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# A VARIATIONAL APPROACH FOR ONE-DIMENSIONAL PRESCRIBED MEAN CURVATURE PROBLEMS

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**Abstract**

We discuss the multiplicity of nonnegative solutions of a parametric one-dimensional mean curvature problem. Our main effort here is to describe the configuration of the limits of a certain function, depending on the potential at zero, that yield, for certain values of the parameter, the existence of infinitely many weak nonnegative and nontrivial solutions. Moreover, thanks to a classical regularity result due to Lieberman, this sequence of solutions strongly converges to zero in $C^1([0, 1])$. Our approach is based on recent variational methods.

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## 1. Introduction

The Minkowski curvature equation

$$\begin{cases} -\text{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = f(t,u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

plays, as is well known, a role in differential geometry and in the theory of relativity (see, for instance, [4] and references therein).

The existence, nonexistence and multiplicity of positive solutions of problem (1.1) have been discussed by several authors in recent decades. See, for instance, the papers of Ni and Serrin [22–24] and Peletier and Serrin [30].

More recently, Obersnel and Omari in [27] studied the existence of positive solutions of the parametric problem

$$\begin{cases} -\text{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \lambda f(t,u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

by comparing $F(t,\xi) := \int_0^\xi f(t,x)\,dx$ with $\xi^2$ near zero and with $\xi$ at infinity. This procedure is motivated by the fact that the curvature operator behaves like the Laplacian operator near zero and the 1-Laplace operator at infinity.

In the paper cited, the authors, requiring a specific configuration of the limits of $(F(t,\xi))/\xi^2$ at zero and of $(F(t,\xi))/\xi$ at infinity, obtain the existence, multiplicity or, in some cases, the existence of infinitely many solutions of (1.1). In particular, combining the lower and upper solutions method, local minimization and critical values estimates, it is proved that if

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty,$$

then, for every $\lambda > 0$, the autonomous analogue of problem (1.2) admits an infinite sequence of positive weak solutions.

We also observe that, treating the general case in [27, Theorem 3.15], one of the key hypotheses is expressed by

$$\limsup_{\xi\to 0^+} \frac{\int_\omega F(x,\xi)\,dx}{\xi^2} = +\infty, \tag{1.3}$$

where $\omega$ is an open subset of $\Omega$. Moreover, in [27, Theorem 3.16], studying problem (1.1), the assumptions on the behaviour of the potential $F$ at zero are replaced by some conditions involving suitable spectral constants $\lambda_1^\sharp$ and $\lambda_1^\star$.

Condition (1.3) has also been used by Coelho *et al.* as an ingredient in the one-dimensional setting, ensuring the existence of a sequence of positive solutions (see [10, Theorem 2.6]).

Further, the existence of infinitely many weak solutions, tending in the $C^1$ norm to zero, has been achieved in [28] via Lusternik–Schnirelmann theory provided the nonlinearity is odd and its primitive is subquadratic at zero. The same thesis, for the one-dimensional case, has been obtained in [25], exploiting a suitable generalized Fučík spectrum and requiring different behaviour of the nonlinearity.

In addition, the one-dimensional problem has been rather thoroughly discussed, using different methods, in recent papers by Bonheure *et al.* [6, 7, 14], Boreanu and Mawhin [5] and Pan [29].

Motivated by this interest, the aim of this paper is to study the one-dimensional prescribed curvature problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = \lambda f(t,u) & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.4}$$

where $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is an $L^1$-Carathéodory function and $\lambda > 0$ is a real parameter. More precisely, the main goal here is to obtain some sufficient conditions to guarantee that, for suitable values of $\lambda$, problem (1.4) has infinitely many nontrivial and nonnegative weak solutions that converge to zero in $C^1$. To this end, we require that the potential $F$ satisfies a suitable oscillatory behaviour at zero finding arbitrarily small solutions (see Theorem 3.1).

Our variational approach is mainly based on Ricceri's variational principle (see [33, Theorem 2.1]). We apply [33, Theorem 2.1] to a suitable regularized problem to obtain a sequence of pairwise distinct critical points for the associated Euler–Lagrange functional and successively we achieve the existence of infinitely many solutions for the original problem by using an original and interesting trick due to Obersnel and Omari (see [28]) and by means of a classical regularity result by Lieberman (see [18]).

We point out that Faraci in [12, Theorem 1.1], using the same variational setting but different technical arguments, ensured the existence of infinitely many solutions for the one-dimensional prescribed curvature problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+u'^2}}\Big)' = h(t)f(u) & \text{in } (0,1),\\ u(0)=u(1)=0, \end{cases}$$

where $h : [0,1] \to \mathbb{R}$ is a positive bounded function with $\operatorname{ess\,inf}_{t\in[0,1]} h(t) > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. In this context the author assumes, among other technical assumptions, that the following condition at zero holds:

$$-\infty < \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} \le \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty.$$

Instead of the above inequality, in Theorem 3.2 we require

$$\liminf_{\xi\to 0^+} \frac{\max_{|x|\le\xi} F(x)}{\xi^2} < \kappa \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2},$$

where

$$\kappa := \frac{7\sqrt{2}\int_{1/4}^{3/4} h(t)\,dt}{128\|h\|_{L^1([0,1])}}.$$

If $h \equiv 1$ in $[0,1]$, the above constant takes the value

$$\kappa := \frac{7}{256}\sqrt{2}.$$

A more precise comparison with the cited result is explained in Remark 4.2. A special case of our main result is as follows.

THEOREM 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative and continuous function such that $f(0) = 0$. Put $F(\xi) := \int_0^\xi f(t)\,dt$ (for every $\xi \in \mathbb{R}$) assuming that*

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = 0 \quad \text{and} \quad 0 < B^\star := \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} \le +\infty.$$

*Then, for every $\lambda > 8/B^\star$, the problem*

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+u'^2}}\Big)' = \lambda f(u) & \text{in } (0,1),\\ u(0)=u(1)=0 \end{cases}$$

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*admits a sequence of nontrivial and nonnegative weak solutions* $\{u_n\} \subset C^1([0,1])$ *which satisfy*

$$\lim_{n\to\infty} \|u_n\|_{C^1([0,1])} = 0.$$

We emphasize that, if

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} < +\infty,$$

Theorem 1.1 is not directly obtainable by the statements of the previously mentioned contributions. In particular, we note that no symmetry assumption is required on the nonlinearity term $f$.

Finally, for completeness, we cite the papers [8, 9, 11, 13, 16, 17, 19, 20, 26] for some relevant contributions related to the subject of this work. See also [3] for other results obtained by using our variational approach.

The plan of this paper is as follows. In Section 2 we introduce our notation and a suitable abstract setting (see Theorem 2.1). In Section 3 our main result (see Theorem 3.1) and some significant consequences (see Theorem 3.2 as well as Corollaries 3.4 and 3.6) are presented. Section 4 is devoted to a special case (see Theorem 4.1) of the main result. A concrete example of an application is given in Example 4.4.

In conclusion, we cite a recent monograph by Kristály *et al.* [15] as a general reference on the variational methods adopted here.

## 2. Preliminaries

We shall prove our theorems by applying the following version of Ricceri's variational principle [33, Theorem 2.1].

THEOREM 2.1. *Let $X$ be a reflexive real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that $\Phi$ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and $\Psi$ is sequentially weakly upper semicontinuous. For every $r > \inf_{u\in X} \Phi(u)$, put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$

*and $\delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r)$. Then, if $\delta < +\infty$, for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either* (c$_1$) *there is a global minimum of $\Phi$ which is a local minimum of $I_\lambda := \Phi - \lambda\Psi$, or* (c$_2$) *there is a sequence $\{u_n\}$ of pairwise distinct critical points* (*local minima*) *of $I_\lambda$ which weakly converges to a global minimum of $\Phi$, with* $\lim_{n\to\infty} \Phi(u_n) = \inf_{u\in X} \Phi(u)$.

REMARK 2.2. We also refer the interested reader to the papers [1, 2, 21] and references therein, in which Ricceri's variational principle and its variants have been successfully used to obtain the existence of infinitely many solutions for different boundary value problems.

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Denote by $X$ the Sobolev space $W_0^{1,2}(]0,1[)$, endowed with the norm

$$\|u\| := \left(\int_0^1 |u'(t)|^2\,dt\right)^{1/2}.$$

It is well known that the space $X$ is compactly embedded into $C^0([0,1])$ and $\|u\|_\infty \leq \|u\|$, where $\|u\|_\infty := \max_{t\in[0,1]} |u(t)|$.

Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be an $L^1$-Carathéodory function, which means that:

(a) $t \mapsto f(t,x)$ is measurable for every $x \in \mathbb{R}$;
(b) $x \mapsto f(t,x)$ is continuous for almost every (a.e.) $t \in [0,1]$;
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,1])$ such that

$$\sup_{|x|\leq\rho} |f(t,x)| \leq l_\rho(t)$$

for a.e. $t \in [0,1]$.

The potential $F$ of $f$ is defined by

$$F(t,\xi) := \int_0^\xi f(t,x)\,dx,$$

for every $(t,\xi) \in [0,1]\times\mathbb{R}$.

We say that a function $u \in X$ is a *weak solution* of problem (1.4), if $u$ satisfies

$$\int_0^1 \frac{u'(t)}{\sqrt{1+u'(t)^2}} v'(t)\,dt - \lambda \int_0^1 f(t,u(t))v(t)\,dt = 0,$$

for every $v \in X$.

## 3. Main results

In this section we establish the main abstract result of this paper. Let

$$B^0 := \limsup_{\xi\to 0^+} \frac{\int_{1/4}^{3/4} F(t,\xi)\,dt}{\xi^2}.$$

With the above notation we are able to prove the following result.

THEOREM 3.1. *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be an $L^1$-Carathéodory function such that*

($a_1$) $f(t,0) = 0$ *for a.e.* $t \in [0,1]$;
($a_2$) $F(t,\xi) \geq 0$ *for all* $(t,\xi) \in ([0,\frac{1}{4}] \cup [\frac{3}{4},1]) \times \mathbb{R}^+$.
*Assume that there exist two sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* $(0,+\infty)$*, with* $\lim_{n\to\infty} b_n = 0$*, such that, for a.e.* $t \in [0,1]$ *and every* $x \in [0,b_1]$,

$$|f(t,x)| \leq k,$$

*for some real constant* $k > 0$*, and*

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(a$_3$) $a_n^2 < \frac{7\sqrt{2}}{128} b_n^2$ *for each* $n \in \mathbb{N}$;

(a$_4$) $\mathcal{A}_0 := \lim_{n\to\infty} (\int_0^1 \max_{|\xi|\le b_n} F(t,\xi)\,dt - \int_{1/4}^{3/4} F(t,a_n)\,dt)/(\frac{7\sqrt{2}}{32} b_n^2 - 4a_n^2) < \frac{B^0}{4}$.

*Then, for each*

$$\lambda \in \Big(\frac{4}{B^0}, \frac{1}{\mathcal{A}_0}\Big),$$

*problem (1.4) admits a sequence of nontrivial and nonnegative weak solutions* $\{u_n\} \subset C^1([0,1])$ *which satisfy*

$$\lim_{n\to\infty} \|u_n\|_{C^1([0,1])} = 0.$$

Proof. Our aim is to apply Theorem 2.1 to problem (1.4). More precisely, to this end, following an idea of Obersnel and Omari in [28], we apply Theorem 2.1 to a modified problem and then, by means of a regularity result (see either [18] or directly [34, Proposition 3.7]), the critical points of the energy are actually solutions of the original problem. Let $s : [0,+\infty) \to (0,+\infty)$ be the $C^{1,1}$ nonincreasing function defined by

$$a(s) := \begin{cases} \dfrac{1}{\sqrt{1+s}} & s \in [0,1) \\ \dfrac{\sqrt{2}}{16}(s-2)^2 + \dfrac{7\sqrt{2}}{16} & s \in [1,2) \\ \dfrac{7\sqrt{2}}{16} & s \in [2,+\infty). \end{cases}$$

Set, for every $s \ge 0$,

$$A(s) := \int_0^s a(t)\,dt.$$

We have

$$\frac{7\sqrt{2}}{16} \le a(s) \le 1$$

and hence

$$\frac{7\sqrt{2}}{16} s \le A(s) \le s \tag{3.1}$$

for every $s \ge 0$. Further, as the function $s \mapsto sa(s^2)$ is increasing, the function $s \mapsto A(s^2)$ is convex in $[0,+\infty)$. Note that $a$ satisfies the structure and the regularity conditions assumed in [18]. For a.e. $t \in [0,1]$, we truncate $f$ as follows:

$$g(t,x) := \begin{cases} 0 & x \in (-\infty,0), \\ f(t,x) & x \in [0,b_1), \\ f(t,b_1) & x \in [b_1,+\infty), \end{cases}$$

where $b_1$ is from the sequence $\{b_n\}$. The function $g$ is $L^1$-Carathéodory and if $G : [0,1]\times\mathbb{R} \to \mathbb{R}$ denotes its primitive, that is, $G(t,\xi) := \int_0^\xi g(t,x)\,dx$ for all $(t,\xi) \in [0,1]\times\mathbb{R}$, then $g$ and $G$ satisfy the assumptions of the theorem. We now introduce the

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auxiliary problem

$$\begin{cases} -(a(|u'|^2)u')' = \lambda g(t,u) & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.2}$$

Let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{2}\int_0^1 A(|u'(t)|^2)\,dt, \quad \Psi(u) := \int_0^1 G(t,u(t))\,dt,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

for every $u \in X$.

Due to (3.1), $\Phi$ is well defined on $X$, continuous and coercive. Moreover, by the convexity of the function $s \mapsto A(s^2)$ in $[0,+\infty)$, $\Phi$ is convex and then sequentially weakly lower semicontinuous. The functional $\Psi$ is well defined and sequentially weakly (upper) continuous. Moreover, $\Phi$ and $\Psi$ are continuously Gâteaux differentiable with derivative given by

$$\Phi'(u)(v) = \int_0^1 a(|u'(t)|^2)u'(t)v'(t)\,dt$$

and

$$\Psi'(u)(v) = \int_0^1 g(t,u(t))v(t)\,dt,$$

for every $u, v \in X$.

Fix $\lambda$ as in the conclusion. First of all, we show that $\lambda < 1/\delta$. To this end, write

$$r_n := \frac{7\sqrt{2}}{32}b_n^2 \quad (\forall n \in \mathbb{N}).$$

Then, for all $u \in X$ with $\Phi(u) < r_n$, taking into account (3.1), we have

$$\frac{7\sqrt{2}}{32}\|u\|^2 \le \Phi(u) < r_n.$$

Thus

$$\|u\|_\infty \le b_n \quad (\forall n \in \mathbb{N})$$

for every $u \in X$ such that $\Phi(u) < r_n$. Then, for every $n \in \mathbb{N}$, it follows that

$$\varphi(r_n) \le \inf_{\Phi(u)<r_n} \frac{\int_0^1 \max_{|\xi|\le b_n} G(t,\xi)\,dt - \int_0^1 G(t,u(t))\,dt}{\frac{7\sqrt{2}}{32}b_n^2 - \frac{1}{2}\int_0^1 A(|u'(t)|^2)\,dt}.$$

Now, let $w_n$ be defined by

$$w_n(t) := \begin{cases} 4a_n t & t \in [0,1/4], \\ a_n & t \in (1/4,3/4], \\ 4a_n(1-t) & t \in (3/4,1], \end{cases}$$

for each $n \in \mathbb{N}$.

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Clearly, $w_n \in X$ and $\Phi(w_n) \le \frac{1}{2}\|w_n\|^2 = 4a_n^2$. Hence, by ($a_3$), we have $\Phi(w_n) < r_n$. Moreover, by ($a_2$), we also have

$$\Psi(w_n) \ge \int_{1/4}^{3/4} G(t, a_n)\, dt,$$

for each $n \in \mathbb{N}$. Therefore, it follows that

$$\varphi(r_n) \le \frac{\int_0^1 \max_{|\xi|\le b_n} G(t,\xi)\, dt - \int_{1/4}^{3/4} G(t, a_n)\, dt}{\frac{7\sqrt{2}}{32} b_n^2 - 4a_n^2},$$

for every $n \in \mathbb{N}$. Hence, bearing in mind ($a_4$), we can write

$$0 \le \delta \le \lim_{n\to\infty} \varphi(r_n) \le \mathcal{A}_0 < +\infty.$$

Taking into account the above relation, since $\lambda < 1/\mathcal{A}_0$, we also have $\lambda < 1/\delta$.

Now, we claim that the functional $I_\lambda$ does not have a local minimum at zero. Since $1/\lambda < B^0/4$, there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to\infty} \eta_n = 0$ and

$$\frac{1}{\lambda} < \tau < \frac{1}{4} \frac{\int_{1/4}^{3/4} G(t, \eta_n)\, dt}{\eta_n^2},$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$, let $s_n \in X$ defined by

$$s_n(t) := \begin{cases} 4\eta_n t & t \in [0, 1/4], \\ \eta_n & t \in (1/4, 3/4], \\ 4\eta_n(1-t) & t \in (3/4, 1]. \end{cases}$$

Note that $\lambda\tau > 1$. Then we obtain

$$\begin{aligned} I_\lambda(s_n) &= \Phi(s_n) - \lambda\Psi(s_n) \\ &\le 4\eta_n^2 - \lambda \int_{1/4}^{3/4} G(t, \eta_n)\, dt \\ &< 4\eta_n^2(1 - \lambda\tau) \\ &< 0 = \Phi(0) - \lambda\Psi(0) \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Thus, the fact that $\|s_n\| \to 0$ implies that $I_\lambda$ does not have a local minimum at zero. This, together with the fact that zero is the only global minimum of $\Phi$, shows that the functional $I_\lambda$ does not have a local minimum at the unique global minimum of $\Phi$. Therefore, by Lemma 2.1, there exists a sequence $\{u_n\}$ of pairwise distinct critical points of $I_\lambda$ that converges weakly to zero. In view of the fact that the embedding $X \hookrightarrow C^0([0, 1])$ is compact, we know that the critical points converge strongly to zero. In particular,

$$\lim_{n\to\infty} \|u_n\|_\infty = 0. \tag{3.3}$$

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Let us prove that the critical points of the energy are nonnegative. Arguing by a contradiction, assume that $u$ is a critical point of $I_\lambda$ and that the set

$$A := \{t \in [0,1] : u(t) < 0\}$$

is nonempty and of positive Lebesgue measure. Put $v := \min\{0, u\}$. Clearly, $v \in X$ and, taking into account that $u$ is a critical point, we have

$$\begin{aligned} 0 &= \Phi'(u)(v) - \lambda\Psi'(u)(v) \\ &= \int_0^1 a(|u'(t)|^2)u'(t)v'(t)\,dt - \lambda\int_0^1 g(t,u(t))v(t)\,dt \\ &= \int_A a(|u'(t)|^2)|u'(t)|^2\,dt \\ &\geq \frac{7\sqrt{2}}{16}\int_A |u'(t)|^2\,dt, \end{aligned}$$

since $a(s) \geq (7\sqrt{2})/16$ for all $s \geq 0$ and $g(t,s) = 0$ for a.e. $t \in [0,1]$ and every $s < 0$. Hence, since $u|_A \in W_0^{1,2}(A)$, $u \equiv 0$ on $A$ which is absurd. Hence, if $u_n$ is a critical point of $I_\lambda$, then it is a weak solution of the auxiliary problem (3.2), it is nonnegative and by (3.3), for $n$ big enough, $u_n(t) \leq b_1$ for every $t \in [0,1]$. Thus

$$\|u_n\|_\infty \leq b_1, \tag{3.4}$$

for $n$ sufficiently large. On the other hand, by our assumption on $f$ in $[0,1]\times[0,b_1]$, and bearing in mind the definition of $g$, it follows that

$$|g(t,x)| \leq k, \tag{3.5}$$

for a.e. $t \in [0,1]$ and $x \in \mathbb{R}$. Now, by (3.4) and (3.5), the regularity theory for problem (3.2) (see either [18] or directly [34, Proposition 3.7]) implies that there are constants $\beta \in (0,1)$ and $\kappa > 0$ such that, for every $n \in \mathbb{N}$, $u_n \in X \cap C^{1,\beta}([0,1])$ and

$$\|u_n\|_{C^{1,\beta}([0,1])} \leq \kappa.$$

Let us prove now that

$$\lim_{n\to\infty} \|u_n\|_{C^1([0,1])} = 0.$$

Indeed, assume by contradiction that there exists a subsequence $\{u_{n_h}\}$ such that

$$\lim_{h\to\infty} \|u_{n_h}\|_{C^1([0,1])} > 0.$$

Then, since (3.3) holds, we must have

$$\lim_{h\to\infty} \|u'_{n_h}\|_\infty > 0. \tag{3.6}$$

The Arzelà–Ascoli theorem yields the existence of a subsequence, still denoted by $\{u_{n_h}\}$, such that $\{u'_{n_h}\}$ is uniformly convergent to zero, in contradiction to (3.6). Accordingly we conclude that, for $n$ big enough, $\|u_n\|_{C^1([0,1])} \leq 1$. This completes the proof. □

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We now point out some consequences of Theorem 3.1. First, by setting

$$A_0 := \liminf_{\xi\to 0^+} \frac{\int_0^1 \max_{|x|\le\xi} F(t,x)\,dt}{\xi^2},$$

we get the following result.

THEOREM 3.2. *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be an $L^1$-Carathéodory function such that assumptions ($a_1$) and ($a_2$) in Theorem 3.1 are satisfied. Assume also that*

($a_5$) *there are two real positive constants $\sigma$ and $k$ such that, for a.e. $t\in[0,1]$ and every $x\in[0,\sigma]$, we have*

$$|f(t,x)|\le k;$$

($a_6$) $A_0 < \frac{7\sqrt{2}}{128}B^0$.

*Then, for each*

$$\lambda\in\Big(\frac{4}{B^0},\frac{7\sqrt{2}}{32A_0}\Big),$$

*problem (1.4) admits a sequence of nontrivial and nonnegative weak solutions $\{u_n\}\subset C^1([0,1])$ which satisfy*

$$\lim_{n\to\infty}\|u_n\|_{C^1([0,1])}=0.$$

PROOF. Let $\{b_n\}$ be a sequence of positive numbers which goes to zero such that

$$\lim_{n\to\infty}\frac{\int_0^1 \max_{|\xi|\le b_n} F(t,\xi)\,dt}{b_n^2}=A_0.$$

Taking $a_n=0$ for every $n\in\mathbb{N}$, by Theorem 3.1 the conclusion follows.

REMARK 3.3. Theorem 1.1 immediately follows by Theorem 3.2.

A special case of Theorem 3.2 is stated in the following corollary.

COROLLARY 3.4. *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be an $L^1$-Carathéodory function such that assumptions ($a_1$) and ($a_2$) in Theorem 3.1 and ($a_5$) in Theorem 3.2 are satisfied. Assume that*

$$A_0<\frac{7\sqrt{2}}{32}\quad\text{and}\quad B^0>4.$$

*Then, the problem*

$$\begin{cases}-\Big(\dfrac{u'}{\sqrt{1+u'^2}}\Big)'=f(t,u) & \text{in } (0,1),\\ u(0)=u(1)=0,\end{cases}$$

*admits a sequence of nontrivial and nonnegative weak solutions $\{u_n\}\subset C^1([0,1])$ which satisfy*

$$\lim_{n\to\infty}\|u_n\|_{C^1([0,1])}=0.$$

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REMARK 3.5. When $f$ is a nonnegative function, condition ($a_6$) becomes

($a'_6$) $A'_0 := \liminf_{\xi\to 0^+} (\int_0^1 F(t,\xi)\,dt)/\xi^2 < \frac{7\sqrt{2}}{128} B^0$.

In this case, ($a'_6$) ensures that for each

$$\lambda \in \Big(\frac{4}{B^0}, \frac{7\sqrt{2}}{32A'_0}\Big),$$

problem (1.4) admits a sequence of nontrivial and nonnegative weak solutions $\{u_n\} \subset C^1([0,1])$ which satisfy

$$\lim_{n\to\infty} \|u_n\|_{C^1([0,1])} = 0.$$

The next result is a consequence of Theorem 3.1 and guarantees the existence of infinitely many solutions to (1.4) for each $\lambda$ lying in a precise half-line.

COROLLARY 3.6. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an $L^1$-Carathéodory function such that assumptions ($a_1$) and ($a_2$) in Theorem 3.1 are satisfied. Assume that there exist a real sequence $\{a_n\}$ and a sequence $\{b_n\}$ in $(0,+\infty)$, with $\lim_{n\to\infty} b_n = 0$, such that, for a.e. $t \in [0,1]$ and every $x \in [0,b_1]$, have*

$$|f(t,x)| \le k,$$

*for some real constant $k > 0$, and ($a_3$) holds. Further, let*

($a_7$) $\int_{1/4}^{3/4} F(t,a_n)\,dt = \int_0^1 \max_{|\xi|\le b_n} F(t,\xi)\,dt$.

*If $B^0 > 0$, then, for each $\lambda > 4/B^0$, problem (1.4) admits a sequence of nontrivial and nonnegative weak solutions $\{u_n\} \subset C^1([0,1])$ which satisfy*

$$\lim_{n\to\infty} \|u_n\|_{C^1([0,1])} = 0.$$

PROOF. By ($a_7$) we obtain $\mathcal{A}_0 = 0$. Hence, since $B^0 > 0$, condition ($a_4$) of Theorem 3.1 holds and the proof is complete. □

## 4. A special case

The following theorem is a significant consequence of Theorem 3.1 that is not directly obtainable by the statements of [12, 25, 27, 28].

THEOREM 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $\inf_{\xi\ge 0} F(\xi) = 0$. Further, let $h \in L^\infty([0,1])$ with $\operatorname{ess\,inf}_{t\in[0,1]} h(t) > 0$. Suppose that there exist two sequences $\{a_n\}$ and $\{b_n\}$ in $(0,+\infty)$, with $a_n < b_n$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} b_n = 0$, such that:*

($a_8$) $\lim_{n\to\infty}(b_n)/(a_n) = +\infty$;

($a_9$) $\max_{x\in[a_n,b_n]} f(x) \le 0$ *for every* $n \in \mathbb{N}$;

($a_{10}$) $4/\int_{1/4}^{3/4} h(t)\,dt < \limsup_{\xi\to 0^+} (F(\xi))/(\xi^2) < +\infty$.

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*Then the problem*

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = h(t)f(u) & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{4.1}$$

*admits a sequence of nontrivial and nonnegative weak solutions* $\{u_n\} \subset C^1([0,1])$ *which satisfy*

$$\lim_{n\to\infty} \|u_n\|_{C^1([0,1])} = 0.$$

Proof. Our aim is to apply Theorem 3.1. If $\{a_n\}$ and $\{b_n\}$ are two real sequences satisfying our assumptions, we have

$$\frac{a_n^2}{b_n^2} < \frac{7\sqrt{2}}{128},$$

for every $n$ sufficiently large. Hence condition ($a_3$) in Theorem 3.1 holds. Now, in order to prove also condition ($a_4$), let us define the real sequence $\{h_n\}$ given by

$$h_n := \|h\|_{L^1([0,1])} \frac{\max_{|\xi|\le b_n} F(\xi)}{a_n^2} - \left(\int_{1/4}^{3/4} h(t)\,dt\right)\frac{F(a_n)}{a_n^2},$$

for every $n \in \mathbb{N}$. At this point observe that hypothesis ($a_9$) yields

$$\max_{|\xi|\le b_n} F(\xi) = \max_{|\xi|\le a_n} F(\xi). \tag{4.2}$$

Thus, since

$$\frac{\int_{1/4}^{3/4} h(t)\,dt}{\|h\|_{L^1([0,1])}} \le 1 \quad \text{and} \quad F(a_n) \ge 0,$$

and bearing in mind (4.2), we can write

$$\begin{aligned} \frac{\max_{|\xi|\le b_n} F(\xi)}{a_n^2} &= \frac{\max_{|\xi|\le a_n} F(\xi)}{a_n^2} \\ &\ge \frac{F(a_n)}{a_n^2} \\ &\ge \frac{\int_{1/4}^{3/4} h(t)\,dt}{\|h\|_{L^1([0,1])}} \frac{F(a_n)}{a_n^2}. \end{aligned}$$

for every $n \in \mathbb{N}$. Hence, since $h_n > 0$ for every $n \in \mathbb{N}$, we have

$$0 \le \limsup_{n\to\infty} h_n.$$

Further, by ($a_{10}$) we clearly have

$$0 < \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} < +\infty, \tag{4.3}$$

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and consequently (note that $a_n \searrow 0^+$ as $n \to \infty$) we obtain

$$0 < \limsup_{n\to\infty} \frac{F(a_n)}{a_n^2} < +\infty.$$

On the other hand, let $\xi_n \in (0, a_n]$ be a sequence such that $F(\xi_n) := \max_{|\xi|\le a_n} F(\xi)$ for every $n \in \mathbb{N}$. Thus

$$\begin{aligned}\limsup_{n\to\infty} \frac{\max_{|\xi|\le b_n} F(\xi)}{a_n^2} &= \limsup_{n\to\infty} \frac{\max_{|\xi|\le a_n} F(\xi)}{a_n^2}\\ &= \limsup_{n\to\infty} \frac{F(\xi_n)}{a_n^2}\\ &\le \limsup_{n\to\infty} \frac{F(\xi_n)}{\xi_n^2}.\end{aligned}$$

The above relations and (4.3) yield

$$0 \le \limsup_{n\to\infty} \frac{\max_{|\xi|\le b_n} F(\xi)}{a_n^2} \le \limsup_{n\to\infty} \frac{F(\xi_n)}{\xi_n^2} < +\infty.$$

Hence, there exists $\beta$ such that

$$0 \le \limsup_{n\to\infty} h_n = \beta. \tag{4.4}$$

Then, by ($a_8$) and (4.4), we have

$$\mathcal{A}_0 = \limsup_{n\to\infty} h_n \left(\frac{7\sqrt{2}}{32} \frac{b_n^2}{a_n^2} - 4\right)^{-1} = 0.$$

In conclusion, condition ($a_4$) holds. Finally, bearing in mind ($a_{10}$), we have

$$1 \in \left(\frac{4}{B^0}, +\infty\right).$$

Thanks to Theorem 3.1, the theorem is proved. □

REMARK 4.2. We observe that, in contrast to Theorem 4.1, studying problem (4.1), one of the key assumptions required by Faraci, is that

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty;$$

see [12, Theorem 1.1]. Moreover, we do not assume here that

$$\limsup_{n\to\infty} \frac{\max_{\xi\in[0,a_n]} F(\xi)}{b_n^2} < \frac{7\sqrt{2}}{32\|h\|_{L^1([0,1])}},$$

see [12, Remark 2.2].

The next result is a direct consequence of Theorem 4.1.

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Proposition 4.3. *Let* $h \in L^\infty([0,1])$ *with* $\operatorname{ess\,inf}_{t\in[0,1]} h(t) > 0$ *and consider* $\{a_n\}$ *and* $\{b_n\}$ *to be two sequences in* $(0,+\infty)$ *such that* $b_{n+1} < a_n < b_n$ *for every* $n \in \mathbb{N}$, $\lim_{n\to\infty} b_n = 0$, *and* $\lim_{n\to\infty} b_n/(a_n) = +\infty$. *Moreover, let* $\varphi \in C^1([0,1])$ *a nonnegative function such that* $\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0$ *and*

$$\max_{s\in[0,1]} \varphi(s) > \frac{4}{\int_{1/4}^{3/4} h(t)\,dt}.$$

*Further, let* $g : \mathbb{R} \to \mathbb{R}$ *be the function defined by*

$$g(t) := \begin{cases} \varphi\Big(\dfrac{t - b_{n+1}}{a_n - b_{n+1}}\Big) & if\ t \in \bigcup_{n\geq n_0} [b_{n+1}, a_n], \\ 0 & otherwise. \end{cases}$$

*Then, the problem*

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+u'^2}}\Big)' = h(t)y(u) & in\ (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.5}$$

*where*

$$y(u) := |u|(2g(u) + ug'(u)),$$

*admits a sequence of nontrivial and nonnegative weak solutions* $\{u_n\} \subset C^1([0,1])$ *which satisfy*

$$\lim_{n\to\infty} \|u_n\|_{C^1([0,1])} = 0.$$

Proof. Let $\{a_n\}$ and $\{b_n\}$ be two positive sequences satisfying our assumptions. We claim that all the hypotheses of Theorem 4.1 are verified. Indeed, we have

$$F(\xi) := \int_0^\xi y(t)dt = \xi^2 g(\xi), \quad \forall \xi \in \mathbb{R}^+.$$

Moreover, direct computations ensure that

$$\max_{x\in[a_{n+1},b_{n+1}]} y(x) = 0,$$

for every $n \in \mathbb{N}$, and

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = \limsup_{\xi\to 0^+} g(\xi) = \max_{s\in[0,1]} \varphi(s) > \frac{4}{\int_{1/4}^{3/4} h(t)\,dt}.$$

The conclusion follows by Theorem 4.1. □

Finally, we present a concrete example of the application of Proposition 4.3.

Example 4.4. Let $h \in L^\infty([0,1])$ with $\operatorname{ess\,inf}_{t\in[0,1]} h(t) > 0$ and take

$$a_n := \frac{1}{n!n} \quad \text{and} \quad b_n := \frac{1}{n!},$$

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for every $n \geq 2$. Define $\varphi \in C^1([0,1])$ by

$$\varphi(s) := \alpha e^4 e^{1/s(s-1)}, \quad (\forall s \in [0,1])$$

and set

$$g(t) := \begin{cases} \varphi\left(\dfrac{t - 1/(n+1)!}{1/(n!n) - 1/(n+1)!}\right) & \text{if } t \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where $A := \bigcup_{n\geq 2}[1/(n+1)!, 1/(n!n)]$. If

$$\alpha > \frac{4}{\int_{1/4}^{3/4} h(t)\,dt},$$

then problem (4.5) admits a sequence of nontrivial and nonnegative weak solutions $\{u_n\} \subset C^1([0,1])$ which satisfy

$$\lim_{n\to\infty} \|u_n\|_{C^1([0,1])} = 0.$$

REMARK 4.5. For some nice and interesting applications related to certain nonlocal problems, see the papers [31, 32].

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