# Interchange of vector valued integrals when the measures are Bochner or Pettis indefinite integrals

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Necessary and sufficient conditions for the interchange of two Bochner integrals and for the interchange of two Pettis integrals are obtained. These conditions are different from those generally required in classical Fubini theorems since they do not require the construction of the cross product measure. The proof makes use of the Vitali-Hahn-Saks Theorem. It should be noted that while Fubini theorems use the cross product measure, one of the difficulties encountered is that the product measure fails to be countable additive - this is pointed out in M. Bhaskara Rao (*Indiana Univ. Math. J.* 21 (1972), 847-848) and Charles Swartz (*Bull. Austral. Math. Soc.* 8 (1973), 359-366). Most applications require the interchange of the two integrals rather than integration with respect to the product measure.

In recent years much work has been done in generalizing Fubini's Theorem. Articles [1], [2], [3], [8], [11], [12] are a small sample of such work. In [1] it is shown that

$$\int_H dK_2 \int_H dK_1 [f \cdot x] = \int_{H \times H} d(K_1 \times K_2)[f \cdot x] \ ,$$

where $H$ is a compact space and where $K_1$ has values in $B(X, Y^{**})$ and $K_2$ has values in $B(Y^{**}, Z^{**})$. A similar result is shown in [2] where

$f \cdot x$ is replaced by a function $f$ from $H \times T$ into $E$ which is integrable with respect to $v(K_1) \times v(K_2)$ where $v(m)$ represents the variation of the vector valued measure $m$ . Results of [8] show that if $\mu$ and $\nu$ are regular vector valued measures defined on Borel sets then $\mu \times \nu$ is well defined and the Fubini Theorem holds provided a separability condition is assumed on $\nu$ . In [11] it is assumed that $\mu$ and $\nu$ are bounded and finitely additive and that $f$ is a real valued function which is bounded. Under certain assumptions it is shown that

$\iint f d\mu d\nu = \iint f d\nu d\mu$ . Finally in [12] the following is shown. Let $M$ and $N$ denote two $\sigma$-algebras of subsets of a set $S$ and $T$ respectively. Let $X$ and $Y$ be two Banach spaces and let $\langle \, , \, \rangle$ denote a continuous bilinear map of $X \times Y$ into the scalars. Let $\lambda$ be a set function defined on rectangles $A \times B$ where $A \in M$ and $B \in N$ , by

$$\lambda(A \times B) = \langle m(A), n(B) \rangle \ ,$$

where $m$ is a vector measure from $M$ into $X$ and $n$ is a vector measure from $N$ into $Y$ . Then under certain conditions

$$\int_T \left\langle \int_S H(s, t) dm(s), dn(t) \right\rangle = \int_S \left\langle \int_T H(s, t) dn(t), dm(s) \right\rangle$$
$$= \iint_{S \times T} H(s, t) d\lambda(s, t) \ ,$$

where $H$ is scalar valued and satisfies some conditions and the first two integrals are in the sense of [5].

It is important to note that in general $\lambda$ as defined above does not have a countable additive extension to the $\sigma$-algebra generated by rectangles. In [10] an example illustrating this is constructed. In [12] Swartz considers the case in which $m$ and $n$ are of the form

$$m(E) = \int_E f d\alpha \quad \text{and} \quad n(F) = \int_F g d\beta$$

where $f$ (and $g$ ) are strongly measurable $X$ (and $Y$ ) valued and Pettis integrable with respect to the $\sigma$-finite positive measure $\alpha$ (and $\beta$ ). The reader is referred to [9] for basic properties of the Pettis integral. Let

$$P(s, t) = \langle f(s), g(t) \rangle \ .$$

It is shown that if $P$ is $\alpha \times \beta$ integrable, then $\lambda$ can be extended to
a countably additive set function over the $\sigma$-algebra generated by
rectangles.  In this case the Fubini Theorem as stated above holds for
$H : S \times T \to R$ where $H$ is bounded and scalar valued.  Furthermore, if $f$
and $g$ are Bochner integrable rather than Pettis integrable, the
requirement that $H$ be bounded may be replaced by the requirement that $H$
be integrable with respect to $v(m) \times v(n)$ .

The main purpose of the present article is to provide a necessary and
sufficient condition for

$$\int_T \left\langle \chi_F(t) \int_S \chi_E(s) H(s, t) dm(s), \, dn(t) \right\rangle$$

$$= \int_S \left\langle \chi_E(s) \int_T \chi_F(t) H(s, t) dn(t), \, dm(s) \right\rangle$$

where $E \in M$ , $F \in N$ , and where $\chi_A$ denotes the characteristic function
of $A$ .  Of course if $f$ and $g$ are Pettis integrable and $H$ is bounded
or if $f$ and $g$ are Bochner integrable and $H$ is $v(m) \times v(n)$
integrable, the above equality is true by $[12]$.  It should be stressed that
we do not assume these conditions on $H$ .  Our first result pertains to the
case in which $f$ and $g$ are Bochner integrable and our second result
assumes that $f$ and $g$ are Pettis integrable.  For elementary properties
of such integrals see $[5]$, $[6]$, $[7]$, and $[9]$.  Finally the reader is
referred to $[4]$ for comparative properties of the Bochner, Pettis, and
Gelfand-Pettis integral.

THEOREM 1.  *Assume $P$ is integrable with respect to $\alpha \times \beta$ and that
$m$ (and $n$ ) are Bochner indefinite integrals of $\alpha$ (and $\beta$ ) ; then*

$$\int_T \left\langle \chi_F(t) \int_S \chi_E(s) H(s, t) dm(s), \, dn(t) \right\rangle$$

$$= \int_S \left\langle \chi_E(s) \int_T \chi_F(t) H(s, t) dn(t), \, dm(s) \right\rangle$$

*for all $E \in M$ and $F \in N$ if, and only if,*

(1)  $\int H(s, t) dm(s)$ *is integrable with respect to $n$ , and*

(2)  $H(s, t)$ *is integrable with respect to $n$ (m almost
everywhere).*

Proof. If the above equality holds it is quite clear that by making $E = S$ and $F = T$, (1) and (2) hold.

We now assume (1) and (2). From (1) it follows that

$$\left\| \int_S H(s,\ t)dm(s) \right\| < \infty\ ,\quad n\ \text{ almost everywhere.}$$

Thus

$$\int\ |H(s,\ t)|\,dv(m)(s) < \infty\ ,\quad n\ \text{ almost everywhere.}$$

For every positive integer $k$ let

$$T(k) = \left\{ t \in T \mid \int\ |H(s,\ t)|\,dv(m)(s) \leq k,\ n\ \text{ almost everywhere} \right\}\ .$$

Then $T(k) \uparrow T$ almost everywhere $(n)$ and $v(n)[T(k)] < \infty$ since $f$ and $g$ Bochner integrable implies that $m$ and $n$ have finite variation. Now

$$\chi_{F \cap T(k)} \chi_E |H(s,\ t)|$$

is integrable with respect to $v(m) \times v(n)$, since

$$\iint\ \chi_{F \cap T(k)}(t)\chi_E(s)\,|H(s,\ t)|\,d[v(m) \times v(n)](s,\ t)$$

$$= \int_T\ \chi_{F \cap T(k)}(t)\left[\int_S\ \chi_E(s)\,|H(s,\ t)|\,dv(m)(s)\right]dv(n)(t)$$

$$\leq \int_T\ k\chi_{F \cap T(k)}(t)\,dv(n)(t) \leq kv(n)[T(k)] < \infty\ .$$

The above steps are justified by Tonelli's Theorem – see [6]. Now by [12] it follows that

$$\int_T\ \left\langle \chi_{F \cap T(k)}(t)\ \int_S\ \chi_E(s)H(s,\ t)dm(s),\ dn(t) \right\rangle$$

$$= \int_S\ \left\langle dm(s),\ \chi_E(s)\ \int_T\ \chi_{F \cap T(k)}(t)H(s,\ t)dn(t) \right\rangle$$

$$= \iint_{S \times T}\ \chi_{F \cap T(k)}(t)\chi_E(s)H(s,\ t)d\lambda(s,\ t) = h_k(E)\ .$$

Clearly $h_k$ are scalar measures defined on $M$. Also $h_k \ll \text{var}(m)$. Let

$$f_k(t) = \chi_{F \cap T(k)}(t)\ \int_S\ \chi_E(s)H(s,\ t)dm(s)\ .$$

Then   $h_k(E) = \int_T \langle f_k, dn \rangle$ .   Now the sequence   $\{f_k\}$   converges   $n$   almost

everywhere to

$$\chi_F \int_S \chi_E(s)H(s, t)dm(s) ,$$

for each   $f_k$   is an   $n$-integrable function from   $T$   into   $X$ .   Also

$$\|f_k\|_X \leq \left\| \int_S \chi_E(s)H(s, t)dm(s) \right\|_X .$$

Thus each   $f_k$   is dominated by a function that is Bochner integrable with

respect to   $n$ .   Thus

$$\chi_F(t) \int_S \chi_E(s)H(s, t)dm(s)$$

is Bochner integrable with respect to   $n$   and   $h_k(E)$   converges to

$$\int_T \left\langle \chi_F(t) \int_S \chi_E(s)H(s, t)dm(s), dn(t) \right\rangle ,$$

which we denote by   $h(E)$ .

Now let

$$g_k(s) = \int_T \chi_{F \cap T(k)}(t)H(s, t)dn(t) .$$

Then all   $g_k$   are functions from   $S$   into   $Y$   and

$$h_k(E) = \int_E \langle dm(s), g_k(s) \rangle .$$

Since

$$v(h_k)(E) = \int_E \|g_k(s)\|_Y dv(m)(s) ,$$

it follows that   $v(h_k) \ll v(m)$ .   By the Vitali-Hahn-Saks Theorem [6,

p. 150] it follows by (2) and the dominated convergence theorem that since

$g_k(s)$   converges   $m$   almost everywhere to

$$\int_T \chi_F(t)H(s, t)dn(t) ,$$

it converges in the $L^1$ norm to that function. Indeed since $h_k(E)$

converges to $h(E)$ by the Vitali-Hahn-Saks Theorem the $h_k$ are uniformly

absolutely continuous with respect to $v(m)$, and since $h_k$ is scalar

valued it follows that $v(h_k)$ are uniformly absolutely continuous with

respect to $v(m)$. (Note that since $f$ is Bochner integrable,
$v(m)(S) < \infty$.) Thus

$$\lim h_k(E) = \int_S \left\langle dm(s), \; \chi_E(s) \int_T \chi_F(t) H(s, \; t) dn(t) \right\rangle$$

$$= \int_T \left\langle \chi_F(t) \int_S \chi_E(s) H(s, \; t) dm(s), \; dn(t) \right\rangle .$$

This finishes the proof of the theorem.

We now assume that $f$ and $g$ are Pettis-integrable. Of course it is
not true anymore that $v(m)$ and $v(n)$ are finite. It is shown in [12]
that in this case the Fubini Theorem holds provided $H$ is bounded. It is
also noted that while the restriction that $H$ be bounded is undesirable,
it is difficult to give conditions that insure that $\int H(s, \; t) f(s) d\alpha(s)$

exists as a Pettis-integral.

THEOREM 2. *Assume $P$ is $\alpha \times \beta$ integrable, that $m$ and $n$ are
Pettis indefinite integrals and that a sequence $\{F_j\}$ may be found in $N$
such that $F_j \uparrow T$ , $n$ almost everywhere, $\chi_{F_j}(t) H(s, \; t)$ is bounded, and
$v(n)(F_j) < \infty$ . Then the conclusion of the previous theorem holds.*

Proof. Again the integrals with respect to $m$ and $n$ are taken in
the sense of [5] or [6]. For each positive integer $k$ define

$$T(k) = \left\{ t \in \bigcup_{j=1}^k F_j \; \Big| \; \int |H(s, \; t)| dv(m)(s) \leq k \right\} .$$

Again $T(k) \uparrow T$ , $n$ almost everywhere, and $v(n)[T(k)] < \infty$ . The
function

$$\chi_{F \cap T(k)}(t) \chi_E(s) |H(s, \; t)|$$

is bounded. Thus by [12] we obtain $\{h_k\}$ as in Theorem 1 such that $h_k$

are finite measures.  Again the use of the Vitali-Hahn-Saks Theorem
completes the proof.  Since $v(m)$ is $\sigma$-finite we can find $E_\varepsilon \in M$ such
that $v(h_k)(S-E_\varepsilon) < \varepsilon$ with $v(m)(E_\varepsilon) < \infty$ .  The rest of the proof follows
as in Theorem 1.

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