrak{g}$. Explicitly, one has

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y), \quad \text{for all } x, y \in \mathfrak{g},$$

where $\nu\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{m}$ is the $\mathfrak{h}$-invariant symmetric mapping uniquely determined by

$$2g\big(\nu(x,y),z_{\mathfrak{m}}\big) = g\big(x_{\mathfrak{m}},[z,y]_{\mathfrak{m}}\big) + g\big(y_{\mathfrak{m}},[z,x]_{\mathfrak{m}}\big), \quad \text{for all } x,y,z\in\mathfrak{g}.$$

The curvature tensor is then determined by the mapping $R\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{gl}(\mathfrak{m})$ such that $R(x,y)=[\Lambda(x),\Lambda(y)]-\Lambda([x,y])$ for all $x,y\in\mathfrak{m}$.

Finally, the Ricci tensor ϱ of g, described in terms of its components with respect to $\{u_i\}$, is given by

$$\varrho(u_i,u_j)=\sum_{r=1}^{4}R_{ri}(u_r,u_j),\quad i,j=1,\dots,4.$$

The *Ricci operator* Q is then defined by condition $g(Q(X),Y)=\varrho(X,Y)$. As ϱ is symmetric, Q is self-adjoint. Contrary to the Riemannian case, this does not necessarily imply that Q is diagonalizable in pseudo-Riemannian settings. Indeed, Q may take any of four different canonical forms [22] .

(A1) We refer to the basis $E=\{h_1,u_1,\dots,u_4\}$ of the Lie algebra $\mathfrak{a}_1$ described in the previous section. Let g be an arbitrary invariant pseudo-Riemannian metric on $\mathfrak{m}$, determined by real coefficients a,b,c,d, with $a(a-4d)\neq 0$. Putting $\Lambda[i]:=\Lambda(u_i)$ for all indices $i=1,\dots,4$, we find

$$\Lambda[1]=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&-\frac{b}{a}&-\frac{c}{a}&-1\end{pmatrix},\quad \Lambda[2]=\begin{pmatrix}0&-\frac{8bd}{a(a-4d)}&\frac{c}{a}&1\\-1&0&\frac{1}{2}&0\\0&-\frac{4bc}{a(a-4d)}&0&0\\-\frac{b}{a}&\frac{4bc}{a(a-4d)}&-\frac{b}{2a}&0\end{pmatrix},$$
$$\Lambda[3]=\begin{pmatrix}0&\frac{c}{a}&0&0\\0&\frac{1}{2}&0&0\\0&0&0&0\\-\frac{c}{a}&-\frac{b}{2a}&0&-\frac{1}{2}\end{pmatrix},\quad \Lambda[4]=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \tag{3.1}$$

Consequently, the curvature tensor is completely determined by

$$R_{12}=\begin{pmatrix}0&\frac{b(a+20d)}{\alpha\,(a-4d)}&-\frac{c}{a}&-1\\1&0&-\frac{1}{2}&0\\0&\frac{12b}{a-4d}&0&0\\\frac{4b}{a}&-\frac{12bc}{a(a-4d)}&\frac{b}{a}&0\end{pmatrix},\qquad R_{13}=\begin{pmatrix}0&-\frac{c}{a}&0&0\\0&0&0&0\\0&0&0&0\\\frac{c}{a}&0&-\frac{c}{2a}&0\end{pmatrix},$$

$$R_{23}=\begin{pmatrix}0&-\frac{b(a+4d}{2a(a-4d)}&-\frac{c}{2a}&-\frac{1}{2}\\\frac{1}{2}&-\frac{c}{a}&-\frac{1}{4}&0\\0&-\frac{2b}{a-4d}&0&0\\-\frac{b}{a}&-\frac{bc(3a-4d)}{a^2(a-4d)}&\frac{c^2}{a^2}&\frac{c}{a}\end{pmatrix},\qquad R_{24}=\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&\frac{b}{a}&\frac{c}{a}&1\end{pmatrix},$$

$$R_{14}=R_{34}=0.$$

Therefore, with respect to $\{u_i\}$, the Ricci tensor ϱ and the Ricci operator Q are given by

(3.2)
$$\varrho = \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{2b(a+12d)}{a(a-4d)} & -\frac{2c}{a} & -2 \\ \frac{1}{2} & -\frac{2c}{a} & -\frac{1}{4} & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} -a^{-1} & 0 & 0 & 0 \\ 0 & -2\,a^{-1} & 0 & \frac{4b(a+4d)}{a^2(a-4d)} \\ \frac{1}{2}a^{-1} & 0 & 0 & -\frac{2c}{a^2} \\ 0 & 0 & 0 & -2\,a^{-1} \end{pmatrix},$$

respectively. In particular, it is easily seen that the Ricci eigenvalues are then given by

$$-\frac{2}{a}, \quad -\frac{2}{a}, \quad -\frac{1}{a}, \quad 0$$

and that Q is diagonalizable if and only if either $b = 0$ or $a + 4d = 0$.

(A2) Let $E = \{h_1, u_1, \dots, u_4\}$ be the basis of the Lie algebra $\mathfrak{a}_2$ used in the previous section and let g be an arbitrary invariant pseudo-Riemannian metric on $\mathfrak{m}$, determined by real coefficients a, b, c, d, with $ad \neq 0$. A direct calculation gives

(3.3)
$$\Lambda[1] = \begin{pmatrix} 0 & 0 & \frac{\alpha c}{d} & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha a}{d} & 0 \end{pmatrix}, \qquad \Lambda[2] = \begin{pmatrix} 0 & -\frac{\alpha c}{d} & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha a}{d} & 0 & 0 \end{pmatrix},$$

$$\Lambda[3] = \begin{pmatrix} \frac{\alpha c}{d} & 0 & -\frac{(\alpha-1)bc}{ad} & -\frac{\alpha c^2-bd}{ad} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ \frac{\alpha a}{d} & 0 & -\frac{(\alpha-1)b}{d} & -\frac{\alpha c}{d} \end{pmatrix}, \quad \Lambda[4] = \begin{pmatrix} -1 & 0 & -\frac{\alpha c^2-bd}{ad} & -\frac{(\alpha-1)c}{a} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\alpha c}{d} & 0 \end{pmatrix}.$$

We then calculate the curvature matrices R_{ij}, and we find

$$R_{12} = \begin{pmatrix} 0 & -\frac{\alpha^2 a}{d} & 0 & 0 \\ 0 & 0 & -\frac{\alpha^2 a}{d} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{13} = \begin{pmatrix} \frac{\alpha^2 a}{d} & 0 & -\frac{\alpha^2 b}{d} & -\frac{\alpha^2 c}{d} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\alpha^2 a}{d} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{14} = \begin{pmatrix} 0 & 0 & -\frac{\alpha^2 c}{d} & -\alpha^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\alpha^2 a}{d} & 0 \end{pmatrix}, \quad R_{23} = \begin{pmatrix} 0 & \frac{\alpha b}{d} & 0 & 0 \\ \frac{\alpha^2 a}{d} & 0 & -\frac{\alpha(\alpha-1)b}{d} & -\frac{\alpha^2 c}{d} \\ 0 & \frac{\alpha^2 a}{d} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\alpha^2 c}{d} & -\alpha^2 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha^2 a}{d} & 0 & 0 \end{pmatrix}, \quad R_{34} = \begin{pmatrix} 0 & 0 & -\frac{2(\alpha-1)bc}{ad} & -\frac{2(\alpha-1)\,b}{a} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\alpha^2 c}{d} & -\alpha^2 \\ \frac{(1-2\alpha)\alpha a}{d} & 0 & \frac{(\alpha^2-2\alpha+2)b}{d} & \frac{\alpha^2 c}{d} \end{pmatrix}.$$

Then the Ricci tensor ϱ and the Ricci operator Q with respect to $\{u_i\}$ are described by

(3.4)
$$\varrho = \begin{pmatrix} 0 & 0 & \frac{3\alpha^2 a}{d} & 0 \\ 0 & -\frac{3\alpha^2 a}{d} & 0 & 0 \\ \frac{3\alpha^2 a}{d} & 0 & -\frac{b(3\,\alpha^2-3\,\alpha+2)}{d} & -\frac{3\alpha^2 c}{d} \\ 0 & 0 & -\frac{3\alpha^2 c}{d} & -3\,\alpha^2 \end{pmatrix},$$
$$Q = \begin{pmatrix} -\frac{3\alpha^2}{d} & 0 & 0 & 0 \\ 0 & -\frac{3\alpha^2}{d} & 0 & 0 \\ -\frac{b(3\,\alpha-2)}{ad} & 0 & -\frac{3\alpha^2}{d} & 0 \\ 0 & 0 & 0 & -\frac{3\alpha^2}{d} \end{pmatrix},$$

The Ricci eigenvalues are all equal to $-\frac{3\alpha^2}{d}$ and Q is diagonalizable if and only if either $b = 0$ or $\alpha = 2/3$.

(A3) All the remaining cases will be treated in the same way. For this reason, unless it is particularly relevant (see Proposition 4.3 below), we shall omit to report the curvature matrices R_{ij}. Let $E = \{h_1, u_1, \ldots, u_4\}$ be the basis of the Lie algebra $\mathfrak{a}_3$ we introduced in the previous section and let g be an invariant pseudo-Riemannian metric on $\mathfrak{m}$, determined by real coefficients a, b, c, d, satisfying $ab \neq 0$. We find

(3.5)
$$\Lambda[1] = \begin{pmatrix} 0 & 0 & 1 & \frac{c}{b} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{b} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Lambda[2] = \begin{pmatrix} 0 & \frac{c}{b} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{a}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\Lambda[3] = \begin{pmatrix} -1 & 0 & 0 & \frac{c^2-bd}{ab} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{c}{ab} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda[4] = \begin{pmatrix} \frac{c}{b} & 0 & \frac{c^2-bd}{ab} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a}{b} & 0 & -\frac{c}{b} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and for the Ricci tensor and the Ricci operator

(3.6)
$$\varrho = \begin{pmatrix} 0 & 0 & 0 & -\frac{3a}{b} \\ 0 & -\frac{3a}{b} & 0 & 0 \\ 0 & 0 & -3 & -\frac{3c}{b} \\ -\frac{3a}{b} & 0 & -\frac{3c}{b} & -\frac{2d}{b} \end{pmatrix},$$
$$Q = \begin{pmatrix} -3\,b^{-1} & 0 & 0 & 0 \\ 0 & -3\,b^{-1} & 0 & 0 \\ 0 & 0 & -3\,b^{-1} & 0 \\ \frac{d}{ab} & 0 & 0 & -3\,b^{-1} \end{pmatrix}.$$

The Ricci eigenvalues are all equal to $-3\,b^{-1}$, and Q is diagonalizable if and only if $d = 0$.

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(A4) Consider the basis $E = \{h_1, h_2, u_1, \dots, u_4\}$ of the Lie algebra $\mathfrak{a}_4$ as in the previous section and an invariant pseudo-Riemannian metric g on $\mathfrak{m}$, determined by real coefficients a, b with $a \neq 0$. We get

$$(3.7)\qquad \begin{gathered} \Lambda[1] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{b}{a} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda[2] = \begin{pmatrix} 0 & \frac{2b}{a} & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -\frac{b}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Lambda[3] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Lambda[4] = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \end{gathered}$$

Thus, the Ricci curvature is described as follows:

$$(3.8)\qquad \varrho = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -\frac{8b}{a} & -3 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$Q = \begin{pmatrix} -2\,a^{-1} & 0 & 0 & 0 \\ 0 & -3\,a^{-1} & -\frac{5b}{a^2} & 0 \\ 0 & 0 & -3\,a^{-1} & 0 \\ 0 & 0 & 0 & -2\,a^{-1} \end{pmatrix}.$$

The Ricci eigenvalues are

$$-\frac{2}{a}, \quad -\frac{2}{a}, \quad -\frac{3}{a}, \quad -\frac{3}{a},$$

and Q is diagonalizable if and only if $b = 0$.

(A5) Let $E = \{h_1, h_2, h_3, u_1, \dots, u_4\}$ be the basis of the Lie algebra $\mathfrak{a}_5$ that we introduced in the previous section and let g be an invariant pseudo-Riemannian metric on $\mathfrak{m}$, determined by a real coefficient $a \neq 0$. For the Levi-Civita connection and curvature, we find

$$(3.9)\qquad \begin{gathered} \Lambda[1] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda[2] = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Lambda[3] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda[4] = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} \end{gathered}$$

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and

(3.10)

$$\varrho = \begin{pmatrix} -8 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} -8\,a^{-1} & 0 & 0 & 0 \\ 0 & -8\,a^{-1} & 0 & 0 \\ 0 & 0 & -8\,a^{-1} & 0 \\ 0 & 0 & 0 & -8\,a^{-1} \end{pmatrix}.$$

(B1) We now consider the cases with metrics having signature $(2,2)$. Let $E = \{h_1, u_1, \dots, u_4\}$ be the basis of the Lie algebra $\mathfrak{b}_1$ we used in the previous section and let g be an invariant pseudo-Riemannian metric on $\mathfrak{m}$, determined by real coefficients a, b, c, d, with $a \neq 0$. Then we obtain

(3.11)

$$\Lambda[1] = \begin{pmatrix} -1 & -\frac{c}{2a} & -\frac{d}{a} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{b}{a} & -\frac{3c}{2a} & -1 \end{pmatrix}, \quad \Lambda[2] = \begin{pmatrix} -\frac{c}{2a} & \frac{c^2-2bd}{a^2} & -\frac{cd}{a^2} & -\frac{d}{2a} \\ -1 & -\frac{c}{a} & -\frac{d}{2a} & 0 \\ 0 & \frac{2b}{a} & \frac{3c}{2a} & 1 \\ -\frac{b}{a} & -\frac{bc}{a^2} & \frac{bd-3\,c^2}{2a^2} & 0 \end{pmatrix},$$

$$\Lambda[3] = \begin{pmatrix} -\frac{d}{a} & -\frac{cd}{a^2} & -\frac{d^2}{a^2} & 0 \\ 0 & -\frac{d}{2a} & 0 & 0 \\ 0 & \frac{3c}{2a} & \frac{d}{a} & 0 \\ -\frac{3c}{2a} & \frac{bd-3\,c^2}{2a^2} & -\frac{cd}{a^2} & \frac{d}{2a} \end{pmatrix}, \quad \Lambda[4] = \begin{pmatrix} 0 & -\frac{d}{2a} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{d}{2a} & 0 \end{pmatrix}.$$

Then the curvature is described by

$$R_{12} = \begin{pmatrix} \frac{3c}{2a} & \frac{22\,bd-15\,c^2}{4a^2} & \frac{3cd}{2a^2} & 0 \\ 0 & \frac{3c}{2a} & 0 & 0 \\ 0 & -\frac{3b}{a} & -\frac{3c}{2a} & 0 \\ \frac{3b}{a} & \frac{3bc}{2a^2} & \frac{5\left(3c^2-2\,bd\right)}{4a^2} & -\frac{3c}{2a} \end{pmatrix}, \; R_{13} = \begin{pmatrix} \frac{d}{a} & \frac{5cd}{4a^2} & \frac{d^2}{a^2} & 0 \\ 0 & \frac{d}{2a} & 0 & 0 \\ 0 & -\frac{3c}{2a} & -\frac{d}{a} & 0 \\ \frac{3c}{2a} & \frac{3\,c^2-bd}{2a^2} & \frac{3cd}{4a^2} & -\frac{d}{2a} \end{pmatrix},$$

$$R_{23} = \begin{pmatrix} -\frac{cd}{4a^2} & -\frac{d\left(3c^2+5bd\right)}{4a^3} & 0 & \frac{d^2}{4a^2} \\ \frac{d}{2a} & \frac{cd}{4a^2} & \frac{d^2}{4a^2} & 0 \\ 0 & \frac{9c^2-10bd}{4a^2} & -\frac{cd}{4a^2} & -\frac{d}{2a} \\ \frac{8bd-9c^2}{4a^2} & \frac{9c\left(bd-c^2\right)}{4a^3} & \frac{3d\left(bd-c^2\right)}{2a^3} & \frac{cd}{4a^2} \end{pmatrix}, \; R_{14} = \begin{pmatrix} 0 & \frac{d}{2a} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{d}{2a} & 0 \end{pmatrix},$$

$$R_{24} = \begin{pmatrix} \frac{d}{2a} & \frac{3cd}{4a^2} & \frac{d^2}{2a^2} & 0 \\ 0 & \frac{d}{a} & 0 & 0 \\ 0 & -\frac{3c}{2a} & -\frac{d}{2a} & 0 \\ \frac{3c}{2a} & \frac{3\,c^2-2bd}{2a^2} & -\frac{d}{a} & \end{pmatrix}, \quad R_{34} = \begin{pmatrix} 0 & \frac{d^2}{4a^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{d^2}{4a^2} & 0 \end{pmatrix}$$

and so the Ricci curvature is described as follows:

$$(3.12)\quad \varrho = \begin{pmatrix} 0 & 0 & \frac{3d}{2a} & 0 \\ 0 & \frac{3(6bd-5c^2)}{2a^2} & \frac{3cd}{2a^2} & \frac{3d}{2a} \\ \frac{3d}{2a} & \frac{3cd}{2a^2} & \frac{3d^2}{2a^2} & 0 \\ 0 & \frac{3d}{2a} & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{3d}{2a^2} & 0 & 0 & 0 \\ 0 & \frac{3d}{2a^2} & 0 & \frac{15(bd-c^2)}{2a^3} \\ 0 & 0 & \frac{3d}{2a^2} & 0 \\ 0 & 0 & 0 & \frac{3d}{2a^2} \end{pmatrix}.$$

Hence, the Ricci eigenvalues are all equal to $\frac{3d}{2a^2}$, and Q is diagonalizable if and only if $bd - c^2 = 0$.

(B2) For the case of the Lie algebra $\mathfrak{b}_2$, given an invariant pseudo-Riemannian metric g on $\mathfrak{m}$ (determined by real coefficients $a \neq 0, b$), we find, with respect to the basis $E = \{h_1, h_2, u_1, \ldots, u_4\}$ introduced in the previous section,

$$(3.13)\quad \begin{aligned} \Lambda[1] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{b}{a} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda[2] = \begin{pmatrix} 0 & \frac{2b}{a} & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -\frac{b}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Lambda[3] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda[4] = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and for the Ricci curvature

(3.14)

$$\varrho = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -\frac{8b}{a} & -3 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} -2\,a^{-1} & 0 & 0 & 0 \\ 0 & -3\,a^{-1} & -\frac{5b}{a^2} & 0 \\ 0 & 0 & -3\,a^{-1} & 0 \\ 0 & 0 & 0 & -2\,a^{-1} \end{pmatrix}.$$

The Ricci eigenvalues are

$$-\frac{2}{a}, \quad -\frac{2}{a}, \quad -\frac{3}{a}, \quad -\frac{3}{a},$$

and Q is diagonalizable if and only if $b = 0$.

(B3) For the Lie algebra $\mathfrak{b}_3$, let $E = \{h_1, h_2, u_1, \ldots, u_4\}$ be the basis as in the previous section and g an invariant pseudo-Riemannian metric on $\mathfrak{m}$, determined by real coefficients $a \neq 0, b$. Then we get

$$(3.15)\quad \begin{aligned} \Lambda[1] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda[2] = \begin{pmatrix} 1 & 0 & \frac{b}{a} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ \Lambda[3] &= \begin{pmatrix} 0 & \frac{b}{a} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -\frac{2b}{a} & 0 \end{pmatrix}, \quad \Lambda[4] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

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and the curvature is described by

$$R_{12} = R_{13} = R_{14} = R_{24} = R_{34} = 0, \quad R_{23} = \begin{pmatrix} 0 & \frac{3b}{a} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3b}{a} & 0 \end{pmatrix}.$$

The Ricci tensor and operator are then given by

$$\varrho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4 Einstein-like Metrics

Einstein-like metrics were introduced and first studied in [16] by A. Gray. A pseudo-Riemannian manifold (M, g):

- belongs to class $\mathcal{A}$ if and only if its Ricci tensor ϱ is *cyclic-parallel*, that is,

(4.1) $$(\nabla_X \varrho)(Y,Z) + (\nabla_Y \varrho)(Z,X) + (\nabla_Z \varrho)(X,Y) = 0,$$

  for all vector fields X, Y, Z tangent to M. Condition (4.1) is equivalent to requiring that ϱ is a *Killing tensor*, that is,

(4.2) $$(\nabla_X \varrho)(X,X) = 0.$$

- belongs to class $\mathcal{B}$ if and only if its Ricci tensor is a *Codazzi tensor*, that is,

(4.3) $$(\nabla_X \varrho)(Y,Z) = (\nabla_Y \varrho)(X,Z).$$

Any manifold belonging to either class $\mathcal{A}$ or $\mathcal{B}$ has constant scalar curvature. Moreover, denoting respectively by $\mathcal{P}$ and $\mathcal{E}$ the class of Ricci-parallel manifolds and the one of Einstein spaces, one has $\mathcal{A} \cap \mathcal{B} = \mathcal{P} \supset \mathcal{E}$. However, $\mathcal{P} \neq \mathcal{E}$. In particular, in pseudo-Riemannian settings, there exist some manifolds with parallel Ricci tensor but neither Einstein nor locally decomposable. Some interesting Lorentzian examples can be found in [6].

In Riemannian settings, Einstein-like metrics of low-dimensional manifolds are related to natural reductivity and symmetry [4]. Consequently, to find some non-reductive homogeneous pseudo-Riemannian manifolds with Einstein-like metrics would point out a deep difference between these geometries. We will show that this is the case, classifying Einstein-like metrics on four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifolds.

4.1 Einstein Examples

In the previous sections, we explicitly described the metric tensor g and the Ricci tensor ϱ of each four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold, in terms of a suitable global basis of vector fields. Checking when these homogeneous spaces satisfy the Einstein equation $\varrho - \lambda g = 0$, we obtain the following classification result.

Theorem 4.1 *Let g be an invariant pseudo-Riemannian metric on a four-dimensional non-reductive homogeneous pseudo-Riemannian manifold $M = G/H$. Then g is Einstein and not of constant curvature if and only if one of the following cases occurs:*

(i) $\mathfrak{g} = \mathfrak{a}_2$, *$g$ is given by* (2.2) *and either* $\alpha = \frac{2}{3}$ *or* $b = 0 \neq \alpha$. *In both cases,* $\lambda = \frac{3\alpha^2}{d}$.
(ii) $\mathfrak{g} = \mathfrak{a}_3$, *$g$ is given by* (2.3) *and* $d = 0$. *In this case,* $\lambda = -\frac{3}{b}$ *and so g is never Ricci-flat.*
(iii) $\mathfrak{g} = \mathfrak{a}_5$ *and g is given by* (2.5). *In this case,* $\lambda = -\frac{8}{a}$ *and so g is never Ricci-flat.*
(iv) $\mathfrak{g} = \mathfrak{b}_1$, *$g$ is given by* (2.6) *and* $c^2 - bd = 0$ *but* $(b, c, d) \neq (0, 0, 0)$. *In this case,* $\lambda = \frac{3d}{2a^2}$ *and so, g is Ricci-flat if and only if* $(c =)d = 0$.
(v) $\mathfrak{g} = \mathfrak{b}_3$ *and g is given by* (2.8) *with* $b \neq 0$. *In this case,* $\lambda = 0$, *that is, g is Ricci-flat.*

Remark 4.2 (a) The classification of Einstein examples given in [14, Theorem 2.6] for the simply connected case (which excludes examples corresponding to Lie algebra $\mathfrak{a}_5$) only reported cases (ii), with $\alpha = \frac{2}{3}$, and (v). Theorem 4.1 shows that other cases occur.

(b) In the statement of Theorem 4.1, following [14, Theorem 2.6], we excluded the cases of constant sectional curvature. To be more precise, starting from the explicit description of the curvature of these spaces we made in the previous section, standard calculations lead to the following result.

Proposition 4.3 *Let g be an invariant pseudo-Riemannian metric on a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold $M = G/H$. Then the following conditions are equivalent:*

(i) (M, g) *is flat;*
(ii) (M, g) *has constant sectional curvature;*
(iii) (M, g) *is conformally flat;*
(iv) *one of the following cases occurs:*
 (a) $\mathfrak{g} = \mathfrak{a}_2$ *and g is given by* (2.2) *with* $\alpha = b = 0$*;*
 (b) $\mathfrak{g} = \mathfrak{b}_1$ *and g is given by* (2.6) *with* $b = c = d = 0$*;*
 (c) $\mathfrak{g} = \mathfrak{b}_3$ *and g is given by* (2.8) *with* $b = 0$.

In particular, conformally flat, four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifolds are necessarily flat.

4.2 Ricci-parallel, Class $\mathcal{A}$ and $\mathcal{B}$ Examples

We now classify Einstein-like metrics of four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifolds, starting from the classification of the corresponding Lie algebras $\mathfrak{a}_i, \mathfrak{b}_j$ and the description of the Levi-Civita connection and Ricci tensor of their invariant metrics.

Because of their tensorial character, conditions $\nabla\varrho = 0$ (characterizing Ricci-parallel examples), (4.2) and (4.3) can be checked with respect to a chosen basis of vector fields. With respect to the global bases $\{u_i\}$ described in Section 2, in Section 3 we described the Levi-Civita connection and the Ricci tensor of four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds, calculating $\Lambda[i] := \Lambda(u_i)$ and $\varrho_{ij} := \varrho(u_i, u_j)$. Thus, we can now calculate the components of the covariant derivative of the Ricci tensor given by

$$\nabla_i \varrho_{jk} = \sum_{r=1}^{4} \big(-\Lambda[k]_{ri}\varrho_{rj} - \Lambda[k]_{rj}\varrho_{ri} \big).$$

(A1) For a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{a}_1$, from (3.1), (3.2) we easily find that the only possibly non-vanishing components of $\nabla\varrho$ with respect to $\{u_i\}$ are

$$\begin{array}{llll} \nabla_1\varrho_{22} = -\frac{8b(a+4d)}{a(a-4d)}, & \nabla_1\varrho_{22} = \frac{2b(a+12d)}{a(a-4d)}, & \nabla_1\varrho_{13} = -\frac{c}{a}, & \nabla_1\varrho_{14} = -1, \\ \nabla_2\varrho_{23} = -\frac{b(3a+4d)}{a(a-4d)}, & \nabla_2\varrho_{33} = \frac{c}{a}, & \nabla_2\varrho_{34} = \frac{1}{2}, & \nabla_3\varrho_{12} = \frac{c}{a}, \\ \nabla_3\varrho_{22} = -\frac{4b(a+4d)}{a(a-4d)}, & \nabla_3\varrho_{22} = \frac{c}{2a} & & \end{array} \tag{4.4}$$

and the ones obtained by using the symmetries of $\nabla\varrho$. Then (4.4) easily yields that *four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifolds corresponding to Lie algebra $\mathfrak{a}_1$ do not belong to either class $\mathcal{A}$ or $\mathcal{B}$. In particular, they are not Ricci-parallel.*

(A2) Equations (3.3) and (3.4) yield that in this case, $\nabla\varrho$ is completely determined by the possibily non-vanishing components

$$\nabla_3\varrho_{34} = -\frac{b\alpha\ (3\,\alpha - 2)}{d}, \quad \nabla_4\varrho_{33} = -2\,\frac{b\,(3\,\alpha - 2)}{d}.$$

Thus, also taking into account Theorem 4.1 and Proposition 4.3, we have the following theorem.

Theorem 4.4 *A four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{a}_2$*

(i) *is Ricci-parallel if and only if it is Einstein;*
(ii) *belongs to class $\mathcal{A}$ but is not Ricci-parallel if and only if $\alpha = -1$ and $b \neq 0$;*
(iii) *belongs to class $\mathcal{B}$ but is not Ricci-parallel if and only if $\alpha = 2$ and $b \neq 0$.*

(A3) We deduce from equations (3.5) and (3.6) that $\nabla\varrho$ is now determined by

$$\nabla_3\varrho_{44} = -\frac{2d}{b}, \quad \nabla_4\varrho_{34} = -\frac{d}{b}.$$

Therefore, we have the following theorem.

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Theorem 4.5 *For a four-dimensional non-reductive homogeneous pseudo-Riemannian manifold corresponding to Lie algebra* $\mathfrak{a}_3$*, the following conditions are equivalent:*

(i) *the manifold belongs to class* $\mathcal{A}$*;*
(ii) *the manifold belongs to class* $\mathcal{B}$*;*
(iii) *the manifold is Ricci-parallel;*
(iv) *the manifold is Einstein;*
(v) *the pseudo-Riemannian metric g described in* (2.3) *satisfies* $d = 0$.

(A4) By (3.7) and (3.8), we have that tensor $\nabla\varrho$ is determined by

$$\nabla_1\varrho_{22} = \frac{10b}{a}, \quad \nabla_2\varrho_{12} = -\frac{7b}{a}, \quad \nabla_2\varrho_{13} = -1.$$

Hence, we easily deduce that *four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds corresponding to case* $\mathfrak{a}_4$ *never belong to either class* $\mathcal{A}$ *or* $\mathcal{B}$*. In particular, they are not Ricci-parallel.*

(A5) By Theorem 4.1, non-reductive, homogeneous, pseudo-Riemannian, manifolds corresponding to Lie algebra $\mathfrak{a}_5$ are Einstein. In particular, they are Ricci-parallel. The same conclusion can be obtained by direct calculation using equations (3.9) and (3.10).

(B1) For non-reductive, homogeneous, pseudo-Riemannian, manifolds corresponding to Lie algebra $\mathfrak{b}_1$, by (3.11) and (3.12) we get that tensor $\nabla\varrho$ is determined by

$$\begin{aligned} &\nabla_1\varrho_{22} = \tfrac{15(c^2-bd)}{a^2}, \qquad \nabla_2\varrho_{12} = \tfrac{15(bd-c^2)}{2a^2}, \qquad \nabla_2\varrho_{22} = \tfrac{15c\left(bd-c^2\right)}{a^3}, \\ &\nabla_2\varrho_{23} = \tfrac{15d\left(bd-c^2\right)}{4a^3}, \quad \nabla_3\varrho_{22} = \tfrac{15d\left(bd-c^2\right)}{2a^3}. \end{aligned}$$

Taking into account Theorem 4.1 and Proposition 4.3, we then easily prove the following.

Theorem 4.6 *For a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra* $\mathfrak{b}_1$*, the following conditions are equivalent:*

(i) *the manifold belongs to class* $\mathcal{A}$*;*
(ii) *the manifold belongs to class* $\mathcal{B}$*;*
(iii) *the manifold is Ricci-parallel;*
(iv) *the manifold is Einstein;*
(v) *the pseudo-Riemannian metric g described in* (2.6) *satisfies* $c^2 - bd = 0$.

(B2) Equations (3.13) and (3.14) yield that $\nabla\varrho$ is determined by

$$\nabla_1\varrho_{22} = \frac{10\,b}{a}, \quad \nabla_2\varrho_{12} = -\frac{7\,b}{a}, \quad \nabla_2\varrho_{13} = -1,$$

from which we easily see that *four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds corresponding to Lie algebra* $\mathfrak{b}_2$ *do not belong to either class* $\mathcal{A}$ *or* $\mathcal{B}$*. In particular, they are not Ricci-parallel.*

(B3) Theorem 4.1 and Proposition 4.3 yield that non-reductive homogeneous pseudo-Riemannian manifolds corresponding to Lie algebra $\mathfrak{b}_3$ are Ricci-flat. In particular, they are Ricci-parallel.

5 Invariant Killing Vector Fields and Ricci Solitons

We shall now look for homogeneous solutions of equation (1.1) for four-dimensional non-reductive, homogeneous, pseudo-Riemannian manifolds corresponding to Lie algebras $\mathfrak{a}_i, \mathfrak{b}_j$.

As we already described the Ricci tensor of these manifolds, it suffices to determine, with respect to the same bases, the Lie derivative $\mathcal{L}_V g$ of the metric tensor g in the direction of an arbitrary vector field $V \in \mathfrak{m}$, pointing out the invariant examples. This also gives us the opportunity to investigate the existence of invariant Killing vector fields, which is interesting in itself under several points of view [22].

In the different cases corresponding to Lie algebras $\mathfrak{a}_i, \mathfrak{b}_j$, we shall always refer to the global bases $\{u_i\}$ introduced in Section 2. With respect to $\{u_i\}$, the Lie derivative $\mathcal{L}_V g$ of the metric tensor with respect to a vector field $V = V_i u_i \in \mathfrak{m}$ is determined by the matrix

$$a_{ij} := \sum_{k=1}^{4}\Big(V_k \Big(\sum_{r=1}^{4} \big(\Lambda[i]_{rk} g_{rj} + \Lambda[j]_{rk} g_{ri} \big) \Big) \Big).$$

(A1) Let $(M = G/H, g)$ be a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{a}_1$ and $V = V_i u_i \in \mathfrak{m}$. Then from (2.1) and (3.1) we obtain

$$\mathcal{L}_V g = \begin{pmatrix} 0 & 2bV_2 & 2cV_2 & aV_2 \\ 2bV_2 & -4bV_1 & -2cV_1 - aV_4 & -aV_1 + \frac{1}{2}aV_3 \\ 2cV_2 & -cV_1 - aV_4 & 0 & \frac{1}{2}aV_2 \\ aV_2 & -aV_1 + \frac{1}{2}aV_3 & \frac{1}{2}aV_2 & 0 \end{pmatrix}.$$

Thus, we can easily prove the following theorem.

Theorem 5.1 *Consider a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{a}_1$. A vector field $V \in \mathfrak{m}$ is Killing if and only if $b = 0$ and $V = k(u_1 + 2\,u_2 - 2\frac{c}{a}\,u_3)$, for a real constant k. Except for the trivial case $V = 0$, V is not invariant.*

(A2) Equations (2.2) and (3.3) imply that in this case,

$$\mathcal{L}_V g = \begin{pmatrix} 0 & 0 & -2\alpha a V_4 & a(\alpha-1)V_3 \\ 0 & 2\alpha a V_4 & 0 & -\alpha a V_2 \\ -2\alpha a V_4 & 0 & 2b(\alpha-1)V_4 & \begin{matrix} a(\alpha+1)V_1 \\ -(\alpha-1)(bV_3 - cV_4) \end{matrix} \\ a(\alpha-1)V_3 & -\alpha a V_2 & \begin{matrix} a(\alpha+1)V_1 \\ -(\alpha-1)(bV_3 - cV_4) \end{matrix} & -2c(\alpha-1)V_3 \end{pmatrix}.$$

Then, solving equation $\mathcal{L}_V g = 0$, we obtain the following theorem.

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Theorem 5.2 *Consider a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{a}_2$. A vector field $V \in \mathfrak{m}$ is Killing if and only if $\alpha = 0$ and either $b = 0$ and*

$$V = k\Big(\frac{c}{a}\,u_1 + u_4\Big) + \mu\, u_2,$$

or $b \neq 0$ and $V = \mu\, u_2$, where k, μ are real constants. In particular, if $b = 0$, then V is an invariant Killing vector field if and only if $\mu = 0$, while when $b \neq 0$, there are not invariant Killing vector fields (besides $V = 0$).

(A3) From equations (2.3), (3.5) we obtain

$$\mathcal{L}_V g = \begin{pmatrix} 0 & 0 & 0 & 2aV_3 \\ 0 & 2aV_3 & -aV_2 & 0 \\ 0 & -aV_2 & 0 & -2aV_1 \\ 2aV_3 & 0 & -2aV_1 & 0 \end{pmatrix}.$$

As $a \neq 0$ so that g is nondegenerate, we have at once the following theorem.

Theorem 5.3 *Consider a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{a}_3$. A vector field $V \in \mathfrak{m}$ is Killing if and only if $V = k\, u_4$, where k is a real constant. Unless $k = 0$, V is not invariant.*

(A4) Equations (2.4), (3.7) yield

$$\mathcal{L}_V g = \begin{pmatrix} 0 & 2bV_2 & aV_2 & \frac{1}{2}aV_4 \\ 2bV_2 & -4bV_1 & -aV_1 & 0 \\ aV_2 & -aV_1 & 0 & 0 \\ \frac{1}{2}aV_4 & 0 & 0 & -aV_1 \end{pmatrix}.$$

As $a \neq 0$ so that g is nondegenerate, we obtain the following theorem.

Theorem 5.4 *Consider a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{a}_4$. A vector field $V \in \mathfrak{m}$ is Killing if and only if $V = k\, u_3$, where k is a real constant. All these Killing vector fields are invariant.*

(A5) We start from equations (2.5) and (3.9), and we find

$$\mathcal{L}_V g = \begin{pmatrix} 0 & \frac{1}{2}aV_3 & 0 & \frac{1}{4}aV_4 \\ \frac{1}{2}aV_3 & 0 & -\frac{1}{2}aV_1 & 0 \\ 0 & -\frac{1}{2}aV_1 & 0 & 0 \\ \frac{1}{4}aV_4 & 0 & 0 & -\frac{1}{2}aV_1 \end{pmatrix},$$

and so we have the following theorem.

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Theorem 5.5 *Consider a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{a}_5$. Then $V \in \mathfrak{m}$ is Killing if and only if $V = k\,u_2$, where k is a real constant. Unless $k = 0$, V is not invariant.*

(B1) Now consider four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifolds corresponding to Lie algebra $\mathfrak{b}_1$. Then equations (2.6) and (3.11) yield

$$\mathcal{L}_V g = \begin{pmatrix} 2aV_3 & 2bV_2 + cV_3 & -aV_1 + 2cV_2 + dV_3 & aV_2 \\ 2bV_2 + cV_3 & -4bV_1 + 2cV_4 & -3cV_1 + dV_4 & -aV_1 - cV_2 \\ -aV_1 + 2cV_2 + dV_3 & -3cV_1 + dV_4 & -2dV_1 & -dV_2 \\ aV_2 & -aV_1 - cV_2 & -dV_2 & 0 \end{pmatrix}$$

and solving equation $\mathcal{L}_V g = 0$, we prove the following theorem.

Theorem 5.6 *Consider a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{b}_1$. A vector field $V \in \mathfrak{m}$ is Killing if and only if $c = d = 0$ and $V = k\,u_4$, for any real constant k. All these Killing vector fields are invariant.*

(B2) From equations (2.7) and (3.13) we get

$$\mathcal{L}_V g = \begin{pmatrix} 0 & 2bV_2 & aV_2 & -\frac{1}{2}aV_4 \\ 2bV_2 & -4bV_1 & -aV_1 & 0 \\ aV_2 & -aV_1 & 0 & 0 \\ -\frac{1}{2}aV_4 & 0 & 0 & aV_1 \end{pmatrix}.$$

As $a \neq 0$, we obtain the following theorem.

Theorem 5.7 *Consider a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{b}_2$. A vector field $V \in \mathfrak{m}$ is Killing if and only if $V = k\,u_3$, for any real constant k. All these Killing vector fields are invariant.*

(B3) Equations (2.8) and (3.15) yield

$$\mathcal{L}_V g = \begin{pmatrix} 0 & -aV_3 & aV_2 & 0 \\ -aV_3 & -2aV_4 & -2bV_3 & aV_2 \\ aV_2 & -2bV_3 & 4bV_2 & 0 \\ 0 & aV_2 & 0 & 0 \end{pmatrix}.$$

Since $a \neq 0$, we get at once the following theorem.

Theorem 5.8 *Consider a four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{b}_3$. A vector field $V \in \mathfrak{m}$ is Killing if and only if $V = k\,u_1$, for any real constant k. All these Killing vector fields are invariant.*

We can now classify homogeneous Ricci solitons among four-dimensional, non-reductive, pseudo-Riemannian, homogeneous manifolds. We solve equation (1.1) using the descriptions of invariant pseudo-Riemannian metrics g, their Ricci tensor ϱ, and the Lie derivative $\mathcal{L}_V g$ obtained above. Then we check whether these solutions correspond to invariant vector fields. Ruling out the trivial solutions of (1.1), corresponding to the case of Einstein manifolds (as classified in Theorem 4.1 and Proposition 4.3) with a vector field $V \in \mathfrak{m}$ which is either Killing or conformally Killing, we obtain the following theorem.

Theorem 5.9 *Let $(M = G/H, g)$ be a four-dimensional non-reductive homogeneous pseudo-Riemannian manifold.*

(i) *If M corresponds to one of Lie algebras $\mathfrak{a}_1, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{b}_2, \mathfrak{b}_3$, then (M, g) is not a non-trivial homogeneous Ricci soliton (for which equation (1.1) is satisfied by a vector field $V \in \mathfrak{m}$).*

(ii) *If M corresponds to the Lie algebra $\mathfrak{a}_2$ (with $\alpha \neq \frac{2}{3}$ and $b \neq 0$, otherwise is Einstein), then (M, g) is a non-trivial homogeneous Ricci soliton if and only if one of the following cases occurs:*

(a) *$\alpha \neq 0, \pm 1$ and $d \neq 0$. In this case, (1.1) holds for*

$$V = \frac{3\alpha - 2}{2d}\Big(-\frac{c(\alpha - 3)}{a(\alpha + 1)} u_1 + u_4 \Big) \quad \text{and} \quad \lambda = -\frac{2\alpha}{d}.$$

The Ricci soliton is either expanding, steady, or shrinking, depending on the sign of α and d, and V is invariant.

(b) *$\alpha = -1$ and $d \neq 0 = c$. In this case, (1.1) holds for*

$$V = k\, u_1 - \frac{5}{2d} u_4 \quad \text{and} \quad \lambda = \frac{2}{d},$$

for any $k \in \mathbb{R}$. The Ricci soliton is either expanding or shrinking, depending on the sign of d, and all vector fields V are invariant.

(c) *$\alpha = 1$ and $d \neq 0$. In this case, (1.1) holds for*

$$V = \frac{1}{2d}\Big(\frac{c}{a}\, u_1 + u_4 \Big) + k\, u_3 \quad \text{and} \quad \lambda = -\frac{2}{d},$$

for any $k \in \mathbb{R}$. The Ricci soliton is either expanding or shrinking, depending on the sign of d, and V is invariant if and only if $k = 0$.

(d) *$\alpha = 0$ and $d \neq 0$. In this case, (1.1) holds for*

$$V = -\frac{1}{d}\Big(\frac{3c}{a}\, u_1 + u_4 \Big) + k\, u_2 \quad \text{and} \quad \lambda = 0,$$

for any $k \in \mathbb{R}$. The Ricci soliton is steady, and V is invariant if and only if $k = 0$.

(iii) *If M corresponds to the Lie algebra $\mathfrak{b}_1$, then (M, g) is a non-trivial homogeneous Ricci soliton if and only if $d = 0 \neq c$. In this case, (1.1) holds for*

$$V = \frac{15c}{4a^2}\, u_4 \quad \text{and} \quad \lambda = 0.$$

Thus, the Ricci soliton is steady and V is invariant.

Remark 5.10 In all the non-trivial examples of three-dimensional, left-invariant, homogeneous Lorentzian Ricci solitons [3], the Ricci operator is not diagonalizable and has three equal eigenvalues. This could suggest that the existence of more invariant Ricci solitons in pseudo-Riemannian settings than in the Riemannian case is related to this special form of the Ricci operator.

As one can see from Section 3, also with regard to four-dimensional, non-reductive, invariant, pseudo-Riemannian Ricci solitons, in all examples we classified in Theorem 5.9 (cases $\mathfrak{a}_2, \mathfrak{b}_1$), the Ricci operator is again not diagonalizable with four equal eigenvalues. Four-dimensional, reductive, invariant Ricci solitons with different types of the Ricci operators (including the diagonalizable one) are presented in [9].

6 Invariant Symplectic, Complex and Kähler Structures

Invariant symplectic structures on a homogeneous space $M = G/H$ are in a one-to-one correspondence with non-degenerate, invariant, skew-symmetric, bilinear forms Ω on $\mathfrak{m}$ such that

$$d\Omega(x, yz) = -\Omega\big([x,y]_{\mathfrak{m}}, z\big) + \Omega\big([x,z]_{\mathfrak{m}}, y\big) - \Omega\big([y,z]_{\mathfrak{m}}, x\big) = 0,$$

for all $x, y, z \in \mathfrak{m}$ [20]. Hence, we can determine the matrices (Ω_{ij}) associated with invariant skew-symmetric bilinear forms Ω on $\mathfrak{m}$, with respect to the corresponding basis $\{u_i\}$, for every non-reductive four-dimensional homogeneous space. By a direct calculation, we get

$$\begin{pmatrix} 0 & \Omega_{12} & 0 & 0 \\ -\Omega_{12} & 0 & \Omega_{23} & 0 \\ 0 & -\Omega_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } (\mathbf{A1}), \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_{23} & 0 \\ 0 & -\Omega_{23} & 0 & \Omega_{34} \\ 0 & 0 & -\Omega_{34} & 0 \end{pmatrix} \text{ for } (\mathbf{A2}), (\mathbf{B3}),$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_{24} \\ 0 & 0 & 0 & \Omega_{34} \\ 0 & -\Omega_{24} & -\Omega_{34} & 0 \end{pmatrix} \text{ for } (\mathbf{A3}), \quad \begin{pmatrix} 0 & \Omega_{12} & 0 & 0 \\ -\Omega_{12} & 0 & 0 & \Omega_{24} \\ 0 & 0 & 0 & 0 \\ 0 & -\Omega_{24} & 0 & 0 \end{pmatrix} \text{ for } (\mathbf{A4}), (\mathbf{B2}),$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } (\mathbf{A5}), \quad \begin{pmatrix} 0 & \Omega_{12} & \Omega_{13} & 0 \\ -\Omega_{12} & 0 & \Omega_{23} & \Omega_{13} \\ -\Omega_{13} & -\Omega_{23} & 0 & \Omega_{3,4} \\ 0 & -\Omega_{1,3} & -\Omega_{34} & 0 \end{pmatrix} \text{ for } (\mathbf{B1}).$$

Therefore, there exist invariant non-degenerate 2-forms only on the non-reductive homogeneous space corresponding to case (**B1**). Moreover, in such a case we find that $d\Omega = 0$ if and only if $\Omega_{2,3} = 0$. Thus, we have the following theorem.

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Theorem 6.1 *A four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold $M = G/H$ admits an invariant symplectic form if and only if $\mathfrak{g} = \mathfrak{b}_1$. In this case, with respect to the basis $\{u_i\}$ of $\mathfrak{m}$, invariant symplectic forms are given by*

$$\Omega = \begin{pmatrix} 0 & \Omega_{12} & \Omega_{13} & 0 \\ -\Omega_{12} & 0 & 0 & \Omega_{13} \\ -\Omega_{13} & 0 & 0 & \Omega_{34} \\ 0 & -\Omega_{13} & -\Omega_{34} & 0 \end{pmatrix},$$

with $\Omega_{12}\Omega_{34} - \Omega_{13}^2 \neq 0$.

Next, invariant complex structures on a homogeneous spaces $M = G/H$ are in a one-to-one correspondence with complex structures J on $\mathfrak{m}$ that commute with the isotropy representation. We now prove the following theorem.

Theorem 6.2 *A four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold $M = G/H$ admits an invariant almost complex structure if and only if $\mathfrak{g} = \mathfrak{b}_1$. In particular, in this case there exists a two-parameter family of invariant complex structures $J_{t,s}$ described, with respect the basis $\{u_i\}$ of $\mathfrak{m}$, by*

$$J_{t,s} = \begin{pmatrix} -s & 0 & 0 & t \\ 0 & -s & -t & 0 \\ 0 & \frac{(1+s^2)}{t} & s & 0 \\ -\frac{(1+s^2)}{t} & 0 & 0 & s \end{pmatrix},$$

for arbitrary real constants s and $t \neq 0$.

Proof We start by determining, for every non-reductive four-dimensional homogeneous space G/H, the linear maps $J\colon \mathfrak{m} \to \mathfrak{m}$ such that $J \circ \rho(x) = \rho(x) \circ J$, for every $x \in \mathfrak{m}$. Then we can check whether these linear maps satisfy $J^2 = -id$ and the integrability conditions

$$[Ju_l, Ju_k] = [u_l, u_k] - J[Ju_l, u_k] - J[u_l, Ju_k],$$

for every $l, k = 1, \dots 4$. A direct calculation shows that there are no invariant almost complex structures unless $\mathfrak{g} = \mathfrak{b}_1$. For example, if $\mathfrak{g} = \mathfrak{a}_1$, then $J \circ \rho = \rho \circ J$ yields

$$J = \begin{pmatrix} a_{11} & 0 & 0 & \frac{1}{2}a_{33} - \frac{1}{2}a_{11} \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & a_{43} & a_{11} \end{pmatrix}$$

with respect to the basis $\{u_i\}$ of $\mathfrak{m}$, and none of these linear maps satisfies $J^2 = -id$. On the other hand, when $\mathfrak{g} = \mathfrak{b}_1$, we find that $J \circ \rho = \rho \circ J$ if and only if

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11} & -a_{14} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ -a_{32} & a_{42} & a_{43} & a_{33} \end{pmatrix},$$

and, imposing $J^2 = -id$, we obtain that in this case, invariant almost complex structures are of the form

$$\begin{pmatrix} -a_{33} & a_{43} & a_{13} & a_{14} \\ 0 & -a_{33} & -a_{14} & 0 \\ 0 & \frac{(1+a_{33}^2)}{a_{14}} & a_{33} & 0 \\ -\frac{(1+a_{33}^2)}{a_{14}} & -\frac{(a_{13}+a_{13}a_{33}^2-2a_{43}a_{33}a_{14})}{a_{14}^2} & a_{43} & a_{33} \end{pmatrix}.$$

Finally, it is easy to check that such an almost complex structure is integrable if and only if $a_{13} = a_{43} = 0$, and this completes the proof. ■

Results above imply that invariant pseudo-Hermitian (in particular, pseudo-Kähler) structures may only exist on the non-reductive four-dimensional homogeneous space $M = G/H$ corresponding to Lie algebra $\mathfrak{b}_1$. We shall now classify such structures on M.

Consider the complex structures $J_{t,s}$ described in Theorem 6.2 and the pseudo-Riemannian metrics g given by (2.6). Requiring the compatibility of $J_{t,s}$ with g, that is,

$$g(Ju_i, Ju_k) = g(u_i, u_k),$$

for every i, k, we get the following system of algebraic equations:

$$\begin{cases} bt^2s^2 + d(1+s^2)^2 - 2c(1+s^2)ts - bt^2 = 0, \\ bt^2s - 2c(1+s^2)t + s(1+s^2)d = 0, \\ bt^2 + ds^2 - 2cts - d = 0. \end{cases} \tag{6.1}$$

As $t \neq 0$, a straightforward calculation proves that the solutions of system (6.1) are given by $c = \frac{sd}{t}$, $b = \frac{(1+s^2)d}{t^2}$.

Next, let $\{u^i\}$ denote the basis dual to the basis $\{u_i\}$ of $\mathfrak{m}$. By Theorem 6.1, the generic invariant symplectic form on $\mathfrak{m}$ is given by

$$\Omega = \Omega_{12}u^1 \wedge u^2 + \Omega_{13}(u^1 \wedge u^3 + u^2 \wedge u^4) + \Omega_{34}u^3 \wedge u^4.$$

In order to have a pseudo-Kähler structure, we must require the compatibility of Ω with $J_{t,s}$, that is,

$$\Omega(J_{t,s}u_l, J_{t,s}u_k) = \Omega(u_l, u_k),$$

for every $l, k = 1, \dots, 4$. We get the equations

$$\begin{cases} -2t(1+s^2)\Omega_{13} + t^2s\Omega_{12} + s(1+s^2)\Omega_{34} = 0, \\ t^2(s^2-1)\Omega_{12} - 2ts(1+s^2)\Omega_{13} + (1+s^2)^2\Omega_{34} = 0, \\ (s^2-1)\Omega_{34} + t^2\Omega_{12} - 2st\Omega_{13} = 0, \end{cases}$$

which easily yield $\Omega_{12} = \frac{(1+s^2)}{t^2}\Omega_{34}$, $\Omega_{13} = \frac{s}{t}\Omega_{34}$. Note that

$$\Omega(u_3, J_{t,s}u_3) = \Omega(u_3, -tu_2 + su_3) = 0, \quad \Omega(u_2, J_{t,s}u_3) = s\Omega_{23} = 0,$$

and for the associated pseudo-Kähler metric g we have

$$b = c = d = 0, \quad a = \frac{\Omega_{34}}{t}.$$

Hence, taking into account Proposition 4.3, we proved the following theorem.

Theorem 6.3 *Let $M = G/H$ denote the four-dimensional, non-reductive, homogeneous, pseudo-Riemannian manifold corresponding to Lie algebra $\mathfrak{g} = \mathfrak{b}_1$ and let g, $J_{t,s}$, and Ω be the invariant pseudo-Riemannian metrics, complex structures, and symplectic forms over M, respectively. Then $(M, g, J_{t,s})$ is a homogeneous pseudo-Hermitian manifold if and only if*

$$c = \frac{sd}{t}, \quad b = \frac{(1+s^2)d}{t^2}.$$

Hence, any invariant complex structure $J_{t,s}$ on M determines a two-parameter family of pseudo-Hermitian structures, depending on two arbitrary real constants $a \neq 0$ and d.

In particular, each complex structure $J_{t,s}$ corresponds to a one-parameter family of pseudo-Kähler structures $(g_a, J_{t,s}, \Omega_a)$, where g_a is the (flat) pseudo-Riemannian metric determined by conditions $b = c = d = 0$, and Ω is completely determined by coefficients $\Omega_{12} = \frac{a(1+s^2)}{t}, \Omega_{34} = at, \Omega_{13} = as$.

Moreover, from Theorems 5.9 and 6.3 we get at once the following corollary.

Corollary 6.4 *Four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds do not admit (non-trivial) pseudo-Hermitian homogeneous Ricci solitons.*

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