ALTERNATIVE METRIZATION PROOFS

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Alternative methods of proving several classical metrization theorems are offered in this paper, showing that they follow by elementary methods from an early theorem of Alexandroff and Urysohn. A simplified proof of the latter theorem is also given. Theorem 5 and a corollary to Theorem 3 state the main results.

A metric d for a set X is a non-negative function on $X \times X$ satisfying, for any x, y, z in X:

$$d(x, y) = 0 \Leftrightarrow x = y, \tag{1}$$

$$d(x, y) = d(y, x), \text{ and} \tag{2}$$

$$d(x, y) + d(y, z) \geqslant d(x, z). \tag{3}$$

For $r > 0$ and $x \in X$, the set $S_d(x, r) = \{y: d(x, y) < r\}$ is called the sphere about x of radius r. The collection of all spheres is a basis for a unique topology on X, called the metric topology induced by d. (A subcollection B of a topology T is a *basis* for T if and only if each member of T, i.e. open set, is the union of some subcollection of B. Equivalently, whenever $x \in U \in T$, a member of B contains x and is a subset of U.) A topological space (X, T) is said to be *metrizable* if there exists a metric for X that induces the topology T.

In 1925 Urysohn **(11)** established the important result that a Hausdorff space with a countable basis is metrizable if and only if it is normal; Tychonoff **(10)** showed the next year that "normal" may be replaced by "regular." Urysohn's proof depended on his famous "lemma" stating that, in a normal space, for any disjoint pair of closed sets there exists a continuous function on the space into the unit interval which takes the value 0 on one closed set and 1 on the other. Thus a class of continuous functions was available to define a metric.

Similar methods were later used by R. H. Bing **(2)** to obtain a necessary and sufficient condition for metrizability of arbitrary topological spaces (see Theorem 5(iii)), and by J. Nagata **(7)** and Yu. Smirnov **(8)** in different ways to improve this condition (Theorem 5(iv)).

The use of Urysohn's lemma and the step of establishing normality are bypassed in the present proofs, which exploit the fact that these conditions involve certain countability requirements on the basis for the topology. By manipulating the basis elements directly, we link the conditions to the

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Alexandroff–Urysohn theorem **(1)** of 1923 (Theorem 5(i)), which in turn has a straightforward "geometric" proof.

1. Developments for a topological space. Let (X, T) be a topological space, $x \in X$, $A \subset X$, and K be a collection of sets covering X. Then $\mathrm{Star}(A, K)$ denotes the union of all those members of K that intersect A, and $\mathrm{Star}(x, K)$ denotes $\mathrm{Star}(\{x\}, K)$.

A sequence will be understood to be a function whose domain is ω, the set of non-negative integers. A sequence G is a *development* for (X, T), or simply X, provided (i) for each $n \in \omega$, G_n is a collection of open sets covering X, and (ii) for every $U \in T$ and $x \in U$ there exists $n \in \omega$ such that

$$\mathrm{Star}(x, G_n) \subset U.$$

Thus in particular $\bigcup\{G_n : n \in \omega\}$ is a basis for the topology. If such a sequence exists, X is said to admit the development G.

The following special types of developments G will be considered:

Regular (after Alexandroff–Urysohn). For each $n \in \omega$, whenever two members of G_{n+1} intersect, their union is a subset of some member of G_n.

Neighbourhood-star. For each $U \in T$ and $x \in U$, there exist $n \in \omega$ and a neighbourhood N of x such that $\mathrm{Star}(N, G_n) \subset U$.

Any development G can be improved in that another development H for the space may be constructed such that $H_0 \supset H_1 \supset \ldots$ and also H_n refines G_n for each n. (A covering K *refines* a covering K' provided each member of K is a subset of some member of K'. To obtain H, first let $F_0 = G_0$ and

$$F_{n+1} = \{g \cap f : g \in G_{n+1}, f \in F_n\} \qquad \text{for } n \in \omega,$$

so that F_{n+1} refines both F_n and G_{n+1}. Then define $H_m = \bigcup\{F_i : i \geqslant m\}$ for each $m \in \omega$.) Thus if G is a neighbourhood-star development, so is H.

THEOREM 1. *A topological space admits a regular development if and only if it admits a neighbourhood-star development.*

Proof. Any regular development H for (X, T) is also a neighbourhood-star development. For if $x \in U \in T$, an n may be chosen so that $\mathrm{Star}(x, H_{n-1}) \subset U$. But since H is regular,

$$\mathrm{Star}(\mathrm{Star}(x, H_n), H_n) \subset \mathrm{Star}(x, H_{n-1}) \subset U.$$

Thus the required neighbourhood of x is $\mathrm{Star}(x, H_n)$.

Now suppose that G is a neighbourhood-star development for X. It may be assumed that $G_0 \supset G_1 \supset \ldots$. For convenience, we say that a member g of G_0 is "little" in a covering K provided whenever a member g' of $\bigcap\{G_n : g \in G_n\}$ intersects g, $g \cup g'$ is a subset of an element of K. Define $H_0 = G_0$ and for each $n \in \omega$, let H_{n+1} consist of all members of G_{n+1} that are little in H_n. If it is granted that each H_n covers X, H is clearly a development for X. But H is

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regular, for suppose $n \in \omega$ and h, h' are members of H_{n+1} that intersect. Since both h and h' are little in H_n and either

$$h' \in \bigcap \{G_m : h \in G_m\} \quad \text{or} \quad h \in \bigcap \{G_m : h' \in G_m\}$$

(or both) hold, it follows that $h \cup h'$ is a subset of a member of H_n.

The proof is completed by showing inductively that each H_n is a covering. Clearly H_0 covers X. Assume that $n \in \omega$ and H_n covers X, and choose any $x \in X$. Then a member h of H_n contains x. Since $G_0 \supset G_1 \supset \ldots$, there exists a neighbourhood N of x and an integer m larger than n such that

$$\operatorname{Star}(N, G_i) \subset h$$

whenever $i \geqslant m$. Then an integer $j \geqslant m$ may be chosen such that

$$\operatorname{Star}(x, G_j) \subset N.$$

Finally choose a g_x in G_j such that $x \in g_x$. Now g_x is little in H_n, for if

$$g' \in \bigcap \{G_k : g_x \in G_k\}$$

and g' intersects g_x, we have $g' \in G_j$ and

$$g_x \cup g' \subset \operatorname{Star}(\operatorname{Star}(x, G_j), G_j) \subset \operatorname{Star}(N, G_j) \subset h \in H_n.$$

But also $g_x \in G_j \subset G_{n+1}$; so g_x is a member of H_{n+1} containing x. Thus H_{n+1} covers X, completing the induction.

2. The Alexandroff–Urysohn theorem. In 1917, Chittenden **(3)** proved that a space is metrizable if and only if the topology can be defined by a uniformly regular écart, i.e. a distance function d satisfying (1) and

(4) for some fixed function $f(\epsilon)$ that approaches 0 with ϵ, if $d(x, y) \leqslant \epsilon$ and $d(z, y) \leqslant \epsilon$, then $d(x, z) \leqslant f(\epsilon)$.

The following theorem was proved first as an easy corollary (with $f(\epsilon) = 2\epsilon$), but Frink **(5)** gave a greatly simplified direct proof in 1937. The proof here differs only in having a perhaps more natural method of showing that "chains" of open sets which have been assigned a sufficiently small numerical "length" really are short; still another method was given by Marion Smith **(9)**.

THEOREM 2 (Alexandroff–Urysohn). *A topological space is metrizable if and only if it is Hausdorff and admits a regular development.*

Proof. Trivially any metric space is Hausdorff, and if, for $n \in \omega$, G_n is the set of all spheres of radius 2^{-n}, the sequence G is clearly a regular development for the space.

To prove the converse, suppose (X, T) is a Hausdorff space and that G is a regular development for X. It is convenient to define a new development F as follows. Let

$$E_m = \bigcup \{G_i : i \geqslant m\}, \qquad m \in \omega.$$

Then let $F_0 = E_0 \cup \{X\}$ and $F_n = E_{2n}$ if $n > 0$. Any finite subcollection C of F_0 will be called a *chain* if its distinct elements may be ordered $f_1, \ldots, f_k$ in such a way that f_i intersects f_{i+1} whenever $0 \leqslant i < k$. Let C^* denote the union of the members of C. It is evident that E is a regular development and that $E_0 \supset E_1 \supset \ldots$. Thus F is a regular development such that (i) $F_0 \supset F_1 \supset \ldots$ and (ii) for each $n \in \omega$, if C is a chain of four or fewer members of F_{n+1}, then C^* is a subset of some member of F_n.

A "size" $\mu(f)$ is assigned to each $f \in F_0$ as follows: if $f \in F_n$ for every $n \in \omega$, let $\mu(f) = 0$; otherwise let m be the largest integer such that $f \in F_m$ and define $\mu(f) = 2^{-m}$. Notice that $\mu(f) \leqslant 2^{-n}$ if and only if $f \in F_n$, and since the space is Hausdorff, only a singleton can have size zero. Further, define the "length" $\lambda(C)$ of a chain C to be the sum of the sizes of its distinct members, i.e. $\lambda(C) = \sum_{f \in C} \mu(f)$.

LEMMA 1. *Suppose C is a chain and $\lambda(C) \leqslant 2^{-n}$ for some $n \in \omega$. Then C^* is a subset of some member of F_n.* (Proof postponed.)

Now to define the metric, for any x and y in X let

$$d(x, y) = \text{g.l.b.}\ \{\lambda(C) \colon C \text{ is a chain}, x \in C^*, y \in C^*\}. \tag{5}$$

Since $\{X\}$ is a chain, d is well defined (and bounded above by 1).

For each $x \in X$, $r > 0$, and $n \in \omega$, it follows from (5) that

$$S_d(x, r) = \bigcup \{C^* \colon C \text{ is a chain}, \lambda(C) < r, x \in C^*\}, \tag{6}$$

and hence by Lemma 1,

$$x \in S_d(x, 2^{-n}) \subset \operatorname{Star}(x, F_n). \tag{7}$$

We now show that d is a metric. Condition (2) is obvious. To see that (1) holds, suppose $x = y$. Then for each $n \in \omega$, some member f of F_n contains x (and hence y). But $\lambda(\{f\}) \leqslant 2^{-n}$, so $d(x, y) \leqslant 2^{-n}$ for all n. Thus $d(x, y) = 0$. Now suppose that $x \neq y$. Then since X is Hausdorff, $y \notin \operatorname{Star}(x, F_m)$ for some $m \in \omega$. By (7), $y \notin S_d(x, 2^{-m})$, and so $d(x, y) \geqslant 2^{-m} > 0$.

To verify (3), let $x, y, z \in X$ and suppose that $\epsilon > 0$. Then there exist chains C_1 and C_2 such that C_1^* contains x and y, C_2^* contains y and z, $\lambda(C_1) < d(x, y) + \epsilon$, and $\lambda(C_2) < d(y, z) + \epsilon$. It is evident that some subcollection C of $C_1 \cup C_2$ is a chain such that C^* contains x and z. But then

$$\lambda(C) \leqslant \lambda(C_1) + \lambda(C_2) < d(x, y) + d(y, z) + 2\epsilon.$$

Thus $d(x, z) < d(x, y) + d(y, z) + 2\epsilon$ for every $\epsilon > 0$, and (3) follows.

Finally, the set of spheres is a basis for T. For (6) implies that every sphere is an open set, and by (7) and the fact that F is a development, whenever $x \in U \in T$ some sphere contains x and lies in U.

Proof of Lemma 1 (by induction on the number of members of C). If C has one member, the lemma is immediate. Assume that C has exactly m

members, $m > 1$, and that the lemma has been established for all chains of fewer members. Let $f_1, \ldots, f_m$ be an ordering of the members of C, where f_i intersects f_{i+1} if $i < m$. Supposing that $\lambda(C) \leqslant 2^{-n}$, we distinguish three cases.

Case 1: for some i, $\mu(f_i) = 2^{-n}$. Then the remaining members of C must be singletons, so $C^* = f_i \in F_n$.

In the following non-trivial cases, C will be divided into three *proper* subcollections, C_1, C_2, C_3, such that each is a chain of length at most $2^{-(n+1)}$, $C_2{}^*$ intersects both $C_1{}^*$ and $C_3{}^*$, and $C = C_1 \cup C_2 \cup C_3$. Then by the inductive assumption, there exist members g_j of F_{n+1} such that $C_j{}^* \subset g_j$ $(j = 1, 2, 3)$. But $\{g_1, g_2, g_3\}$ is then a chain of members of F_{n+1}; so by property (ii) of F it follows that $g_1 \cup g_2 \cup g_3$ (and hence C^*) is a subset of some member of F_n, which is the desired situation.

Case 2: for some i, $\mu(f_i) = 2^{-(n+1)}$. Here define $C_2 = \{f_i\}$, $C_1 = \{f_1, \ldots, f_{i-1}\}$, and $C_3 = \{f_{i+1}, \ldots, f_m\}$, so that each has length no more than $2^{-(n+1)}$. (In case i is 1 or m, let C_1 or C_3, respectively, be $\{f_i\}$ instead.)

Case 3: $\mu(f_i) \leqslant 2^{-(n+2)}$ for every i. Let p be the largest integer strictly less than m such that $\lambda(C_1) \leqslant 2^{-(n+1)}$ if $C_1 = \{f_1, \ldots, f_p\}$; then let q be the least integer greater than 1 for which $\lambda(C_3) \leqslant 2^{-(n+1)}$, where $C_3 = \{f_q, \ldots, f_m\}$. If $q \leqslant p + 1$, define $C_2 = \{f_p\}$. On the other hand, if $1 < p < p + 1 < q < m$, we see that C_1 and C_3 must have length at least $2^{-(n+2)}$, because their maximality ensures that no more of the small f_i's could have been annexed without exceeding the length of $2^{-(n+1)}$. Defining, in this situation,

$$C_2 = \{f_{p+1}, \ldots, f_{q-1}\}$$

it follows that

$$\lambda(C_2) \leqslant 2^{-n} - \lambda(C_1) - \lambda(C_3) \leqslant 2^{-n} - 2^{-(n+2)} - 2^{-(n+2)} = 2^{-(n+1)}.$$

This completes the last case and the induction.

3. Other metrization theorems.

The next theorem is a special case of Theorem 4. But its proof, suggested by J. Martin, is more transparent and motivates the latter's proof.

THEOREM 3. *Any regular space with a countable basis admits a neighbourhood-star development.*

Proof. Suppose $\{b_0, b_1, \ldots\}$ is a countable basis for the regular space (X, T). Whenever $\mathrm{Cl}(b_i) \subset b_j$, define

$$E(i,j) = \{b_j, X - \mathrm{Cl}(b_i)\},$$

a pair of open sets covering X (Cl = closure). Let G be a sequence with range $\{E(i,j)\colon \mathrm{Cl}(b_i) \subset b_j\}$.

Now if $x \in U \in T$, we can choose j such that $x \in b_j \subset U$; and, by regularity, an i may be chosen such that $x \in b_i$ and $\mathrm{Cl}(b_i) \subset b_j$. If $G_n = E(i,j)$, then $\mathrm{Star}(b_i, G_n) = b_j \subset U$. So G is a neighbourhood-star development for X.

COROLLARY (Urysohn–Tychonoff). *A space with a countable basis is metrizable if and only if it is regular and Hausdorff;* cf. **(10)**.

Proof. Metric spaces are regular and Hausdorff, and Theorems 1, 2, and 3 supply the converse.

The following terminology will be used in the remaining theorems. A collection of point sets will be called *discrete* (*locally finite*) provided each point has a neighbourhood that intersects at most one (only finitely many) of the members of the collection. A collection is *σ-discrete* or *σ-locally finite* if it is the union of countably many collections, each being discrete or locally finite, respectively. Clearly discrete collections are locally finite; and if S is any subcollection of a locally finite collection, it is easy to check that $\cup\{\mathrm{Cl}(s): s \in S\}$ is closed.

A direct extension of the proof of Theorem 3 shows that any regular space with a σ-discrete basis admits a neighbourhood-star development. To prove the more general σ-locally finite case, the next lemma supplies first a convenient improvement to any σ-locally finite basis.

LEMMA 2. *Suppose (X, T) is a topological space with a σ-locally finite basis. Then there exists a sequence D with the following two properties:*

(i) *If $m \in \omega$, D_m is a locally finite collection of open sets.*

(ii) *For each $U \in T$ and $x \in U$, there exists $n \in \omega$ such that some subset of U is a member of D_n and is the only member of D_n that contains x.*

Proof. Let $\cup\{A_k: k \in \omega\}$ be a basis for T for which each A_k is a locally finite collection. For each $i, j \in \omega$, let $B(i,j)$ consist of all sets of the form $a_0 \cap \ldots \cap a_j$, where $a_0, \ldots, a_j$ are distinct members of A_i. Thus $B(i,j)$ is a collection of (perhaps empty) open sets. To see that $B(i,j)$ is locally finite, notice that each point has a neighbourhood intersecting only finitely many, say m, members of A_i. But then the number of members of $B(i,j)$ that intersect the neighbourhood is at most the number of combinations of m objects taken $j+1$ at a time, i.e. finite. If D is any sequence whose range is $\{B(i,j): i, j \in \omega\}$, D satisfies (i).

To verify (ii), suppose that $x \in U \in T$. Since $\cup\{A_k: k \in \omega\}$ is a basis, for some $i \in \omega$ at least one member a' of A_i contains x and is a subset of U. If $j+1$ denotes the number of members of A_i containing x, $b = \cap\{a: x \in a \in A_i\}$ is the only member of $B(i,j)$ that contains x. Further, $b \subset a' \subset U$. This completes the proof of the lemma.

THEOREM 4. *Any regular space with a σ-locally finite basis admits a neighbourhood-star development.*

Proof. Suppose (X, T) is regular and has a σ-locally finite basis. Then there is a sequence D satisfying (i) and (ii) of Lemma 2. For $i, j, k \in \omega$ we define the following collections. $Q(i,j,k)$: all ordered triples (a, b, c) such that $a \in D_i$, $b \in D_j$, $c \in D_k$, $\mathrm{Cl}(a) \subset b$, and $\mathrm{Cl}(b) \subset c$. $A(i,j,k)$ and $B(i,j,k)$: all first and second members, respectively, of the triples in $Q(i, j, k)$. $E(i, j, k)$: the collection $B(i, j, k)$ together with the single open set

$$X - \bigcup \{\mathrm{Cl}(a) \colon a \in A(i, j, k)\}.$$

Then $E(i, j, k)$ is an open covering of X.

Finally, letting G be a sequence with $\{E(i, j, k) \colon i, j, k \in \omega\}$ as its range, it will be shown that G is a neighbourhood-star development for X.

Suppose $x \in U \in T$. Then by property (ii) of D and regularity of the space, integers k, j, and i may be chosen, in that order, such that $c' \subset U$, $\mathrm{Cl}(b') \subset c'$, and $\mathrm{Cl}(a') \subset b'$, where each of a', b', c' is the unique member of D_i, D_j, D_k, respectively, that contains the point x. Note that (a', b', c') is a member of $Q(i, j, k)$. Now let

$$K = \bigcup \{\mathrm{Cl}(b) \colon b \in B(i, j, k), \mathrm{Cl}(b) \not\subset c'\},$$

a closed set. Also $x \notin K$, for otherwise some member of D_k other than c' would contain x, which is impossible. Hence $N = a' - K$ is a neighbourhood of x, and

$$\mathrm{Star}(N, E(i, j, k)) \subset \bigcup \{b \colon b \in B(i, j, k), \mathrm{Cl}(b) \subset c'\} \subset c' \subset U.$$

Since for some $n \in \omega$, $G_n = E(i, j, k)$, we have $\mathrm{Star}(N, G_n) \subset U$, just as required.

The metrization results are now collected.

THEOREM 5. *Each of the following is a necessary and sufficient condition for a topological space to be metrizable:*

(i) *The space is Hausdorff and admits a regular development.*

(ii) *The space is Hausdorff and admits a neighbourhood-star development.*

(iii) *The space is regular, Hausdorff, and has a σ-discrete basis.*

(iv) *The space is regular, Hausdorff, and has a σ-locally finite basis.*

Proof. The equivalence of (ii) to (i) and (i) to metrizability follows from Theorems 1 and 2. But (iii) implies (iv) and by Theorem 4, (iv) implies (ii). The proof is completed by noting that metrizability implies (iii), using Stone's paracompactness result (**6**, p. 129) and the fact that metric spaces are regular and Hausdorff.

Remark. A space is automatically regular if it admits either a regular or neighbourhood-star development. For suppose G is a neighbourhood-star development and x is a point in the open set U. Then there is an open set N such that $x \in N$ and $\mathrm{Star}(N, G_n) \subset U$. But $\mathrm{Cl}(N) \subset \mathrm{Star}(N, G_n)$.

Thus the condition of Hausdorff in Theorem 5 may be replaced in each case by T_0, which merely requires that for any two points there be an open set containing just one, since a regular T_0-space is Hausdorff (A. Mazur).

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