COMPOSITIO MATHEMATICA

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Compositio Math. **149** (2013), 1856–1870.

doi:10.1112/S0010437X12000322

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Abstract

In this paper, we develop a method of solving the Poincaré–Lelong equation, mainly via the study of the large time asymptotics of a global solution to the Hodge–Laplace heat equation on $(1,1)$-forms. The method is effective in proving an optimal result when M has nonnegative bisectional curvature. It also provides an alternate proof of a recent gap theorem of the first author.

1. Introduction

Solving the Poincaré–Lelong equation amounts to finding, for a given real $(1,1)$-form ρ, a smooth function u such that $\sqrt{-1}\partial\bar{\partial}u=\rho$. Motivated by geometric considerations, on a complete noncompact Kähler manifold (M,g), this was first studied by Mok *et al.* [MSY81] under some restrictive conditions including a point-wise quadratic decay on $\|\rho\|$, nonnegative bisectional curvature and maximum volume growth on M. There have been many publications since then (e.g. [Fan06, Ni98, NST01]). Finally in [NT03], Theorem 1.1 below was proved. Before we state the result we fix some notations. Let (M,g) be a complete noncompact Riemannian manifold and let $o\in M$ be a fixed point. For a smooth function f on M, let

$$k_f(x,r)=\frac{1}{V_x(r)}\int_{B_x(r)}|f| \tag{1.1}$$

where $B_x(r)$ is the geodesic ball of radius r with center at x and $V_x(r)$ is the volume of $B_x(r)$. If we fix a point $o\in M$, we also denote $k_f(o,r)$ by $k_f(r)$. In [NT03, Theorem 6.1] we proved the following result.

Theorem 1.1. *Let M^n (with $m=2n$ being the real dimension) be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let ρ be a real d-closed $(1,1)$-form. Suppose that*

$$\int_0^\infty k_{\|\rho\|}(s)\,ds<\infty, \tag{1.2}$$

and

$$\liminf_{r\to\infty}\left[\exp(-\alpha r^2)\cdot\int_{B_o(r)}\|\rho\|^2(y)\,d\mu(y)\right]<\infty \tag{1.3}$$

for some $\alpha>0$. Then there is a solution u of the Poincaré–Lelong equation

$$\sqrt{-1}\partial\bar{\partial}u=\rho.$$

Received 13 September 2011, accepted 15 March 2012, revised 25 March 2013, accepted in final form 27 March 2013, published online 9 September 2013.
2010 Mathematics Subject Classification 53C21 (primary), 32Q15, 35B40, 35K05 (secondary).
Keywords: Poincaré–Lelong equation, Hodge-Laplacian heat equation, Kähler manifolds, Convex exhaustion.
The first author is partially supported by NSF grant DMS-1105549. The second author is partially supported by Hong Kong RGC General Research Fund #CUHK 403011

Moreover, for any $0<\epsilon<1$, u *satisfies*

$$\alpha_1 r\int_{2r}^{\infty} k_{\|\rho\|}(s)\,ds+\beta_1\int_0^{2r} s\,k_{\|\rho\|}(s)\,ds\geqslant u(x)$$
$$\geqslant \beta_3\int_0^{2r} s\,k_{\|\rho\|}(s)\,ds-\alpha_2 r\int_{2r}^{\infty} k_{\|\rho\|}(s)\,ds-\beta_2\int_0^{\epsilon r} s\,k_{\|\rho\|}(x,s)\,ds \tag{1.4}$$

for some positive constants $\alpha_1(m)$, $\alpha_2(m,\epsilon)$ *and* $\beta_i(m)$, $1\leqslant i\leqslant 3$, *where* $r=r(x)$.

Due to the technical nature of the assumption (1.3), which arises from the parabolic method employed in [NT03], and is related to the uniqueness of the heat equation, it is desirable to be able to remove it. The purpose of this paper is to prove the following theorem.

Theorem 1.2. *Let* (M^n,g) *be a complete noncompact Kähler manifold (of complex dimension* n*) with nonnegative Ricci curvature and nonnegative quadratic orthogonal bisectional curvature. Suppose that* ρ *is a smooth* d*-closed real* $(1,1)$*-form on* M *and let* $f=\|\rho\|$ *be the norm of* ρ*. Suppose that*

$$\int_0^{\infty} k_f(r)\,dr<\infty. \tag{1.5}$$

Then there is a smooth function u *so that* $\rho=\sqrt{-1}\partial\bar\partial u$*. Moreover, for any* $0<\epsilon<1$*, the estimate (1.4) holds.*

Note that in the theorem we only require (M^n,g) has nonnegative Ricci curvature and nonnegative quadratic orthogonal bisectional curvature (see § 3 for the definition), which is weaker than the nonnegativity of the bisectional curvature (also see the Appendix for the discussion on the relations between various curvature conditions). The current result is also more general than those previous versions of our work with a less involved proof.

The solution space to a Poincaré–Lelong equation clearly is an affine space consisting u_s+u_h with u_s being a special solution and u_h being any element of the linear space of the pluriharmonic functions. The estimate (1.4) selects the minimum one among them. In the view that on Kähler manifolds with nonnegative Ricci curvature, the sublinear growth is the optimal necessary condition to imply the constancy of a pluriharmonic function, the assumption (1.2) is almost the optimal condition which one can expect to ensure that estimate (1.4) selects the unique (up to a constant) solution.

The use of the Hodge–Laplace heat equation here is motivated by that of [Ni12]. On the other hand, here we establish the closedness of the global $(1,1)$-form solution to the Hodge–Laplace heat equation, and in § 5 we obtain an alternate proof of the gap theorem in [Ni12] without appealing to the relative monotonicity.

We organize the paper as follows. In § 2 we state and prove a general result which reduces solving the Poincaré–Lelong to solving the Hodge–Laplace heat equation and heat equation on functions, together with certain assumed estimates and d-closedness of the solution. In § 3, we construct certain exhaustion functions and cut-off functions. Then we establish estimates for solutions to the Hodge–Laplace heat equation with initial data being the $(1,1)$-forms obtained by multiplying the constructed cut-off functions to a fixed $(1,1)$-form. Section 4 contains the estimates needed and the proof of the d-closedness of the global solution, which is obtained by taking the limit of the approximating ones obtained in § 3, of the Hodge–Laplace heat equation. In § 5 we give the proof of Theorem 1.2 and an alternate proof of the gap theorem.

2. A general method of solving the Poincaré–Lelong equation

Let (M^n, g) be a complete noncompact Kähler manifold with complex dimension n with Kähler form. Let ρ be a real $(1,1)$ form. If in a local coordinate $\rho = \sqrt{-1}\rho_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}$ we denote the trace of ρ, $\sum_{i,j=1}^n g^{i\bar{j}}\rho_{i\bar{j}}$, by $\mathrm{tr}(\rho)$, which is equal to $\Lambda\rho$, where Λ is the adjoint of L with $L\sigma = \omega \wedge \sigma$.

Let $\Delta = -\Delta_d = -(d\delta + \delta d)$ be the Hodge Laplacian for forms. On Kähler manifolds it is well known that $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.

In this section we shall prove the result below, which reduces the solving of the Poincaré–Lelong equation to the study of two parabolic equations, namely the heat equation and Hodge–Laplace heat equation, as well as obtaining relevant estimates.

Theorem 2.1. *Suppose M has nonnegative Ricci curvature. Suppose further that the following are true.*

(a) *There is an $(1,1)$-form $\eta(x,t)$ satisfying*

$$\begin{cases} \eta_t(x,t) - \Delta\eta(x,t) = 0, & \text{in } M \times [0,\infty), \\ \eta(x,0) = \rho(x), & x \in M, \end{cases} \tag{2.1}$$

such that $\eta(x,t)$ is closed for all t, and for some $p > 0$

$$\lim_{R\to\infty} \frac{1}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} \|\eta\|^p(x,t)\, d\mu(x)\, dt = 0 \tag{2.2}$$

for all $T > 0$. Moreover, $\lim_{t\to\infty} \eta(x,t) = 0$.

(b) *There is a function $u(x)$ solving $\Delta u(x) = \mathrm{tr}(\rho)(x)$, and a solution $v(x,t)$ of*

$$\begin{cases} v_t(x,t) - \Delta v(x,t) = 0, & \text{in } M \times [0,\infty), \\ v(x,0) = 2u(x), & x \in M, \end{cases} \tag{2.3}$$

such that for the same $p > 0$,

$$\lim_{R\to\infty} \frac{1}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} |v|^p(x,t) + |u(x)|^p\, d\mu(x)\, dt = 0 \tag{2.4}$$

for all $T > 0$ and $\lim_{t\to\infty} \partial\bar{\partial} v(x,t) = 0$.

Then $2\sqrt{-1}\partial\bar{\partial}u = \rho$.

Before we prove the theorem, let us first recall the following consequence of Kähler identities.

Lemma 2.1. *For a $(1,1)$-form η, we have*

$$\partial\bar{\partial}\Lambda\eta = \sqrt{-1}\Delta_{\bar{\partial}}\eta - \bar{\partial}\Lambda\partial\eta + \partial\Lambda\bar{\partial}\eta - \Lambda\partial\bar{\partial}\eta. \tag{2.5}$$

Hence if $\Lambda\partial\eta = \Lambda\bar{\partial}\eta = \Lambda\partial\bar{\partial}\eta = 0$, then

$$\partial\bar{\partial}\Lambda\eta = \sqrt{-1}\Delta_{\bar{\partial}}\eta = \sqrt{-1}\Delta_\partial\eta. \tag{2.6}$$

In particular, if η is d-closed, then (2.6) is true.

Proof. Recall the Kähler identities: $\partial\Lambda - \Lambda\partial = -\sqrt{-1}\bar{\partial}^*$, $\bar{\partial}\Lambda - \Lambda\bar{\partial} = \sqrt{-1}\partial^*$. $\Delta_\partial = \Delta_{\bar{\partial}}$, we have

$$\begin{aligned} \partial\bar{\partial}\Lambda\eta &= \partial\Lambda\bar{\partial}\eta + \sqrt{-1}\partial\partial^*\eta \\ &= \sqrt{-1}\Delta_\partial\eta + \partial\Lambda\bar{\partial}\eta - \sqrt{-1}\partial^*\partial\eta \\ &= \sqrt{-1}\Delta_\partial\eta + \partial\Lambda\bar{\partial}\eta - (\bar{\partial}\Lambda - \Lambda\bar{\partial})\partial\eta. \end{aligned}$$

From this, (2.5) follows. □

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We also need the following maximum principle for solutions of the heat equation.

LEMMA 2.2. *Suppose (M^m, g) is a complete noncompact Riemannian manifold with nonnegative Ricci curvature and u is a smooth nonnegative subsolution of the heat equation on $M \times [0, T]$. Assume that there exists a sequence $R_i \to \infty$ and $p > 0$ such that*

$$\lim_{i\to\infty} \frac{1}{R_i^2 V_o(R_i)} \int_0^T \int_{B_o(R_i)} u^p(y, s)\, d\mu(y)\, ds = 0. \tag{2.7}$$

Then $u(x, t) \leqslant \sup_{y\in M} u(y, 0)$. In particular, if u_1 and u_2 are two solutions of the heat equation such that $|u_1|$ and $|u_2|$ satisfy the decay conditions (2.7) and if $u_1 = u_2$ at $t = 0$, then $u_1 \equiv u_2$.

Proof. By [LT91, Theorem 1.2], for any $\epsilon > 0$, there is a constant $C > 0$ independent of R such that

$$\sup_{B_o(\frac{1}{2}R)\times[0,T]} u^p \leqslant \frac{C}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} u^p(y, s)\, d\mu(y)\, ds + (1 + \epsilon)\Big(\sup_{y\in M} u(y, 0)\Big)^p$$

if $R^2 \geqslant 4T$. Let $R \to 0$ and then let $\epsilon \to 0$; the first assertion follows.

To prove the second assertion, apply the above argument to $(|u_1 - u_2|^2 + \epsilon)^{\frac{1}{2}}$ for $\epsilon > 0$ and let $\epsilon \to 0$. □

Proof of Theorem 2.1. Let η be as in assertion (a). Let $\phi = \mathrm{tr}(\eta)$, namely $\Lambda(\eta)$. Then ϕ satisfies the heat equation in $M \times [0, \infty)$ with initial value $\mathrm{tr}(\rho)$. Let

$$w(x, t) = -2 \int_0^t \phi(x, s)\, ds.$$

Then

$$w_t - \Delta w = -2\mathrm{tr}(\rho), \quad w(x, 0) = 0.$$

Hence $\tilde{v}(x, t) = 2u(x) - w(x, t)$ satisfies

$$\tilde{v}_t - \Delta\tilde{v} = 0, \quad \tilde{v}(x, 0) = 2u(x).$$

By (2.2), (2.4) and Lemma 2.2, we conclude that $v = 2u - w$.

On the other hand, by Lemma 2.1 and the fact that η is closed, we have

$$\begin{aligned} \frac{d}{dt}(\eta + \sqrt{-1}\partial\bar{\partial}w) &= -2\Delta_{\bar{\partial}}\eta + \sqrt{-1}\partial\bar{\partial}w_t \\ &= -2\Delta_{\bar{\partial}}\eta - 2\sqrt{-1}\partial\bar{\partial}\Lambda\eta \\ &= 0. \end{aligned}$$

At the same time, at $t = 0$, $\eta + \sqrt{-1}\partial\bar{\partial}w(\cdot, t) = \rho$. Hence this equation holds for all t. That is to say,

$$\eta + 2\sqrt{-1}\partial\bar{\partial}u - \sqrt{-1}\partial\bar{v} = \rho.$$

Since $\lim_{t\to\infty} \eta(x, t) = \lim_{t\to\infty} \sqrt{-1}\partial\bar{\partial}v(x, t) = 0$, we have $2\sqrt{-1}\partial\bar{\partial}u = \rho$. □

Remark 2.1. From the proof it is easy to see that if in assertion (a) we only assume that $\Lambda\partial\eta = \Lambda\bar{\partial}\eta = \Lambda\partial\bar{\partial}\eta = 0$, then the conclusion of the theorem is still true. Moreover from the proof we see that η is closed. In fact, one can check that if η satisfies the equation $(\partial/\partial t)\eta + \Delta_{\partial}\eta = 0$, then

$$\Phi = \eta(t) + \int_0^t (\bar{\partial}^*\partial + \partial^*\bar{\partial})\eta - \sqrt{-1} \int_0^t \Lambda\partial\bar{\partial}\eta$$

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is closed, provided that $\rho(x,0)$ is closed. Hence, in particular, $\Lambda\partial\eta=\Lambda\bar\partial\eta=\Lambda\partial\bar\partial\eta=0$ implies that $d\eta=0$.

Hence in order to solve the Poincaré–Lelong equation, it is sufficient to find η and u, v as in the theorem.

By the previous work [NST01, NT03], under a certain average growth condition on $\|\rho\|$ we can find u and v satisfying assertion (b) of Theorem 2.1. We now list this known result and a useful estimate below. First recall the following result from [NT03, Lemma 1.1].

Lemma 2.3. *Let (M,g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let $o\in M$ be a fixed point. Let $h\geqslant 0$ be a continuous function. Let $H(x,y,t)$ be the heat kernel and let*

$$v(x,t)=\int_M H(x,y,t)h(y)\,d\mu(y).$$

Assume that v is defined on $M\times[0,T]$ with

$$\liminf_{r\to\infty}\exp\left(-\frac{r^2}{20T}\right)\int_{B_o(r)}h=0. \tag{2.8}$$

Then for any $r^2\geqslant t>0$ and $p\geqslant 1$

$$\frac{1}{V_o(r)}\int_{B_o(r)}v^p(x,t)\,dx\leqslant C(n,p)\left[k_{h^p}(4r)+t^{-p}\left(\int_{4r}^{\infty}s\exp\left(-\frac{s^2}{20t}\right)k_h(s)\,ds\right)^p\right].$$

Proof. Note the proof in [NT03, pp. 467–468] can be carried over because only (2.8) is needed for the integration by parts. □

Proposition 2.1. *Let (M,g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let $o\in M$ be a fixed point. Let f be a smooth function on M such that*

$$\int_0^\infty k_f(r)\,dr<\infty.$$

Then we can find functions u and v with $\Delta u=f$, v satisfying (2.3) such that (2.4) is true for $p=1$, and $\lim_{t\to\infty}\partial\bar\partial v(x,t)=0$. Moreover u satisfies (1.4). In fact, u and v are given by

$$u(x)=\int_M(G(o,y)-G(x,y))f(y)\,d\mu(y),\qquad \text{if } M \text{ is nonparabolic}$$

$$v(x,t)=\int_M H(x,y,t)u(y)\,d\mu(y).$$

Here $G(x,y)$ is the minimal positive Green's function of M (if M is nonparabolic) and $H(x,y,t)$ is the heat kernel of M.

Proof. First consider the case that M is nonparabolic. The existence of u and v given by the expressions in the proposition so that $\lim_{t\to\infty}\partial\bar\partial v(x,t)=0$ follows from [NT03, Lemma 6.1]. For the sake of the completeness we note here that the argument of [NT03, Lemma 6.1] shows that for any fixed r, on $B_o(r)\times[t_0-1,t_0+1]$, the function $w(x,t)\doteqdot v(x,t)-v(o,t_0)$ is a solution of the heat equation, and $|w|(x,t)$ is bounded by a constant $C(t_0)$, which satisfies $\lim_{t_0\to\infty}C(t_0)=0$. The interior Schauder estimates then enable us to conclude that $\lim_{t\to\infty}\partial\bar\partial v(x,t)=0$ uniformly on $B_0(r/2)$.

By the proof of [NT03, Lemma 6.1], we have

$$k_u(r) = o(r^2).$$

From this and Lemma 2.3, (2.4) is true for $p = 1$ in this case.

In general, by considering $\widetilde{M} = M \times \mathbb{R}^4$, we can find $\widetilde{u}$ as above. By [Fan06, pp. 458–460], $\widetilde{u}$ is independent of $y \in \mathbb{R}^4$ for $(x, y) \in M \times \mathbb{R}^4$. Let $u(x) = \widetilde{u}(x, y)$. Then $\Delta u = f$ and $k_u(r) = o(r^2)$. The existence of v is as in the previous case. □

To apply Theorem 2.1 we also need a *d-closed* solution of the Hodge–Laplace heat equation with estimate (2.2). The existence of such solutions is the content of the next two sections.

3. Solution of the Hodge–Laplace heat equation: preliminary results

In this section we collect some basic results needed for the later discussions and construct a solution of the Hodge–Laplace heat equation (2.1) via a suitable approximation.

First we need the existence of an exhaustion function on complete manifolds with nonnegative Ricci curvature with the property that it has a bounded complex Hessian if additionally (M^n, g) is a Kähler manifold with nonnegative quadratic orthogonal bisectional curvature.

Recall that a Kähler manifold (M^n, g) is said to have *nonnegative quadratic orthogonal bisectional curvature* if, at any point and any unitary frame $\{e_i\}$,

$$\sum_{i,j} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geqslant 0 \tag{3.1}$$

for all real numbers a_i. Let $o \in M$ be a fixed point, $r(o, x)$ be the distance function to o.

Let

$$\zeta(x, t) = \int_M H(x, y, t) r(o, y)\, d\mu(y)$$

be the positive solution of the heat equation with $r(o, x)$ being the initial value. Here $H(x, y, t)$ is the heat kernel.

Proposition 3.1. (i) *Assume that (M, g) has nonnegative Ricci curvature. Then, for any $t > 0$, $\zeta(x, t)$ is a smooth exhaustion function. Moreover $|\nabla\zeta|(x, t) \leqslant 1$.*

(ii) *Assume additionally that (M, g) has nonnegative quadratic orthogonal bisectional curvature. Then, for any $t > 0$, there exists C such that $\|\partial\bar{\partial}\zeta\|(x, t) \leqslant C$. Furthermore C can be chosen with $C = 1/\sqrt{2t}$.*

Proof. Part (i) follows from [NT03, Corollary 1.1 and Lemma 1.4]. For part (ii), note that the Bochner formula implies that

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla\zeta|^2 \geqslant 2\|\nabla^2\zeta\|^2. \tag{3.2}$$

As in [NT03, proof of Lemma 3.1], by multiplying a suitable cut-off function to (3.2) and performing the integration by parts we have that for any $x \in M$, and for any $T, R > 0$,

$$\int_0^T \fint_{B_x(R)} \|\nabla^2\zeta\|^2 \leqslant C\left(R^{-2}\int_0^T \fint_{B_x(2R)} |\nabla\zeta|^2 + \fint_{B_x(2R)} |\nabla r|^2\right) \leqslant C(1 + R^{-2}T) \tag{3.3}$$

for some constant C independent of x and R. Now one can apply [LT91, Theorem 1.1] to conclude the bound of $\|\partial\bar{\partial}\zeta\|$, since by [NT03, proof of Lemma 2.1] (see also [GK67], [Wil13, Lemma 5.1]) one only needs the nonnegativity of the quadratic orthogonal bisectional curvature to conclude that $\|\partial\bar{\partial}\zeta\|(x,t)$ is a subsolution to the heat equation (see also (3.8) below). More precisely, under the curvature assumption of part (ii) we have that

$$\left(\Delta - \frac{\partial}{\partial t}\right)\|\partial\bar{\partial}\zeta\|^2(x,t) \geqslant 2\|\nabla\partial\bar{\partial}\zeta\|^2(x,t). \tag{3.4}$$

Putting (3.2) and (3.4) together we have that

$$\left(\Delta - \frac{\partial}{\partial t}\right)(|\nabla\zeta|^2 + 2t\|\partial\bar{\partial}\zeta\|^2) \geqslant 0. \tag{3.5}$$

The estimate $|\nabla\zeta|(x,t) \leqslant 1$ and (3.3) enable one to apply the maximum principle to $|\nabla\zeta|^2 + 2t\|\partial\bar{\partial}\zeta\|^2$, and conclude that

$$|\nabla\zeta|^2(x,t) + 2t\|\partial\bar{\partial}\zeta\|^2(x,t) \leqslant 1.$$

This gives the precise upper bound claimed in part (ii). □

The Bochner formulae we have applied above are the special cases of the general formulae described below. Let η be a (p,q)-form valued in a holomorphic Hermitian vector bundle E with local frame $\{E_\alpha\}$ and locally $\eta = \sum \eta^\alpha E_\alpha$.

$$\eta^\alpha = \frac{1}{p!q!}\sum \eta^\alpha_{I_p\bar{J}_q}\, dz^{I_p} \wedge dz^{\bar{J}_q}.$$

Here $I_p = (i_1, \ldots, i_p)$, $\bar{J}_q = (\bar{j}_1, \ldots, \bar{j}_q)$ and $dz^{I_p} = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, $dz^{\bar{J}_q} = dz^{\overline{j_1}} \wedge \cdots \wedge dz^{\overline{j_q}}$. The Kodaira–Bochner formulae include the following two identities:

$$\begin{aligned}(\Delta_{\bar{\partial}}\eta)^\alpha_{I_p\bar{J}_q} = {} & -\sum_{ij} g^{\bar{j}i}\nabla_i\nabla_{\bar{j}}\eta^\alpha_{I_p\bar{J}_q} + \sum_{\nu=1}^{q} \Omega^{\alpha\bar{l}}_{\beta\,\bar{j}_\nu}\,\eta^\beta_{I_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q} \\ & + \sum_{\nu=1}^{q} R^{\bar{l}}{}_{\bar{j}_\nu}\,\eta^\alpha_{I_p\bar{j}_1\cdots(\bar{l})\cdots\bar{j}_q} - \sum_{\mu=1}^{p}\sum_{\nu=1}^{q} R_{i_\mu}{}^{k\bar{l}}{}_{\bar{j}_\nu}\,\eta^\alpha_{i_1\cdots(k)_\mu\cdots i_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}(\Delta_{\bar{\partial}}\eta)^\alpha_{I_p\bar{J}_q} = {} & -\sum_{ij} g^{i\bar{j}}\nabla_{\bar{j}}\nabla_i\eta^\alpha_{I_p\bar{J}_q} + \sum_{\nu=1}^{q} \Omega^{\alpha\bar{l}}_{\beta\bar{j}_\nu}\,\eta^\beta_{I_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q} - \sum_{\beta}\Omega^\alpha_\beta\eta^\beta_{I_p\bar{J}_q} \\ & + \sum_{\mu=1}^{p} R_{i_\mu}{}^{k}\eta^\alpha_{i_1\cdots(k)_\mu\cdots i_p\bar{J}_q} - \sum_{\mu=1}^{p}\sum_{\nu=1}^{q} R_{i_\mu}{}^{k\bar{l}}{}_{\bar{j}_\nu}\,\eta^\alpha_{i_1\cdots(k)_\mu\cdots i_p\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_q},\end{aligned} \tag{3.7}$$

where $(k)_\mu$ means that the index in the μth position is replaced by k. Here

$$\Theta^\alpha_\beta = \frac{\sqrt{-1}}{2\pi}\sum \Omega^\alpha_{\beta\, i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

is the curvature of E and $\Omega^\alpha_\beta = \Lambda\Theta^\alpha_\beta$ is the mean curvature.

Lemma 3.1. *Let (M,g) be a Kähler manifold.*

(i) *Suppose that M has nonnegative quadratic bisectional curvature. Let $\eta(x,t)$ be a $(1,1)$-form satisfying*

$$\frac{\partial}{\partial t}\eta(x,t) - \Delta\eta(x,t) = \xi(x,t)$$

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with $\xi(x,t)$ being another $(1,1)$-form. Then

$$\left(\frac{\partial}{\partial t}-\Delta\right)\|\eta\|^2(x,t)\leqslant -2\|\nabla\eta\|^2(x,t)+2\langle\xi,\eta\rangle(x,t). \tag{3.8}$$

In particular, for any $\epsilon>0$,

$$\left(\frac{\partial}{\partial t}-\Delta\right)(\|\eta\|^2(x,t)+\epsilon)^{\frac{1}{2}}\leqslant\|\xi\|(x,t).$$

(ii) *Suppose that M has nonnegative Ricci curvature. Let $\sigma(x,t)$ be a $(1,0)$-form satisfying*

$$\frac{\partial}{\partial t}\sigma(x,t)-\Delta\sigma(x,t)=0.$$

Then

$$\left(\frac{\partial}{\partial t}-\Delta\right)\|\sigma\|^2(x,t)\leqslant -2\|\nabla\sigma\|^2(x,t). \tag{3.9}$$

In particular, for any $\epsilon>0$,

$$\left(\frac{\partial}{\partial t}-\Delta\right)(\|\sigma\|^2(x,t)+\epsilon)^{\frac{1}{2}}\leqslant 0.$$

Proof. Applying (3.6) and (3.7) to $\eta(x,t)$, $\sigma(x,t)$ and using the curvature assumption, the results follow. □

Our strategy of constructing a global solution η to the Hodge–Laplace heat equation is by two approximations. First we construct $\rho^{(i)}$ by multiplying the initial d-closed $(1,1)$-form ρ by suitable cut-off functions $\phi^{(i)}$, which we describe below.

Let $\kappa(s)$ be a smooth cut-off function on $\mathbb{R}$ such that $\kappa(s)=0$ for $|s|>1$. For a sequence $\{R_i\}$ with $R_i\to\infty$, let

$$\phi^{(i)}(x)=\kappa\left(\frac{\zeta(x,1)}{R_i}\right)$$

where $\zeta(z,t)$ is the exhaustion function constructed by solving the heat equation with distance function as the initial data. Proposition 3.1 implies that there exists an absolute constant $C_1>0$ such that

$$|\nabla\phi^{(i)}|(x)+\|\partial\bar\partial\phi^{(i)}\|\leqslant\frac{C}{R_i}. \tag{3.10}$$

Let $\rho^{(i)}(x)=\phi^{(i)}(x)\rho(x)$. In general one can not expect that $\rho^{(i)}$ is closed. However, we have the following estimate:

$$\|\partial\rho^{(i)}\|(x)+\|\bar\partial\rho^{(i)}\|(x)+\|\partial\bar\partial\rho^{(i)}\|(x)\leqslant\frac{C_1}{R_i}\|\rho\|(x). \tag{3.11}$$

A solution $\eta^{(i)}(x,t)$ of the Hodge–Laplace heat equation with initial data $\rho^{(i)}(x)$ is constructed via the compact exhaustion detailed as follows. Let $\{\Omega_\nu\}$ be a sequence of exhaustion domains. Let Φ_ν be the solution of the following initial boundary value problem:

$$\begin{cases}\Phi_t-\Delta\Phi=0, & \text{in } \Omega_\nu\times(0,\infty),\\ \lim_{t\to 0}\Phi(x,t)=\rho^{(i)}(x), & x\in\Omega_\nu,\\ \mathbf{n}\Phi=0, & \text{on } \partial\Omega_\nu\times(0,\infty),\\ \mathbf{t}\Phi=0, & \text{on } \partial\Omega_\nu\times(0,\infty).\end{cases} \tag{3.12}$$

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Here $\mathbf{n}\Phi$ and $\mathbf{t}\Phi$ denote the normal and tangential parts of Φ (see [Mor66] for the basic definitions and related properties). The solvability of (3.12) is classical. See for example [Eva10, Mor66, LSU67]. For ν large, the solution Φ_ν is smooth since the initial and boundary values are compatible. Let

$$v^{(i)}(x,t) \doteqdot \int_M H(x,y,t)\|\rho^{(i)}\|(y)\,d\mu(y) \quad \text{and} \quad v(x,t) \doteqdot \int_M H(x,y,t)\|\rho\|(y)\,d\mu(y). \tag{3.13}$$

Clearly $v^{(i)}(x,t) \leqslant v(x,t)$, provided v is defined. The Bochner formula (3.8) and the maximum principle imply that

$$\|\Phi_\nu\|(x,t) \leqslant v^{(i)}(x,t).$$

Thus, by the Schauder's estimate, after possibly passing to a subsequence, $\{\Phi_\nu\}$ converges to a limit solution $\eta^{(i)}(x,t)$, which is a $(1,1)$-form with the initial value $\rho^{(i)}(x)$. Note that if ρ is real, then $\eta^{(i)}$ is real by uniqueness. To summarize, we have the following lemma.

Lemma 3.2. *Let $\eta^{(i)}$ be as above. Suppose that $v(x,t)$ in (3.13) is well defined. Then, after possibly passing to a subsequence, $\{\eta^{(i)}\}$ converges to a $(1,1)$-form η satisfying the Hodge–Laplace heat equation on $M \times [0,\infty)$ with initial value ρ. Moreover,*

$$\|\eta^{(i)}\|(x,t) \leqslant v^{(i)}(x,t) \quad \text{and} \quad \|\eta\|(x,t) \leqslant v(x,t). \tag{3.14}$$

In particular, for each i, $\|\eta^{(i)}\|(x,t)$ is bounded on M, and for any compact subset K, $\|\eta^{(i)}\|(x,t)$ is bounded uniformly (in terms of i) on $K \times [0,T]$, for any $T > 0$.

Next we will prove that under certain conditions, η satisfies the conditions in Theorem 2.1(a).

4. Global solutions of the Hodge–Laplace heat equation

In this section we prove the following result on the global solutions of the Hodge–Laplace heat equation.

Theorem 4.1. *Let (M,g) be a complete Kähler manifold with nonnegative quadratic orthogonal bisectional curvature and with nonnegative Ricci curvature. Assume that ρ is a d-closed $(1,1)$-form such that $f = \|\rho\|$ satisfies*

$$\limsup_{R\to\infty} \frac{k_f(R)}{R^2} = 0. \tag{4.1}$$

Then there exists a solution of

$$\begin{cases} \eta_t - \Delta\eta = 0, & \text{in } M \times [0,\infty), \\ \eta(x,0) = \rho(x), & x \in M. \end{cases} \tag{4.2}$$

such that η is a closed $(1,1)$-form. Furthermore we have that:

(a) *for any $T > 0$,*

$$\lim_{R\to\infty} \frac{1}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} \|\eta\|(x,t)\,d\mu(x)\,dt = 0; \tag{4.3}$$

(b) *$\lim_{t\to\infty} \eta(x,t) = 0$ for all $x \in M$ provided that $\lim_{R\to\infty} k_f(R) \to 0$.*

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Proof. By the decay assumption on $\|\rho\|$,

$$v(x,t)=\int_M H(x,y,t)\|\rho\|(y)\,d\mu(y)$$

is well defined by [NT03, Lemma 2.1]. Let $\{\eta^{(i)}(x,t)\}$ and η be the solutions of the Hodge–Laplace heat equation obtained in Lemma 3.2, which are real $(1,1)$-forms. Then $\|\eta\|(x,t)\leqslant v(x,t)$. Now by (4.1) we have that $k_f(4R)/R^2\to 0$ as $R\to\infty$. By Lemma 2.3 and the fact that $\|\eta\|(x,t)\leqslant v(x,t)$, we conclude that (4.3) is true for any $T>0$.

By [LT91, Theorem 1.1], if additionally we assume that $\lim_{R\to\infty} k_f(R)=0$, then we have that $\lim_{t\to\infty} v(x,t)=0$. Hence $\lim_{t\to\infty}\eta(x,t)=0$.

It remains to prove $d\eta=0$. By Remark 2.1, it is sufficient to prove that $\Lambda\partial\eta=\Lambda\bar\partial\eta=\Lambda\partial\bar\partial\eta=0$. These will be established via lemmas below. With them we complete the proof of the theorem. □

Lemma 4.1. *Let $\eta^{(i)}$ and η as in the proof of Theorem 4.1. Let $\sigma_1^{(i)}=\Lambda\bar\partial\eta^{(i)}$ and $\sigma_2^{(i)}=\Lambda\partial\eta^{(i)}$. Then*

$$\|\sigma_j^{(i)}\|(x,t)\leqslant C\int_M H(x,y,t)\|\rho^{(i)}\|(y)\,d\mu(y) \tag{4.4}$$

for $j=1,2$. In particular, for each i, $\|\sigma_j^{(i)}\|$ are bounded on $M\times[0,\infty)$. Moreover, $\Lambda\partial\eta=\Lambda\bar\partial\eta=0$.

Proof. By Lemma 3.2, for each i, $\|\eta^{(i)}\|$ is bounded on $M\times[0,\infty)$. By Lemma 3.1,

$$\left(\frac{\partial}{\partial t}-\Delta\right)\|\eta^{(i)}\|^2\leqslant -2\|\nabla\eta^{(i)}\|^2.$$

As in the proof of Proposition 3.1, there exists a $C>0$, such that for all i, and for $T>0$, $r>0$,

$$\int_0^T ⨍_{B_o(r)}\|\nabla\eta^{(i)}\|^2\leqslant C(1+r^{-2}T).$$

Hence

$$\int_0^T ⨍_{B_o(r)}\|\sigma_1^{(i)}\|^2\leqslant C(1+r^{-2}T).$$

On the other hand, by Lemma 3.1 again, for any $\epsilon>0$,

$$\left(\frac{\partial}{\partial t}-\Delta\right)(\|\sigma_1^{(i)}\|^2+\epsilon)^{\frac12}\leqslant 0.$$

Hence one can apply the maximum principle to conclude that

$$(\|\sigma_1^{(i)}\|^2(x,t)+\epsilon)^{\frac12}\leqslant\int_M H(x,y,t)(\|\Lambda\bar\partial\rho^{(i)}\|(y)+\epsilon)\,d\mu(y).$$

Letting $\epsilon\to 0$, we have

$$\|\sigma_1^{(i)}\|(x,t)\leqslant\int_M H(x,y,t)\|\Lambda\bar\partial\rho^{(i)}\|(y)\,d\mu(y).$$

Similarly, one can prove

$$\|\sigma_2^{(i)}\|(x,t)\leqslant\int_M H(x,y,t)\|\Lambda\partial\rho^{(i)}\|(y)\,d\mu(y).$$

The last assertion follows from (3.11) and the fact that a subsequence of $\eta^{(i)}$ converge locally uniformly to η. □

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Next we want to prove that $\Lambda\partial\bar\partial\eta(x,t)=0$. For that, let

$$\theta=\int_0^t \Lambda\partial\bar\partial\eta(x,\tau)\,d\tau \quad \text{and correspondingly} \quad \theta^{(i)}(x,t)=\int_0^t \Lambda\partial\bar\partial\eta^{(i)}(x,\tau)\,d\tau.$$

For that we need more lemmas.

LEMMA 4.2. *For any $T>0$, there exists $C_i>0$ such that*

$$\int_0^T \fint_{B_o(R)} \|\theta^{(i)}\|^2(x,t)\,d\mu(x)\,dt \leqslant C_i \tag{4.5}$$

for all R.

Proof. Let $w^{(i)}(x,t)=\Lambda\eta^{(i)}(x,t)$. By Lemma 3.2, $w^{(i)}(x,t)$ is bounded on $M\times[0,\infty)$ for each i and is uniformly (in i) bounded on $K\times[0,T]$ for any compact K. It also satisfies the heat equation. Using the differential equation/inequality

$$\begin{aligned}\left(\Delta-\frac{\partial}{\partial t}\right)(w^{(i)})^2(x,t) &= 2|\nabla w^{(i)}|^2(x,t),\\ \left(\Delta-\frac{\partial}{\partial t}\right)|\nabla w^{(i)}|^2(x,t) &\geqslant 2\|\nabla^2 w^{(i)}\|^2(x,t)\end{aligned}$$

integration by parts as in the proof of Proposition 3.1 yields, for $R^2\geqslant T$,

$$\int_0^T \fint_{B_o(R)} (|\nabla w^{(i)}|^2(x,t)+\|\nabla^2 w^{(i)}\|^2(x,t))\,d\mu(x)\,dt \leqslant C_i \tag{4.6}$$

for some $C_i>0$ independent of R. Let $\sigma_1^{(i)}(x,t)=\Lambda\bar\partial\eta^{(i)}$ and $\sigma_2^{(i)}(x,t)=\Lambda\partial\eta^{(i)}(x,t)$. By Lemma 2.1,

$$\theta^{(i)}(x,t)=-\int_0^t \partial\bar\partial w^{(i)}(x,\tau)d\tau+\int_0^t \sqrt{-1}\Delta_{\bar\partial}\eta^{(i)}(x,\tau)+\partial\sigma_1^{(i)}(x,\tau)-\bar\partial\sigma_2^{(i)}(x,\tau)\,d\tau. \tag{4.7}$$

Note that

$$\left\|\int_0^t \sqrt{-1}\Delta_{\bar\partial}\eta^{(i)}(x,\tau)\,d\tau\right\|=\left\|\int_0^t \sqrt{-1}\eta^{(i)}_\tau d\tau\right\|\leqslant v^{(i)}(x,t)+\|\rho^{(i)}\|(x). \tag{4.8}$$

On the other hand, by Lemma 3.1,

$$\left(\Delta-\frac{\partial}{\partial t}\right)\|\sigma_j^{(i)}\|^2(x,t)\geqslant 2\|\nabla\sigma_j^{(i)}\|^2(x,t) \quad \text{for } j=1,2, \tag{4.9}$$

and $\|\sigma_j^{(i)}\|^2$ are bounded on $M\times[0,\infty)$ by Lemma 4.1. As before, multiplying (4.9) by a cut-off function, then integration by parts yields

$$\int_0^T \fint_{B_o(R)} \|\nabla\sigma_1^{(i)}\|^2(x,t)+\|\nabla\sigma_2^{(i)}\|^2(x,t)\,d\mu(x)\,dt \leqslant C_i. \tag{4.10}$$

The claimed result now follows from (4.6), (4.7), (4.8) and (4.10). □

Lemma 4.2 allows us to apply the maximum principle to $\|\theta^{(i)}\|(x,t)$, once we establish that $\|\theta^{(i)}\|(x,t)$ is a subsolution of a heat type equation. To this end we have the following lemma.

LEMMA 4.3. *For any $\epsilon>0$,*

$$\left(\Delta-\frac{\partial}{\partial t}\right)(\|\theta^{(i)}\|(x,t)+\epsilon)^{\frac12}\geqslant -\|\Lambda\partial\bar\partial\rho^{(i)}\|(x). \tag{4.11}$$

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Proof. First, direct calculation shows that

$$\begin{aligned}\frac{\partial}{\partial t}\theta^{(i)}(x,t)-\Delta\theta^{(i)}(x,t)&=\Lambda\partial\bar\partial\eta^{(i)}(x,t)-\int_0^t\Delta\Lambda\partial\bar\partial\eta^{(i)}(x,\tau)\,d\tau\\&=\Lambda\partial\bar\partial\eta^{(i)}(x,t)-\int_0^t\Lambda\partial\bar\partial\Delta\eta^{(i)}(x,\tau)\,d\tau\\&=\Lambda\partial\bar\partial\eta^{(i)}(x,t)-\int_0^t\Lambda\partial\bar\partial(\eta^{(i)})_t(x,\tau)\,d\tau\\&=\Lambda\partial\bar\partial\rho^{(i)}(x).\end{aligned}$$

The claimed inequality follows from (3.8) and Lemma 3.1. □

Lemma 4.4. *Let $\theta(x,t)=\int_0^t\Lambda\partial\bar\partial\eta(x,\tau)\,d\tau$. Then $\theta(x,t)=0$ for all x,t.*

Proof. Let

$$z^{(i)}(x,t)=\int_0^t\int_M H(x,y,\tau)\|\Lambda\partial\bar\partial\rho^{(i)}\|(y)\,d\mu(y)\,d\tau.$$

It is easy to see that

$$\left(\Delta-\frac{\partial}{\partial t}\right)z^{(i)}(x,t)=-\|\Lambda\partial\bar\partial\rho^{(i)}\|(x).$$

Note that $z^{(i)}$ is bounded. By Lemmas 4.2, 4.3 and the maximum principle Lemma 2.2, we conclude that

$$\begin{aligned}\|\theta^{(i)}\|(x,t)&\leqslant z^{(i)}(x,t)\\&\leqslant\frac{C_1}{R_i}\int_0^t\int_M H(x,y,\tau)\|\rho\|(y)\,d\mu(y)\,d\tau\\&\leqslant\frac{C_1}{R_i}\int_0^t v(x,\tau)\,d\tau\\&\to 0\end{aligned}$$

as $i\to\infty$. Here we have used (3.11). □

By the definition of θ, we conclude that $\Lambda\partial\bar\partial\eta=0$. The proof of Theorem 4.1 is completed.

5. Proof of Theorem 1.2 and an alternate proof of the gap theorem

In this section we shall prove the result generalizing [NT03, Theorem 6.1] and provide an alternate proof of the gap theorem of [Ni12]. With the notations of previous sections we can state the main theorem.

Theorem 5.1. *Let (M^n,g) be a complete noncompact Kähler manifold (of complex dimension n) with nonnegative Ricci curvature and nonnegative quadratic orthogonal bisectional curvature. Suppose that ρ is a smooth d-closed real $(1,1)$-form on M and let $f=\|\rho\|$ be the norm of ρ. Suppose that*

$$\int_0^\infty k_f(r)dr<\infty. \tag{5.1}$$

Then there is a smooth function u so that $\rho=\sqrt{-1}\partial\bar\partial u$. Moreover, for any $0<\epsilon<1$, the estimate (1.4) holds.

Proof. Observe that the volume comparison implies $k_f(r) \leqslant 2^{2n} k_f(s)$ for all $s \in (r, 2r)$. Thus (5.1) implies that $\lim_{r\to\infty} k_f(r)\, r = 0$. Thus Theorem 5.1 follows from Theorems 2.1, 4.1 and Proposition 2.1. □

In the rest we shall give an alternate proof of the gap theorem of [Ni12]. The proof here avoids the use of the *relative monotonicity*. It uses a Li–Yau–Hamilton type estimate in [NN11] together with Lemma 4.1, but avoids solving the Poincaré–Lelong equation.

Theorem 5.2. *Let (M, g) be a complete Kähler manifold with nonnegative bisectional curvature. Assume that $\rho \geqslant 0$ is a d-closed $(1,1)$-form. Suppose that*

$$\int_0^r s \fint_{B_o(s)} \|\rho\|(y)\, d\mu(y)\, ds = o(\log r) \tag{5.2}$$

for some $o \in M$. Then $\rho \equiv 0$.

Proof. By the assumption (5.2), we can apply Theorem 4.1. Let $\eta^{(i)}$ and η be as in the proof of Theorem 4.1 with $\|\eta^{(i)}\|$ bounded and

$$\|\eta\|(x,t) \leqslant \int_M H(x,y,t)\|\rho\|(x)\, d\mu.$$

Since ρ_i is nonnegative, by [NT03, Theorem 2.1], we can conclude that $\eta^{(i)}$ is nonnegative. Hence η is nonnegative. By [NN11, Theorem 1.1], we have

$$w_t + \frac{1}{2}(\bar\partial^*\Lambda\bar\partial + \partial^*\Lambda\partial)\eta + \frac{w}{t} \geqslant 0 \tag{5.3}$$

where $w = \Lambda\eta$. By Lemma 4.1, $\Lambda\bar\partial\eta = \Lambda\partial\eta = 0$. Let $u(x)$ be the solution of $\Delta u = \Lambda\rho$ obtained in [NST01], which satisfies the estimates (1.4). Let $v(x,t)$ be the solution of the heat equation with initial data $2u$. By Proposition 2.1 and the estimates of u and η, we conclude that

$$v(x,t) = 2u(x) + \int_0^t w(x,\tau)\, d\tau.$$

By Proposition 2.1, through previous work [NST01], for any $x_0 \in M$ we have that

$$u(x) = o(\log(r(x_0,x)))$$

as $x \to \infty$, where $r(x_0,x)$ denotes the distance function as before. By the moment type estimate (cf. [Ni02, Theorem 3.1]) we have that $v(x_0,t) = o(\log t)$ as $t \to \infty$. This fact, together with (5.3), implies that $\Lambda\eta(x_0,t) = 0$ for any $t > 0$. Otherwise, assume that $\Lambda\eta(x_0,t_0) = \delta > 0$ for some $t_0 > 0$. Note that we have $v_t = 2\Lambda\eta = 2w$. Then by (5.3) we have, for all $t \geqslant t_0$,

$$v_t(x,t) \geqslant \frac{2t_0\delta}{t},$$

which in particular implies that

$$v(x_0,t) \geqslant 2t_0\delta \log\left(\frac{t}{t_0}\right) - v(x_0,t_0).$$

This contradicts $v(x,t) = o(\log(t))$. The contradiction implies that $\Lambda\eta(x_0,t) = 0$ for any $t \geqslant 0$. Hence we have $\Lambda\rho(x) \equiv 0$, which implies the claim $\rho \equiv 0$ by the nonnegativity of ρ. □

https://doi.org/10.1112/S0010437X12000322 Published online by Cambridge University Press

Acknowledgements
The authors would like to thank Gilles Carron for pointing out a mistake in a previous version of this paper. The authors would also like to thank Alexander Grigoryan for useful discussions and the referee for helpful suggestions.

Appendix.

Here we make some comments on the relation of the conditions of *nonnegative bisectional curvature* (NB), *nonnegative orthogonal bisectional curvature* (NOB), *nonnegative quadratic orthogonal bisectional curvature* (NQOB) (3.1), and the following curvature condition on $(2,1)$-forms σ:

$$-\langle E(\sigma),\sigma\rangle \geqslant -a^2|\sigma|^2 \tag{A.1}$$

for some $a>0$, where

$$(E(\sigma))_{ij\bar{k}} = -R^l_i\sigma_{lj\bar{k}} - R^l_j\sigma_{il\bar{k}} - R^{\bar{l}}_{\bar{k}}\sigma_{ij\bar{l}} + 2R^{l\bar{m}}_{i\bar{k}}\sigma_{lj\bar{m}} + 2R^{l\bar{m}}_{j\bar{k}}\sigma_{il\bar{m}}. \tag{A.2}$$

For condition (A.1) with $a=0$, we abbreviate '(nonnegativity associated with certain 3-forms)' as '(NCF)', as well as an invariant representation of them. This condition on $(2,1)$-forms arises from the Bochner formula on $(2,1)$-forms, which is related to a previous method of constructing global d-closed solutions to the Hodge–Laplace heat equation. Algebraically, (NB) is stronger than (NOB), which in turn is stronger than (NQOB). We introduce some notations for convenience. First, the curvature operator of a Kähler manifold can be viewed as bilinear form on $\mathfrak{gl}(n,\mathbb{C})$ (which can be identified with $\wedge^{1,1}(\mathbb{C}^n)$ via the metric) in the sense that for any $X\wedge\overline{Y}$ and $Z\wedge\overline{W}$

$$\langle \mathrm{Rm}(X\wedge\overline{Y}), \overline{Z\wedge\overline{W}}\rangle \doteqdot \mathrm{Rm}(X\wedge\overline{Y}, \overline{Z\wedge\overline{W}}) = R_{X\overline{Y}W\overline{Z}}.$$

Hence, for any $\Omega=(\Omega^{i\bar{j}})$, it is easy to check that $\langle \mathrm{Rm}(\Omega),\overline{\Omega}\rangle = R_{i\bar{j}k\bar{l}}\Omega^{i\bar{j}}\overline{\Omega^{k\bar{l}}}\in\mathbb{R}$. Hence one can identify (cf. [Wil13]) the condition (NB) as

$$\{\mathrm{Rm} \mid \langle \mathrm{Rm}(\Omega),\overline{\Omega}\rangle \geqslant 0,\ \text{ for any } \Omega, \mathrm{rank}(\Omega)=1\}. \tag{A.3}$$

Similarly, condition (NOB) is equivalent to

$$\{\mathrm{Rm} \mid \langle \mathrm{Rm}(\Omega),\overline{\Omega}\rangle \geqslant 0,\ \text{ for any } \Omega, \mathrm{rank}(\Omega)=1, \Omega^2=0\}. \tag{A.4}$$

For the other two conditions we recall two operators from the study of the Ricci flow. The first one is the $\barwedge$ operator on A, B, any two Hermitian symmetric transformations on $T'M$, defined by

$$\begin{aligned}(A\barwedge B)_{i\bar{j}k\bar{l}} &= A_{i\bar{j}}B_{k\bar{l}} + B_{i\bar{j}}A_{k\bar{l}} + A_{i\bar{l}}B_{k\bar{j}} + B_{i\bar{l}}A_{k\bar{j}}\\ &= 2\langle (A\wedge\bar{B} + B\wedge\bar{A})(e_i\wedge e_{\bar{j}}), \overline{e_l\wedge e_{\bar{k}}}\rangle\\ &\quad + 2\langle (A\wedge\bar{B} + B\wedge\bar{A})(e_k\wedge e_{\bar{j}}), \overline{e_l\wedge e_{\bar{i}}}\rangle.\end{aligned}$$

The resulting operator so defined is also a Kähler curvature operator which, in particular, satisfies the first Bianchi identity. Here $(A\wedge B)(X\wedge Y) = \frac{1}{2}(A(X)\wedge B(Y) + B(X)\wedge A(Y))$ as defined in [Wil13]. This operator is the one involved in the $\mathsf{U}(n)$-invariant irreducible decomposition of the space of the Kähler curvature operators. Now the condition (NQOB) is equivalent to

$$\{\mathrm{Rm} \mid \langle \mathrm{Rm}, A^2\barwedge \mathrm{id} - A\barwedge A\rangle \geqslant 0,\ \text{ for all Hermitian symmetric } A\}. \tag{A.5}$$

For (NCF), it can be identified with

$$\mathrm{Ric}\wedge\mathrm{id}\wedge\mathrm{id} - 2\,\mathrm{Rm}\wedge\mathrm{id} \geqslant 0 \tag{A.6}$$

on the space of $(2,1)$-forms. Here, for $X \wedge Y \wedge Z$, $\mathrm{Rm} \wedge \mathrm{id}(X \wedge Y \wedge Z) = \mathrm{Rm}(X \wedge Y) \wedge Z - \mathrm{Rm}(X \wedge Z) \wedge Y + X \wedge \mathrm{Rm}(Y \wedge Z)$, and $\mathrm{Ric} \wedge \mathrm{id} \wedge \mathrm{id}$ is defined as $(\mathrm{Ric} \wedge \mathrm{id}) \wedge \mathrm{id}$.

It is known that (NB) and (NOB) are Ricci flow invariant conditions [Wil13]. It would be interesting to find out about (NQOB) and (NCF). Our speculation is that condition (NCF) follows from (NQOB) and $\mathrm{Ric} \geqslant 0$. If this speculation holds (whose validity can be easily checked when $n = 2$) one can write an alternate proof of the d-closedness result obtained in § 4. This is where (NCF) assumption arises.

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