BANACH SPACES WITH PROPERTY (w)

by DENNY H. LEUNG

(Received 2 December, 1991)

1. Introduction. A Banach space E is said to have *Property* (w) if every operator from E into E' is weakly compact. This property was introduced by E. and P. Saab in **[9]**. They observe that for Banach lattices, Property (w) is equivalent to Property (V*), which in turn is equivalent to the Banach lattice having a weakly sequentially complete dual. Thus the following question was raised in **[9]**.

Does every Banach space with Property (w) have a weakly sequentially complete dual, or even Property (V*)?

In this paper, we give two examples, both of which answer the question in the negative. Both examples are James type spaces considered in **[1]**. They both possess properties stronger than Property (w). The first example has the property that every operator from the space into the dual is compact. In the second example, both the space and its dual have Property (w). In the last section we establish some partial results concerning the problem (also raised in **[9]**) of whether (w) passes from a Banach space E to $C(K, E)$.

We use standard Banach space terminology as may be found in **[6]**. For a Banach space E, E' denotes its dual, and U_E its closed unit ball. If F is also a Banach space, then we let $L(E, F)$ (respectively $K(E, F)$) denote the space of all bounded linear operators (respectively all compact operators) from E into F. The norm in l^p is denoted by $\|\cdot\|_p$. Let us also establish some terminology about sequences. If (e_i) is a sequence in a Banach space, we use $[e_i]$ to denote the closed linear span of (e_i). The sequence is *semi-normalized* if $0 < \inf \|e_i\| \leq \sup \|e_i\| < \infty$. If (f_i) is another sequence in a possibly different Banach space, we say that (e_i) *dominates* (f_i) if there is a constant C such that $\|\sum a_i f_i\| \leq C \|\sum a_i e_i\|$ for all finitely non-zero real sequences (a_i). Two sequences are *equivalent* if each dominates the other. Finally, we use the symbol $\lesssim$ (respectively $\gtrsim$) to indicate $\leq$ (respectively $\geq$) up to a fixed constant. The symbol $\approx$ stands for "$\lesssim$ and $\gtrsim$".

2. James type constructions. We first recall the construction of James type spaces as in **[1]**. Let (e_i) be a normalized basis of a Banach space E. For $(a_i) \in c_{00}$, the space of all finitely non-zero real sequences, let

$$|||\sum a_i u_i||| = \sup\left\{\left\|\sum_{i=1}^{k}\left(\sum_{j=p(i)}^{q(i)} a_j\right)e_{p(i)}\right\| : k \in \mathbb{N}, 1 \leq p(1) \leq q(1) < \ldots < p(k) \leq q(k)\right\}.$$

The completion of the linear span of the sequence (u_i) is denoted by $J(e_i)$. Since we shall only consider unconditional (e_i), we use the equivalent norm

$$\|\sum a_i u_i\| = \sup\left\{\left\|\sum_{i=1}^{k}\left(\sum_{j=p(i)}^{p(i+1)-1} a_j\right)e_{p(i)}\right\| : k \in \mathbb{N}, 1 = p(1) < p(2) < \ldots < p(k+1)\right\}.$$

As in **[1]**, $\sum_{i=1}^{k} (\sum_{j=p(i)}^{p(i+1)-1} a_j)e_{p(i)}$ is called a *representative* of $\sum a_i u_i$ in E. The biorthogonal sequences of (e_i) and (u_i) are denoted by (e_i') and (u_i') respectively. The basis projections with respect to the basis (u_i) are denoted by $(P_n)_{n=0}^{\infty}(P_0 = 0)$. The functional S defined by $S(\sum a_i u_i) = \sum a_i$ is bounded, hence $S \in J(e_i)'$. The following lemma is useful for computing the norms of certain vectors in $J(e_i)$. Recall that a basic sequence (x_i) is *right dominant* **[1]** if it is unconditional and whenever $1 \le m(1) \le n(1) < \ldots < m(i) \le n(i) < \ldots$, we have

$$\|\sum a_{n(i)} x_{m(i)}\| \le \|\sum a_{n(i)} x_{n(i)}\|.$$

LEMMA 1. *Let (e_i) be a right dominant basis of a Banach space. There exists a constant C such that whenever $(v_i)_{i=1}^l$ is a block basis of (u_i) satisfying*

$$v_i = \sum_{j=n(i)}^{n(i+1)-1} a_j u_j, \qquad \sum_{j=n(i)}^{n(i+1)-1} a_j = 0, \qquad 1 \le i \le l, \tag{1}$$

there is a block basis $(w_i)_{i=1}^l$ of (e_i) such that $\|w_i\| \le \|v_i\|$, $1 \le i \le l$, and $\|\sum_{i=1}^l v_i\| \le C\,\|\sum_{i=1}^l w_i\|$.

Proof. Let (v_i) be as given. Choose $p(1) < p(2) < \ldots < p(k+1)$ such that

$$\left\|\sum_{i=1}^{l} v_i\right\| \le \left\|\sum_{i=1}^{k}\left(\sum_{j=p(i)}^{p(i+1)-1} a_j\right) e_{p(i)}\right\|.$$

Let $(q(i))$ be a finite strictly increasing sequence such that

$$\{q(i)\} = \{p(i) : 1 \le i \le k+1\} \cup \{n(i) : 1 \le i \le l+1\}.$$

Let $A = \{i : \text{there exists some } m \text{ with } p(i) < n(m) < p(i+1)\}$. For all $i \in A$, let $m_i = \min\{m : n(m) > p(i)\}$ and $r_i = \max\{m : n(m) < p(i+1)\}$. By (1), for $i \in A$,

$$\sum_{j=p(i)}^{p(i+1)-1} a_j = \sum_{j=p(i)}^{n(m_i)-1} a_j + \sum_{j=n(r_i)}^{p(i+1)-1} a_j \equiv b_i + c_i.$$

Therefore, since (e_i) is right dominant,

$$\begin{aligned}\left\|\sum_{i\in A}\left(\sum_{j=p(i)}^{p(i+1)-1} a_j\right) e_{p(i)}\right\| &\le \left\|\sum_{i\in A} b_i e_{p(i)}\right\| + \left\|\sum_{i\in A} c_i e_{p(i)}\right\| \\ &\le \left\|\sum_{i\in A} b_i e_{p(i)}\right\| + \left\|\sum_{i\in A} c_i e_{n(r_i)}\right\| \\ &\le \left\|\sum\left(\sum_{j=g(i)}^{q(i+1)-1} a_j\right) e_{q(i)}\right\|.\end{aligned}$$

Also, it is clear that

$$\left\|\sum_{i\notin A}\left(\sum_{j=p(i)}^{p(i+1)-1} a_j\right) e_{p(i)}\right\| \le \left\|\sum\left(\sum_{j=q(i)}^{q(i+1)-1} a_j\right) e_{q(i)}\right\|.$$

Hence

$$\left\|\sum_{i=1}^{l} v_i\right\| \le \left\|\sum\left(\sum_{j=q(i)}^{q(i+1)-1} a_j\right) e_{q(i)}\right\|.$$

Now let $B_m = \{i : n(m) \le q(i) < n(m+1)\}$. Then

$$\left\|\sum_{i=1}^{l} v_i\right\| \le \left\|\sum_{m}\left(\sum_{i\in B_m}\left(\sum_{j=q(i)}^{q(i+1)-1} a_j\right) e_{q(i)}\right)\right\| \equiv \|\sum w_m\|.$$

Note that w_m is a representative of v_m in E. Hence $\|w_m\| \le \|v_m\|$. Clearly, (w_m) is a block basis of (e_i).

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LEMMA 2. *Let (e_i) be a shrinking normalized unconditional basic sequence. Assume that $L(E, J(e_i)') \neq K(E, J(e_i)')$ for some subspace E of $J(e_i)$. Then there are semi-normalized block bases (x_i) and (x_i') of (u_i) and (u_i') respectively such that (x_i) dominates (x_i') and $Sx_i = 0$ for all i.*

Proof. Since (e_i) is shrinking, by [**1**, Theorems 2.2 and 4.1], $J(e_i)' = [\{S\} \cup \{u_i'\}_{i=1}^{\infty}]$. Let $T: E \to J(e_i)'$ be non-compact. Define a projection $P: J(e_i)' \to [u_i']$: $P(aS + \sum a_i u_i') = \sum a_i u_i'$. Then $(1-P)T$ has rank 1. Thus PT is non-compact. Replacing T by PT, we may assume without loss of generality that range $T \subseteq [u_i']$. Choose a bounded sequence (y_i) in E so that $\inf_{i,j} \|Ty_i - Ty_j\| > 0$. By [**1**, Theorem 4.1], l^1 does not embed into $J(e_i)$. Hence we may assume that (y_i) is weakly Cauchy. Thus $(y_{2i-1} - y_{2i})$ is weakly null and semi-normalized; hence by Proposition 1.a.12 of [**6**], we may assume that it is equivalent to a semi-normalized block basis (x_i) of (u_i). Since (x_i) is weakly null, $Sx_i \to 0$. By using a perturbation of a subsequence, we may further assume that $Sx_i = 0$ for all i. Similarly, $(T(y_{2i-1} - y_{2i})) \subseteq [u_i']$ is semi-normalized and weakly Cauchy. Using the same argument, we may assume that it is equivalent to a semi-normalized block basis (x_i') of (u_i'). Finally,

$$
\begin{aligned}
\|\sum a_i x_i'\| &\approx \|\sum a_i T(y_{2i-1} - y_{2i})\| \\
&\leq \|T\| \, \|\sum a_i (y_{2i-1} - y_{2i})\| \\
&\approx \|\sum a_i x_i\|,
\end{aligned}
$$

as required.

THEOREM 3. *Let (e_i) be a right dominant, normalized basic sequence which dominates all of its normalized block bases. Then the following are equivalent.*
(a) *For every subspace E of $J(e_i)$, $L(E, J(e_i)') = K(E, J(e_i)')$.*
(b) *The sequence (e_i) does not dominate (e_i').*

Proof. Since (e_i) is unconditional, if it is not shrinking, it has a normalized block basis equivalent to the l^1 basis. Hence (e_i) dominates the l^1 basis. Thus (e_i) is equivalent to the l^1 basis. Both (a) and (b) fail in this case, so they are equivalent. Now assume that (e_i) is shrinking.

(a)$\Rightarrow$(b). Assume (e_i) dominates (e_i'). If $\|\sum b_i u_i\| \leq 1$, then $\|\sum b_i e_i\| \leq 1$. Hence

$$
|\langle \sum b_i u_i, \sum a_i u_{2i}' \rangle| = |\sum a_i b_{2i}| = |\langle \sum b_i e_i, \sum a_i e_{2i}' \rangle| \leq \|\sum a_i e_{2i}'\|.
$$

Therefore, (u_{2i}') is dominated by (e_{2i}'). By [**1**, Proposition 2.3], $(u_{2i-1} - u_{2i})$ is equivalent to (e_{2i}). Hence, if $E = [u_{2i-1} - u_{2i}]$, then the map $T: E \to J(e_i)'$ defined by $T(\sum a_i (u_{2i-1} - u_{2i})) = \sum a_i u_{2i}'$ is bounded. Clearly, T is not compact.

(b)$\Rightarrow$(a). Suppose $L(E, J(e_i)') \neq K(E, J(e_i)')$ for some subspace E of $J(e_i)$. By Lemma 2, there are semi-normalized block bases (x_i) and (x_i') of (u_i) and (u_i') respectively such that (x_i) dominates (x_i') and $Sx_i = 0$. Let $1 \leq n_1 \leq m_1 < n_2 \leq m_2 < \ldots$ be such that

$x_i' \in [u_n']_{n=n_i}^{m_i}$. By using a subsequence, we can assume that

$$m_i - n_i + 1 \le n_{i+1} - m_i - 1, \quad \text{for all} \quad i \ge 1. \tag{2}$$

Let $R: J(e_i) \to J(e_i)$ denote the right shift operator $R(\sum a_i u_i) = \sum a_i u_{i+1}$. Since (e_i) dominates all of its normalized block bases, (R^k) is uniformly bounded. Choose a normalized block basis (y_i) of (u_i) such that $y_i \in [u_n]_{n=n_i}^{m_i}$ and $\langle y_i, x_i' \rangle = \|x_i'\|$ for all i. Now let $z_i = y_i - R^{m_i+1-n_i} y_i$ for all $i \ge 1$. By (2), (z_i) is a semi-normalized block basis of (u_i). Also $\langle z_i, x_i' \rangle = \|x_i'\|$ and $Sz_i = 0$ for all i. Fix $(a_i) \in c_{00}$. By Lemma 1, there is a block basis (w_i) of (e_i) such that $\|w_i\| \le \|a_i z_i\|$ for all i, and

$$\|\sum a_i z_i\| \le \|\sum w_i\| = \left\| \sum \|w_i\| \frac{w_i}{\|w_i\|} \right\| \le \|\sum \|w_i\| e_i\| \le \|\sum a_i e_i\|,$$

since (e_i) dominates $\left(\frac{w_i}{\|w_i\|}\right)$. Computing the norm of a vector of the form $\sum b_i x_i'$ on some $\sum a_i z_i$, we find that (x_i') dominates (e_i'). Since $Sx_i = 0$, for all i, the computation above can also be applied to (x_i). Hence (x_i) is dominated by (e_i). Since (x_i) dominates (x_i') by the choice of the sequences, we see that (e_i) dominates (e_i').

Corollary 4. *Let (e_i) be the unit vector basis of l^p, where $2 < p < \infty$. Then $J(e_i)$ has Property (w) but not a weakly sequentially complete dual.*

Proof. This follows immediately from Theorem 3 and the fact that $J(e_i)$ is quasi-reflexive of order 1 by [**1**, Theorem 4.1].

Remark. Theorem 3 fails without the assumption that (e_i) dominates all of its normalized block bases. In fact, if (e_i) is subsymmetric, then by [**1**, Proposition 2.3], (e_i) is equivalent to $(u_{2i-1} - u_{2i})$. Similarly, (e_i') is equivalent to (u_{2i}') in this case. Now if we let (e_i) be the unit vector basis of the Lorentz space $d(w, 2)$ [**6**], then (e_i) is symmetric and shrinking, and does not dominate (e_i'). However, l^2 embeds into both $d(w,2)$ and its dual, and hence into both $J(e_i)$ and $J(e_i)'$ by the observation made above. Hence condition (a) of Theorem 3 fails.

3. A non-reflexive space whose dual and itself have Property (w). In this section, we give an example of a non-reflexive Banach space E so that both E and E' have Property (w). In fact, neither E nor any of its higher duals is weakly sequentially complete. The example will again be a $J(e_i)$ space with a suitably chosen (e_i).

If (e_i) is a normalized basis of a reflexive Banach space E, then $J(e_i)$ is quasi-reflexive of order 1 by [**1**, Theorem 4.1]. Thus, if we define the functional L on $J(e_i)'$ by $L(aS + \sum a_i u_i') = a$, then $J(e_i)'' = [\{L\} \cup \{u_i\}_{i=1}^{\infty}]$. Using this observation, the following Proposition can be obtained by straightforward perturbation arguments.

Proposition 5. *Let (e_i) be a normalized unconditional basis of a reflexive Banach space E.*

(a) *Let (y_n) be a bounded sequence in $J(e_i)$ with no weakly convergent subsequence. Then there exist a subsequence (y_{n_i}), an element y_0 of $J(e_i)$, a block basis (z_i) of (u_n), and $a \ne 0$* such that

(i) $(y_{n_i} - y_0) \approx (z_i)$,

(ii) $Sz_i = a$ *for all* i.

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(b) *Let* (y'_n) *be a bounded sequence in* $[u'_n]$ *with no weakly convergent subsequence. Then there are a subsequence* (y'_{n_i}), *a vector* $y'_0 \in [u'_n]$, $0 = k_0 < k_1 < \ldots$, $(z'_i) \subseteq [u'_n]$, *and* $b \neq 0$ *such that*

(i) *for all* i, $z'_i \in [u'_n]_{n=k_{i-1}+1}^{k_i}$,

(ii) *the sequence* $(y'_{n_i} - y'_0 - bP'_{k_{i-1}}S - z'_i)$ *is norm null.*

Recall that a Banach lattice E satisfies an *upper p-estimate* if there is a constant K such that $\|\sum x_n\| \leq K(\sum \|x_n\|^p)^{1/p}$ for every finite sequence (x_n) of pairwise disjoint elements in E.

PROPOSITION 6. *Let* (e_i) *be a subsymmetric normalized basis of a reflexive Banach space* E, *and let* $1 < p < \infty$. *Assume that*

(i) E *satisfies an upper* p*-estimate*,

(ii) (e'_i) *does not dominate* (e_i),

(iii) *the* l^p *basis does not dominate* (e'_i).

Then both $J(e_i)$ *and* $J(e_i)'$ *have Property* (w), *but neither has a weakly sequentially complete dual.*

Proof. Since $J(e_i)$ is quasi-reflexive of order 1, neither $J(e_i)$ nor $J(e_i)'$ has a weakly sequentially complete dual. If $T: J(e_i) \to J(e_i)'$ is not weakly compact, then we may assume that range $T \subseteq [u'_i]$, as in the proof of Lemma 2. Now there is a bounded sequence (y_n) in $J(e_i)$ such that (Ty_n) has no weakly convergent subsequence and $\inf_{i,j} \|y_i - y_j\| > 0$. Apply Proposition 5 to (y_n) and $(y'_n) \equiv (Ty_n)$ to yield the various objects identified there. For all i, let $x_i = y_{n_{4i-1}} - y_{n_{4i-3}}$. Then (x_i) is semi-normalized and $(x_i) \approx (z_{4i-1} - z_{4i-3})$ (where (z_i) is as given by Proposition 5). Also $\|Tx_i - x'_i\| \to 0$, where

$$x'_i = b(P'_{k_{4i-1}} - P'_{k_{4i-3}})S + z'_{4i-1} - z'_{4i-3}.$$

Note that (x'_i) is a semi-normalized block basis of (u'_i) such that $x'_i \in [u'_n]_{n=k_{4i-4}+1}^{k_{4i-1}}$ and $\langle u_j, x'_i \rangle = b$ if $k_{4i-3} < j \leq k_{4i-2}$. Without loss of generality, assume $(Tx_i) \approx (x'_i)$. For all i, let $k_{2i-1} < j_i \leq k_{2i}$. By the subsymmetry of (e_i), it is easy to see that $\|\sum b_i(u_{j_{2i-1}} - u_{j_{2i}})\| \approx \|\sum b_i e_i\|$. Computing the norm of a vector of the form $\sum a_i x'_i$ on $\sum b_i(u_{j_{2i-1}} - u_{j_{2i}})$, we see that (x'_i) dominates (e'_i). On the other hand, since $S(z_{4i-1} - z_{4i-3}) = 0$ for all i, Lemma 1 implies that $(x_i) \approx (z_{4i-1} - z_{4i-3})$ is dominated by some semi-normalized block basis of (e_i). But since E satisfies an upper p-estimate, (x_i) is dominated by the l^p basis. Thus

$$\left\|\sum a_i e'_i\right\| \leq \left\|\sum a_i x'_i\right\| \approx \left\|\sum a_i Tx_i\right\| \leq \left\|\sum a_i x_i\right\| \leq \left(\sum |a_i|^p\right)^{1/p},$$

a contradiction. Hence $J(e_i)$ has Property (w).

Since $J(e_i)$ is quasi-reflexive, if $J(e_i)'$ has not Property (w), there is an operator $T: [u'_n] \to J(e_i)$ which is not weakly compact. Choose a bounded sequence (y'_n) in $[u'_n]$ so that $(y_n) \equiv (Ty'_n)$ has no weakly convergent subsequence. As before, apply Lemma 5. In the present situation, we may assume $z'_i = 0$ as (z'_i) is a weakly null sequence. Arguing as before, we find that the sequence $(z_{2i} - z_{2i-1})$ is dominated by $((P'_{k_{2i}} - P'_{k_{2i-1}})S)$. Since $Sz_i = a \neq 0$ for all i, and because (e_i) is subsymmetric, it follows that $(z_{2i} - z_{2i-1})$ dominates (e_i). On the other hand, if $x \in J(e_i)$, $\|x\| \leq 1$, then $\|\sum \langle (P_{k_{2i}} - P_{k_{2i-1}})x, S \rangle e_i\| \leq$

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1. Therefore,

$$|\langle x, \sum a_i(P'_{k_{2i}} - P'_{k_{2i-1}})S\rangle| = |\sum a_i\langle (P_{k_{2i}} - P_{k_{2i-1}})x, S\rangle|$$
$$\leq \|\sum a_i e'_i\|.$$

Hence (e'_i) dominates $((P'_{k_{2i}} - P'_{k_{2i-1}})S)$. Consequently, (e'_i) dominates (e_i). This contradicts assumption (ii). Hence $J(e_i)'$ also has Property (w).

We now construct a sequence (e_i) satisfying the conditions in Proposition 6. For a number $p \in (1, \infty)$, let $p' = p/(p-1)$. Now fix p and r such that $1 < p < 2 < r < \infty$ and $r' < p$. Let $(k_n) \subseteq \mathbb{N}$ be such that

$$1 + 2\sum_{i=1}^{n-1} i^{r/2}k_i^{r/2} \leq k_n^{1-p/2}/n^{p/2}, \tag{3}$$

$$\left(\frac{n+1}{n}\right)^{r/2}\left(\frac{k_n}{k_{n+1}}\right)^{(1/p-1/p')r/2} \leq \tfrac{1}{2}. \tag{4}$$

Finally, let $\alpha_n = \sqrt{n}\, k_n^{(1/p'-1/p)2}$ for all $n \geq 1$.

LEMMA 7. *For all $l, j \in \mathbb{N}$ with*

$$2\sum_{i=1}^{l-1} i^{r/2}k_i^{r/k} \leq j \leq k_l^{1-p/2}/l^{p/2}, \tag{5}$$

and for all $(x_i)_{i=1}^{l}$ with $0 \leq x_i \leq k_i$, $1 \leq i < l$, $0 \leq x_l \leq j - \sum_{i=1}^{l-1} x_i$, we have

$$0 \leq \sum_{i=1}^{l} \alpha_i^r x_i^{r/p} \leq \frac{j}{2} + 2.$$

Proof. Note that $j - \sum_{i=1}^{l-1} k_i \geq 0$ by the choice of j, since $r > 2$. Clearly, there is no loss of generality in assuming that $x_l = j - \sum_{i=1}^{l-1} x_i$. Let $f : \prod_{i=1}^{l-1} [0, k_i] \to \mathbb{R}$ be defined by

$$f(x_1, \ldots, x_{l-1}) = \sum_{i=1}^{l} \alpha_i^r x_i^{r/p}.$$

Clearly, $f \geq 0$ on $\prod_{i=1}^{l-1} [0, k_i]$. Suppose that f attains its maximum at $(j_1, \ldots, j_{l-1}) \in \prod_{i=1} [0, k_i]$. Let $A = \{1 \leq i < l : j_i = 0\}$ and $B = \{1 \leq i < l : j_i = k_i\}$. Then for $i \notin A \cup B$,

$$\frac{\partial f}{\partial x_i}(j_1, \ldots, j_{l-1}) = 0 \Rightarrow \alpha_i^r j_i^{r/p} = \alpha_l^r j_l^{r/p-1} j_i.$$

Hence,

$$f(j_1, \ldots, j_{l-1}) = \sum_{i \in B} \alpha_i^r k_i^{r/p} + \sum_{\substack{1 \leq i < l \\ i \notin A \cup B}} \alpha_l^r j_l^{r/p-1} j_i + \alpha_l^r j_l^{r/p},$$

https://doi.org/10.1017/S0017089500009769 Published online by Cambridge University Press

where $j_l = j - \sum_{i=1}^{l-1} j_i$. Now $\alpha_i^r k_i^{r/p} = i^{r/2} k_i^{r/2}$. Hence

$$\begin{aligned} f(j_1, \ldots, j_{l-1}) &= \sum_{i \in B} i^{r/2} k_i^{r/2} + \alpha_{lj}^r j_l^{r/p} \Big(1 + \sum_{\substack{1 \le i < l \\ i \notin A \cup B}} j_i / j_l\Big) \\ &\le \frac{j}{2} + \alpha_{lj}^r j_l^{r/p} \Big(1 + \sum_{i=1}^{l-1} j_i / j_l\Big) \\ &\le \frac{j}{2} + \alpha_{lj}^r j_l^{r/p} \Big(1 + \sum_{i=1}^{l-1} k_i \Big/ \Big(j - \sum_{i=1}^{l-1} k_i\Big)\Big) \\ &\le \frac{j}{2} + 2\alpha_{lj}^r j_l^{r/p} \quad \text{by choice of } j \\ &\le \frac{j}{2} + 2\alpha_{lj}^r j^{r/p} \\ &\le \frac{j}{2} + 2(\alpha_l k_l^{(1/p - 1/2)} / l^{1/2})^r \quad \text{by choice of } j \\ &= \frac{j}{2} + 2. \end{aligned}$$

Let (t_i) be the normalized basic sequence so that for all $(a_j^i)_{j=1}^{k_i}{}_{i=1}^{\infty} \in c_{00}$,

$$\left\| \sum_{i=1}^{\infty} \Big(\sum_{j=1}^{k_i} a_j^i t_{k_1 + \ldots + k_{i-1} + j} \Big) \right\| = \|(a_j^i)\|_r \vee \left\| \Big(\alpha_i \Big(\sum_{j=1}^{k_i} |a_j^i|^p \Big)^{1/p} \Big)_{i=1}^{\infty} \right\|_r \quad (k_0 = 0).$$

EXAMPLE 8. Let E be the Banach space with a basis (e_i) so that

$$\|\textstyle\sum b_i e_i\| = \sup\{\|\sum b_i t_{\sigma(i)}\| : \sigma \text{ is a permutation of } \mathbb{N}\}.$$

Then E satisfies the conditions of Proposition 6. Hence $J(e_i)$ and $J(e_i)'$ have Property (w), but neither has a weakly sequentially complete dual.

Proof. It is clear that (e_i) is a symmetric basis and E satisfies an upper p-estimate. To show that E is reflexive, it suffices to show that c_0 does not lattice embed into E. This will follow if we show that c_0 does not lattice embed into $[t_n]$. Let (x_i) be a disjoint normalized sequence in $[t_n]$. If $\inf \|x_i\|_r = \epsilon > 0$, then

$$\left\| \sum_{i=1}^{k} x_i \right\| \ge \left\| \sum_{i=1}^{k} x_i \right\|_r \ge \Big(\sum_{i=1}^{k} \|x_i\|_r^r \Big)^{1/r} \ge \epsilon k^{1/r}$$

for all k. Hence (x_i) is not equivalent to the c_0 basis. On the other hand, if $\inf \|x_i\|_r = 0$, we may assume that $\|x_i\|_r \to 0$. For each i, we write $x_i = \sum_j x_i(j)$, with $x_i(j) \in [t_n]_{n=k_1+\ldots+k_{j-1}+1}^{k_1+\ldots+k_j}$. Since $\|x_i\|_r \to 0$, $\lim_i x_i(j) = 0$ for all j. Thus, by perturbation and dropping to a subsequence we may assume that each x_i has the form $\sum_{j=j_{i-1}+1}^{j_i} x_i(j)$, where (j_i) is strictly increasing. But then since (x_i) is normalized and these factors add according

https://doi.org/10.1017/S0017089500009769 Published online by Cambridge University Press

to the l^r norm, we see that (x_i) is not equivalent to the c_0 basis. This shows that E is reflexive.

For all l, choose j as in Equation (5). We estimate the norm of $\sum_{i=1}^{j} e_i$. It is easy to see that there exists $(j_i)_{i=1}^{\infty} \subseteq \mathbb{N} \cup \{0\}$ such that $\sum j_i = j$, $j_i \le k_i$ for all i, and

$$\left\| \sum_{i=1}^{j} e_i \right\| = j^{1/r} \vee \|(\alpha_i j_i^{1/p})_{i=1}^{\infty}\|_r.$$

Now

$$\sum_{i=1}^{l} \alpha_i^r j_i^{r/p} \le \frac{j}{2} + 2$$

by Lemma 7. For $i > l$,

$$\begin{aligned} \alpha_i j_i^{1/p} &\le \alpha_i j^{1/p} \\ &\le \alpha_i (k_l^{1-p/2}/l^{p/2})^{1/p} \\ &= \sqrt{\frac{i}{l}} \left(\frac{k_l}{k_i}\right)^{(1/p-1/p')/2} \\ &= \left(\prod_{m=i}^{l+1} \frac{m}{m-1}\right)^{1/2} \left(\prod_{m=i}^{l+1} \frac{k_{m-1}}{k_m}\right)^{(1/p-1/p')/2} \\ &\le 2^{-(i-l)/r}, \end{aligned}$$

by Equation (4). It follows easily that $\|(\alpha_i j_i^{1/p})_{i=1}^{\infty}\|_r \le j^{1/r}$. Hence

$$\left\| \sum_{i=1}^{j} e_i \right\| \le j^{1/r}.$$

This implies that $\left\| \sum_{i=1}^{j} e_i' \right\| \ge j^{1/r'}$. Since $r' < p$, the l^p basis does not dominate (e_i').

Finally, for all n,

$$\left\| \sum_{i=1}^{k_n} e_i \right\| \ge \alpha_n k_n^{1/p} = \sqrt{n k_n}.$$

Also,

$$\begin{aligned} \left\| \sum_{i=1}^{k_n} b_i e_i \right\| \le 1 \quad &\Rightarrow \quad \alpha_n \left(\sum_{i=1}^{k_n} |b_i|^p \right)^{1/p} \le 1 \\ &\Rightarrow \quad \left\langle \sum_{i=1}^{k_n} b_i e_i, \sum_{i=1}^{k_n} e_i' \right\rangle \le \|(b_i)\|_p \, k_n^{1/p'} \\ &\qquad \le \frac{k_n^{1/p'}}{\alpha_n} = \sqrt{\frac{k_n}{n}}. \end{aligned}$$

Hence $\left\| \sum_{i=1}^{k_n} e_i' \right\| \le (k_n/n)^{1/2}$. Therefore, (e_i') does not dominate (e_i).

https://doi.org/10.1017/S0017089500009769 Published online by Cambridge University Press

4. Property (w) in $C(K, E)$. In this section, we consider the question of whether the Property (w) passes from a Banach space E to $C(K, E)$, the space of all continuous E-valued functions on a compact Hausdorff space K. Let E, F be Banach spaces, and let K be an arbitrary compact Hausdorff space whose collection of Borel subsets is denoted by Σ. The dual of $C(K, E)$ is isometric to the space $M(K, E')$ of all regular E'-valued measures of bounded variation on K [**4**]. In case $E = \mathbb{R}$, we use the notation $C(K)$ and $M(K)$ respectively. For μ in $M(K, E')$, let $|\mu| \in M(K)$ denote its variation ([**4**, p. 2]). Given $T \in L(C(K, E), F)$, it is well known that T can be represented by a vector measure $G: \Sigma \to L(E, F'')$ [**4**]. In fact, G is given by

$$G(A)x = T''(\chi_A \otimes x),$$

for all $A \in \Sigma$ and $x \in E$, where χ_A is the characteristic function of the set A. For all $y' \in F'$, we define $G_{y'} \in M(K, E')$ by

$$\langle x, G_{y'}(A) \rangle = \langle y', G(A)x \rangle$$

for all $x \in E$, $A \in \Sigma$. Then the *semivariation* of G is given by

$$\|G\|(A) = \sup\{|G_{y'}|(A) : \|y'\| \leq 1\}.$$

The following result is well known [**2**].

Proposition 9. *Let $T: C(K, E) \to F$ be weakly compact. Then its representing measure G satisfies the following conditions.*

(a) *G takes values in $L(E, F)$.*

(b) *For every $A \in \Sigma$, $G(A)$ is weakly compact.*

(c) *The semivariation of G is continuous at $\varnothing$, i.e., $\lim_n \|G\|(A_n) = 0$ for every sequence (A_n) in Σ which decreases to $\varnothing$.*

Condition (c) is just the uniform countable additivity of the set $\{|G_{y'}| : \|y'\| \leq 1\}$. By Lemma VI.2.13 of [**4**], this is equivalent to the fact that

$$\lim_n \sup_{\|y'\| \leq 1} |G_{y'}|(A_n) = 0$$

whenever (A_n) is a pairwise disjoint sequence in Σ. For the sake of brevity, we introduce the following *ad hoc* terminology.

Definition. A pair (K, E), where K is a compact Hausdorff space and E is a Banach space, is called *simple* if for every Banach space F, every operator $T: C(K, E) \to F$ whose representing measure G satisfies conditions (a)–(c) of Proposition 9 is weakly compact.

Theorem 10. *Suppose that (K, E) is simple. Then E has Property (w) if and only if $C(K, E)$ does.*

Proof. One direction is trivial. Now assume that E has Property (w). Then E' cannot contain a copy of c_0 [**9**]. We first prove that every (bounded linear) operator from $C(K, E)$ into E' is weakly compact. Let $T: C(K, E) \to E'$ be represented by the measure G. For all $x \in E$, define $T_x: C(K) \to E'$ by $T_x f = T(f \otimes x)$ for all $f \in C(K)$. Since E' does not contain a copy of c_0, T_x is weakly compact [**7**]. Hence G takes values in $L(E, E')$ [**5**]. Since E has Property (w), $G(A)$ is weakly compact for all $A \in \Sigma$. If $\|G\|$ is not continuous at $\varnothing$, then there are $(x''_n) \subseteq U_{E''}$, a pairwise disjoint sequence (A_n) in Σ, and $\epsilon > 0$ such that $|G_{x''_n}(A_n)| > \epsilon$ for all n. Choose Σ-measurable U_E-valued simple functions (f_n) such that $\operatorname{supp} f \subseteq A_n$ and $\int f_n \, dG_{x''_n} > \epsilon$ for all n. Since U_E is $\sigma(E'', E')$-dense in $U_{E''}$ and f_n is simple, there exist $(x_n) \subseteq U_E$ such that $\int f_n \, dG_{x_n} > \epsilon$. Note that for all $x \in E$,

$$\sum_n \left| \int f_n \, dG_x \right| \leq \sum_n |G_x| \, (A_n) \leq |G_x| \, (K) = \|T'y\|.$$

Hence $S: E \to l^1$ defined by

$$Sx = \left(\int f_n \, dG_n \right)_n$$

is bounded. Since $(Sx_n)(n) > \epsilon$ for all n, (Sx_n) is not weakly compact in l^1. Using Proposition 2.a.2 of [**6**], we can find a subsequence (x_{n_m}) such that (Sx_{n_m}) is equivalent to the l^1 basis, and $[Sx_{n_m}]$ is complemented in l^1. From this it follows readily that E contains a complemented copy of l^1 as well. This contradicts the fact that E has Property (w). Since G satisfies conditions (a)–(c) of Proposition 9, and (K, E) is simple, the claim follows.

The proof that $C(K, E)$ has Property (w) proceeds analogously. Let T be an operator from $C(K, E)$ into $M(K, E')$ represented by the measure G; define $T_x \in L(C(K), M(K, E''))$ as above by $T_x f = T(f \otimes x)$ for all $x \in E$, $f \in C(K)$. If $M(K, E')$ contains a copy of c_0, then l^1 embeds complementably in E [**8**], a contradiction. Hence T_x does not fix a copy of c_0, and so is weakly compact [**7**]. Thus G is $L(E, M(K, E'))$-valued [**5**]. For all $A \in \Sigma$, let

$$S = G(A)' \big|_{C(K,E)} : C(K, E) \to E'.$$

S is weakly compact by the argument above. It is easy to see that $S'x = G(A)x$ for all $x \in E$. Thus $G(A) = S' \big|_E$ is weakly compact. Finally, if $\|G\|$ is not continuous at $\varnothing$, we obtain, as in the proof of the claim, a pairwise disjoint sequence (A_n) in Σ, a Σ-measurable U_E-valued sequence of functions (f_n) on K, a sequence $(g_n) \subseteq U_{C(K,E)}$, and $\epsilon > 0$ such that $\operatorname{supp} f_n \subseteq A_n$ and $\int f_n \, dG_{g_n} > \epsilon$ for all n. Now

$$S: C(K, E) \to l^1, \qquad Sg = \left(\int f_n \, dG_g \right)_n$$

is bounded. Since $(Sg_n)(n) > \epsilon$ for all n, we obtain as before that l^1 embeds complementably into $C(K, E)$. By [**8**], this implies that l^1 embeds complementably into E, a contradiction. Since the pair (K, E) is simple, we conclude that T is weakly compact.

The pair (K, E) is simple in either one of the following situations [**2**], [**3**], [**10**]:

(a) E' and E'' both have the Radon-Nikodým Property [**2**];

(b) K is a scattered compact [**3**].

Corollary 11. *If the pair* (K, E) *satisfies one of the conditions* (a) *or* (b) *listed above, then* E *has Property* (w) *if and only if* $C(K, E)$ *does.*

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Department of Mathematics
National University of Singapore
Singapore 0511