IDENTITIES OF NON-ASSOCIATIVE ALGEBRAS

J. MARSHALL OSBORN

In the first part of this paper we define a partial ordering on the set of all homogeneous identities and find necessary and sufficient conditions that an identity does not imply any identity lower than it in the partial ordering (we call such an identity irreducible). Perhaps the most interesting property established for irreducible identities is that they are skew-symmetric in any two variables of the same odd degree and symmetric in any two variables of the same even degree. The results of the first section are applied to commutative algebras in the remainder of the paper. It is proved that any commutative algebra with unity element of characteristic not 2, 3, or 5 satisfying an identity of degree 4 or less not implied by the commutative law is either power-associative or satisfies one of two other identities. A similar, but more complicated theorem is proved for commutative algebras satisfying identities of degree 5.

An application of the results of §1 to non-commutative algebras has already been made in **(1)**.

1. Let A be an algebra (possibly infinite-dimensional) over a field F, and let $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_m$ be a set of positive integers. We shall say that A satisfies a homogeneous identity of type $[n_1, n_2, \ldots, n_m]$ if there exists a polynomial $P(x_1, \ldots, x_m)$ in a set of non-commuting non-associating indeterminates $x_1, \ldots, x_m$ over the field F such that the number of x_i's in each term of $P(x_1, \ldots, x_m)$ is exactly n_i, and such that P vanishes when $x_1, \ldots, x_m$ are replaced by any m elements from A. Here n_i is called the degree of x_i in P, and the sum $n_1 + \ldots + n_m$ is called the degree of P. Note that the symbol $[n_1, n_2, \ldots, n_m]$ is only defined when the integers $n_1, \ldots, n_m$ are ordered so that $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_m$.

Hereafter it will be tacitly assumed that all identities mentioned are homogeneous except when otherwise noted. We shall also find it convenient—and more concise—not to distinguish between the polynomial P and the identity $P = 0$.

We now define a partial ordering on the set of homogeneous identities as follows. Let P and P' be two identities of degree n and n' and of type $[n_1, n_2, \ldots, n_m]$ and $[n'_1, n'_2, \ldots, n'_{m'}]$ respectively. Then P is to be lower than P' in the partial ordering if and only if either (i) $n < n'$ or (ii) $n = n'$ and $n_j > n'_j$ for the first integer j such that $n_j \neq n'_j$. Two identities are incomparable if and only if they have the same order type.

Received August 6, 1963. The research for this paper was supported by National Science Foundation Grant G-19052.

If P and Q are two identities over F, and S is any set of identities over F, we shall say that P *implies* Q *relative to* S if every algebra with unity over F satisfying P and all the identities of S also satisfies Q. An identity will be called *irreducible* relative to S if it does not imply any identity lower than it in the partial ordering which is not already implied by the set S. If an identity is irreducible relative to the null set of identities, it will be called *absolutely irreducible.*

Observe that when an identity P is partially or totally linearized, the type increases; while if two or more variables are set equal, the type decreases. Thus, if two variables are set equal in an irreducible identity, it either vanishes or gives an identity implied by S. It is also sometimes possible to obtain from P an identity lower than P that does not arise just by setting variables equal if one first linearizes and then sets variables equal. For example, an identity of type [2, 2] could be partially linearized to give an identity of type [2, 1, 1], then have two variables set equal to give an identity of type [3, 1].

Another way in which an identity can imply identities of lower type involves the notion of partial derivative. If we formally replace a variable x_i in a polynomial $P(x_1, \ldots, x_m)$ by the expression $x_i + 1$, we get an inhomogeneous polynomial which can be expressed as a sum of homogeneous parts. Thus,

$$\begin{aligned}(1)\qquad P(x_1, \ldots, x_i + 1, \ldots, x_m) &= P(x_1, \ldots, x_i, \ldots, x_m)\\ &+ P_1(x_1, \ldots, x_i, \ldots, x_m) + \ldots + P_{n_i}(x_1, \ldots, 1, \ldots, x_m),\end{aligned}$$

where x_i has degree $n_i - k$ in P_k. By the kth partial derivative of P with respect to x_i (written $\partial_i^k P$) we shall mean the polynomial P_k just defined. For $k > n_i$, $\partial_i^k P$ is defined to be zero, and for $k = 0$ we define $\partial_i^0 P = P$. Our definition of the kth partial derivative has been chosen to differ from the more usual one by a factor of $k!$ in order to avoid difficulties for characteristic p. In our notation for partial derivatives, equation (1) becomes

$$(2)\qquad P(x_1, \ldots, x_i + 1, \ldots, x_n) = \sum_{k=0}^{\infty} \partial^k P.$$

It is easy to verify the relations $\partial_i^k(\partial_j^l P) = \partial_j^l(\partial_i^k P)$ and to derive

$$\begin{aligned}(3)\qquad & P(x_1 + \alpha_1, x_2 + \alpha_2, \ldots, x_m + \alpha_m)\\ &= \sum \alpha_1^{k_1}\alpha_2^{k_2} \ldots \alpha_m^{k_m}\partial_1^{k_1}\partial_2^{k_2} \ldots \partial_m^{k_m} P(x_1, x_2, \ldots, x_m),\end{aligned}$$

where $\alpha_1, \ldots, \alpha_m \in F$ and where the summation is on the k_i's, each running from zero to n_i (or to infinity).

THEOREM 1. *Let A be an algebra over a field F satisfying an identity P of type $[n_1, \ldots, n_m]$ and let the cardinality of F be at least $n_1 + 1$. If A' is the algebra consisting of A with a unity element adjoined, then A' satisfies P if and only if A satisfies all the partial derivatives of P. If A already has a unity element, then it satisfies all the partial derivatives of P.*

COROLLARY 1. *An identity P of type $[n_1, \ldots, n_m]$ over a field of at least $n_1 + 1$*

elements implies all of its partial derivatives. If P is irreducible relative to a set S, its partial derivatives are implied by S. If P is absolutely irreducible, its partial derivatives are all zero.

COROLLARY 2. *If an identity is satisfied by an algebra with unity element, then the sum of its coefficients is zero.*

To prove this theorem, suppose first that A has a unity element. Then the α_i's in (3) may be regarded as multiples of that unity element, and hence as elements of A. Thus, the left side of (3) vanishes for any way of replacing $x_1, \ldots, x_m$ by elements from A, showing that the right side does also. Writing

$$Q_i = \sum \alpha_2^{k_2} \ldots \alpha_m^{k_m} \partial_1^{i} \partial_2^{k_2} \ldots \partial_m^{k_m} P$$

for $i = 0, 1, \ldots, n_1$, we may express (3) as $0 = \sum_i \alpha_1^{i} Q_i$. Choosing $n_1 + 1$ distinct values for α_1 gives $n_1 + 1$ equations in the $n_1 + 1$ expressions Q_i with a non-singular matrix of coefficients. Thus, the Q_i's are all zero, and iterating this process m times shows that all partial derivatives are zero, proving the last part of Theorem 1.

If A' satisfies P, then the proof that we have just given shows that A' satisfies all partial derivatives of P. Hence the same is true for its subalgebra A. Conversely, if A satisfies all partial derivatives of P, the right-hand side of (3) vanishes on A. But every element of A' may be represented uniquely in the form $x + \alpha$ for $x \in A$ and $\alpha \in F$, so that (3) just states that P vanishes on A'.

The first corollary follows immediately from the theorem, while the second follows when F has at least $n_1 + 1$ elements from the remark that the partial derivative $\partial_1^{n_1} \ldots \partial_m^{n_m} P$ is just the sum of the coefficients of P. Corollary 2 can also be proved directly from the relation $P(1, \ldots, 1) = 0$, showing that no restriction on the cardinality of F is needed.

We are now in a position to show that the properties already derived for irreducible identities are, in fact, sufficient to characterize the class of irreducible identities.

THEOREM 2. *The identity P is irreducible relative to a set of identities S if and only if the following two conditions are satisfied:*

(i) *Every partial derivative of P is implied by S.*

(ii) *If certain of the variables in P or in a linearization of P are combined to give an identity lower than P, then the resulting identity will be implied by S.*

If P does not satisfy (i) or (ii), it clearly implies a lower identity, and hence is not irreducible. For the converse we must show that for each identity Q lower than P and not implied by S, there exists an algebra with unity element satisfying S and P, but not Q. To construct such an algebra, let S^* be S augmented by all partial derivatives of identities in S, let Q have type $[t_1, t_2, \ldots, t_r]$, and let E be the free algebra over F on the r symbols $z_1, z_2, \ldots, z_r$

(a basis of E is given by the set of non-associative non-commutative monomials in $z_1, \ldots, z_r$). If J is the ideal of E generated by the set

$$J_0 = \{a | I(a_1, \ldots, a_k) = a \text{ for some } I \in S^*, \text{ and } a_1, \ldots, a_k \in E\},$$

then $C = E/J$ is the free algebra satisfying S^* on r generators, in the sense that every algebra over F satisfying S^* and having r generators is a homomorph of C. Since Q is not implied by S^*, there exists at least one homomorph of C not satisfying Q, and hence C does not satisfy Q. But letting $z'_i = z_i + J$, we observe that $Q(z'_1, \ldots, z'_r) = 0$ would imply that C satisfies Q, since every set of r elements of C generates a subalgebra which is a homomorphic image of C in such a way that these r elements are the images of $z'_1, \ldots, z'_r$ respectively. Hence $Q(z'_1, \ldots, z'_r) \neq 0$ and $Q(z_1, \ldots, z_r) \notin J$.

Definining an element of E to be homogeneous if it is a linear combination of monomials in $z_1, \ldots, z_r$, all of which have the same degree in each z_1, we see that every element of J_0 is homogeneous. From this it is easy to see that J, regarded as a vector space, has a basis consisting of homogeneous elements of E. Letting K be the subspace of E spanned by those monomials in $z_1, \ldots, z_r$ in which the degree of z_i is at least $t_i + 1$ for some i, we see that K is an ideal of E which also has a basis of homogeneous elements. Then any homogeneous element of $J + K$ is the sum of a homogeneous element in J and a homogeneous element in K. In particular, a homogeneous element of degree t_i in z_i for each i can only be in $J + K$ if it is in J. This shows that $Q(z_1, \ldots, z_r) \notin J + K$. Hence, the algebra $B = E/(J + K)$ satisfies S^* but not Q.

We show next that B satisfies P. Letting $y_i = z_i + (J + K)$ for $i = 1, \ldots, r$, we observe that any product of more than

$$t = \sum_{i=1}^{r} t_i$$

elements vanishes, since this is true for monomials in the y_i's. If Q has degree less than P, this implies that P is satisfied trivially. If P also has degree t, let it have type $[n_1, \ldots, n_m]$, and let $b_1, \ldots, b_m$ be any m elements of B. Expressing each b_j as a linear combination of y_i's and products of y_i's, we may express $P(b_1, \ldots, b_m)$ as a linear combination of products of y_i's and observe that each term of $P(b_1, \ldots, b_m)$ which involves a non-linear term from at least one b_j will vanish since the degree is too high. Thus we may assume that the b_j's are linear combinations of the y_i's. Then $P(b_1, \ldots, b_m)$ may be expressed as a linear combination of polynomials, each of which is either P applied to a set of y_i's in some order, or is a polynomial obtained from P by linearization or combining variables applied to a set of y_i's in some order. But any such polynomial is satisfied trivially in B unless y_i has degree no more than t_i for each i, so that we need consider only the polynomial of type $[t_1, \ldots, t_r]$ obtained from P by linearizing and combining variables. However, the condition (ii) states that this polynomial is implied by S, and hence must vanish over B.

Thus B has all the required properties except for the existence of a unity element. But B satisfies all partial derivatives of P and S, so that we may simply adjoin a unity element and the resulting algebra will still satisfy P and S but not Q.

Theorem 2 is useful in several ways in our investigation. First of all, it assures us that partial differentiation, linearization, and setting variables equal are the only methods of deriving one identity from another which need to be considered. Secondly, it gives us a very computable criterion for checking irreducibility or for searching for irreducible identities. And thirdly, it guarantees the existence of non-trivial examples of algebras satisfying conditions (i) and (ii).

It might be remarked concerning condition (i) of Theorem 2 that, if the characteristic of F is either zero or larger than the degree of any x_i in P, then the vanishing of all first partial derivatives of P guarantees that the remaining derivatives will vanish. However, without this restriction on the characteristic, this does not follow. For example, over a field of characteristic 2, the first two derivatives of the polynomial $P(x) = (x^2x)x^2 - (x^2x^2)x$ are

$$\partial_1{}^1P(x) = 0 \quad \text{and} \quad \partial_1{}^2P(x) = xx^2 + x^2x.$$

In general, there seems to be little that one can say about the form of an identity which is irreducible relative to an arbitrary set of identities S. However, one can say much more in certain special cases. Perhaps the most interesting such result is

THEOREM 3. *Let n be a positive integer and let $P(x_1, \ldots, x_m)$ be absolutely irreducible with coefficients in a field F of characteristic not dividing $n!$. Then P is either symmetric or skew-symmetric in its arguments of degree n, depending on whether n is even or odd.*

To prove this theorem, it is sufficient to show that P is respectively symmetric or skew-symmetric in any two arguments of degree n, say x and y. Since P is irreducible, it vanishes if x and y are set equal. Selecting any term T of P, the other terms which combine with it when x and y are set equal look identical with T in the way in which the variables are associated and in the way in which the other variables are placed; the only difference is that certain x's and y's have interchanged positions. Ordering the positions occupied by x's and y's in T from left to right, and letting I denote the set of all distinct ways of order n x's and n y's, we shall let T_σ where $\sigma \in I$, stand for the term derived from T by arranging the x's and y's of T in the order σ. We shall let k_σ denote the coefficient of T_σ in P, and let ω stand for that element of I such that $T_\omega = T$. It will also be useful to define I' to be the set of all ways of ordering $(n+1)$ x's and $(n-1)$ y's, and to let T_σ for $\sigma \in I'$ denote the monomial that arises from T by replacing the n x's and n y's in T by $(n+1)$ x's and $(n-1)$ y's in the order σ. If σ and τ are any two elements of $I \cup I'$, we shall define their inner product $\sigma\cdot\tau$ to be the number of positions in which both σ and τ have a y. Then, it is easy to see that $0 \leqslant \sigma\cdot\tau \leqslant n$, that $\sigma\cdot\tau = n$

if and only if $\sigma = \tau \in I$, and that $\sigma \cdot \tau = 0$ for $\sigma, \tau \in I$ if and only if $\tau = \sigma'$, the arrangement obtained from σ by interchanging the symbols x and y.

Suppose now that the variable y in P is partially linearized to yield an identity P^* of degree $n - 1$ in y, and containing a new variable z which enters linearly. Then setting $z = x$ in P^* gives an identity P' which vanishes identically, since it is lower than P in the partial ordering. During the partial linearization, each term of P gives rise to n different terms of P^*; while, when z is set equal to x in P^*, each term of P' results from combining $n + 1$ distinct terms of P^*. In particular, for each $\sigma \in I$, the term T_σ of P gives rise to the n terms of the form T_τ of P' where τ runs over those elements of I' such that $\sigma \cdot \tau = n - 1$. And for each $\tau \in I'$, the total coefficient of T_τ in P' will be the sum of the $n + 1$ coefficients k_σ where σ runs over all those elements of I such that $\sigma \cdot \tau = n - 1$. Since P' vanishes identically, this gives the relation

$$\sum_\sigma k_\sigma = 0 \text{ (sum over } \sigma \text{ such that } \sigma \cdot \tau = n - 1) \tag{4}$$

for each $\tau \in I'$.

If $i = \tau \cdot \omega$, then every σ occurring in (4) satisfies either $\sigma \cdot \omega = i$ or $\sigma \cdot \omega = i + 1$. Each $\sigma \in I$ satisfying $\sigma \cdot \omega = i$ occurs in exactly $n - i$ equations of the type (4) for which $\tau \cdot \omega = i$, while each $\sigma \in I$ satisfying $\sigma \cdot \omega = i + 1$ occurs in exactly $i + 1$ of the equations (4) for which $\tau \cdot \omega = i$. Defining K_i to be the sum of those k_σ for which $\sigma \cdot \omega = i$, we may then add up all of the equations (4) for which $\tau \cdot \omega = i$ to get

$$(n - i)K_i + (i + 1)K_{i+1} = 0 \qquad \text{for each } i = 0, \ldots, n - 1.$$

Substituting these equations one into another yields

$$\begin{aligned} K_0 &= -\frac{1}{n}K_1 = \frac{1 \cdot 2}{n(n-1)}K_2 = -\frac{1 \cdot 2 \cdot 3}{n(n-1)(n-2)}K_3 = \ldots \\ &= (-1)^n \frac{1 \cdot 2 \cdots n}{n(n-1)\cdots 1}K_n, \end{aligned}$$

or $K_0 = (-1)^n K_n$. But, since $\sigma \cdot \omega = n$ implies $\sigma = \omega$, we have $K_n = k_\omega$. Similarly, $K_0 = k_{\omega'}$, where ω' is the ordering obtained from ω by interchanging all of the x's and y's. Thus, $k_{\omega'} = (-1)^n k_\omega$, which is what was to be proved.

It might be remarked that one cannot hope to prove that identities irreducible relative to a non-vacuous set S satisfy the conclusion of Theorem 3. For, let P be an absolutely irreducible identity in the variables $x_1, \ldots, x_m$, at least two of which are of degree n, and let Q' be any identity in a subset of $x_1, \ldots, x_m$ which is implied by S and which has degree in each x_i less than the degree of P in that x_i. Multiplying Q' on the right by a sufficient number of x_i's for different values of i to obtain an identity Q of the same type as P, we observe that Q is not symmetric or skew-symmetric in its variables of degree n. But then $P + Q$ is an identity irreducible relative to S which does not satisfy the conclusion of Theorem 3.

For certain sets of identities S, one may modify Theorem 3 to get a true statement for identities irreducible relative to S. For example, if S consists of the associative law, we may combine terms of P which differ only in the way in which they are associated and think of P as having associative monomials, in which case the conclusion of Theorem 3 holds (the proof of Theorem 3 may be used for this case without alteration). If S consists of the commutative law, we may combine terms of P which may be transformed into each other using the commutative law, and prove that P is skew-symmetric in its arguments of degree 1 and symmetric in its arguments of degree 2. However, in this case it can be shown that the property (ii) of Theorem 2 (which is all that is used in the proof of Theorem 3) is not strong enough to establish skew-symmetry in variables of degree 3.

2. We now turn to the problem of finding those identities of low degree which are irreducible relative to the commutative law. In this section we prove

THEOREM 4. *Over a field of characteristic not* 2 *or* 3, *an identity of degree* $\leqslant 4$ *is irreducible relative to the commutative law if and only if it is one of the following:*

$$(yx)x = yx^2, \tag{5}$$

$$(x^2x)x = x^2x^2, \tag{6}$$

$$2((yx)x)x + yx^3 = 3(yx^2)x, \tag{7}$$

$$2(y^2x)x - 2((yx)y)x - 2((yx)x)y + 2(x^2y)y - y^2x^2 + (yx)(yx) = 0. \tag{8}$$

Since 5 implies (6) and since (6) is equivalent to power-associativity in a commutative algebra of characteristic not 2, 3, or 5, this theorem has the following

COROLLARY. *Let A be a commutative algebra with unity element over a field F of characteristic not* 2, 3, *or* 5, *and let A satisfy an identity of degree* $\leqslant 4$ *not implied by the commutative law. Then either A is power-associative, or it satisfies at least one of the two identities* (7) *and* (8).

An investigation of algebras satisfying (7) may be found in **(3)**. The identity (8) has so far resisted attack.

The proof of Theorem 4 will consist of checking the most general identity of each type of degree $\leqslant 4$ for irreducibility using the criteria given in Theorem 2. We shall consider only degrees 3 and 4, since it is obvious that there are no non-trivial identities of degree 1 or 2 except for the commutative law itself, using the fact that the sum of the coefficients must be zero. Similarly, since there is only one term of type [3] when the commutative law is assumed, there cannot be an irreducible identity of this type either. For each of the two types [2, 1] and [4] there are exactly two terms, and the requirement that the sum of the coefficients be zero leads to (5) and (6) respectively. It is easy to check that both of these identities satisfy the irreducibility criteria of Theorem 2.

To show that there are no irreducible identities of types [1, 1, 1] or [1, 1, 1, 1], we prove

LEMMA 1. *For characteristic not two, an identity P irreducible relative to the commutative law is skew-symmetric in its variables of degree one. P cannot contain a term which is carried into itself (modulo the commutative law) when two linear variables are transposed. In particular, no multilinear identity is irreducible relative to the commutative law.*

Let T be a term in an irreducible identity P containing the two variables x and y linearly, and let T' be obtained from T by switching the positions of x and y. If $T = T'$, then no other terms of P combine with T when x and y are set equal. But since P vanishes identically when x and y are set equal, the coefficient of T must be zero. If $T \not\equiv T'$, then T and T' combine when x and y are set equal, and no other terms of P combine with them. Thus the coefficient of T' is just the negative of the coefficient of T in this case. For the last statement of the lemma, we observe that, in any term of a multilinear identity, there are two variables that may be interchanged using the commutative law.

Returning to the proof of Theorem 4, we still have the types [3, 1], [2, 2], and [2, 1, 1] to consider. In the first case, the identity must have the form

$$\alpha_1(yx\cdot x)x + \alpha_2(yx^2)x + \alpha_3 yx^3 + \alpha_4(yx)x^2 = 0 \tag{9}$$

for some choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F$. Setting $y = x$ in (9) gives

$$(\alpha_1 + \alpha_2 + \alpha_3)x^3x + \alpha_4 x^2x^2 = 0,$$

which implies $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and $\alpha_4 = 0$ if (9) is irreducible. We may also differentiate (9) with respect to x to get

$$(3\alpha_1 + 2\alpha_2 + 2\alpha_4)(yx)x + (\alpha_2 + 3\alpha_3 + \alpha_4)yx^2 = 0,$$

or $3\alpha_1 + 2\alpha_2 = 0$ and $\alpha_2 + 3\alpha_3 = 0$ using the relation $\alpha_4 = 0$. Thus the only identity of type [3, 1] that could be irreducible is (7). On the other hand, the coefficients of (7) satisfy the two sets of relations just derived, and since the derivative of (7) with respect to y vanishes identically, we have satisfied the criteria of Theorem 2. Thus (7) must be irreducible.

Consider next identities of type [2, 2]. The possible terms in such an identity are $r_1 = (y^2x)x$, $r_2 = ((yx)y)x$, $r_3 = ((yx)x)y$, $r_4 = (x^2y)y$, $r_5 = y^2x^2$, and $r_6 = (yx)(yx)$, so that an identity of type [2, 2] will be of the form

$$\sum_{i=1}^{6} \alpha_i r_i = 0. \tag{10}$$

Linearizing y in (10) and setting one of the new variables equal to x gives an identity of type [3, 1] which must vanish if (10) is irreducible. This gives the relations

$$2\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_2 + \alpha_4 = 0, \quad \alpha_3 + \alpha_4 = 0, \quad \text{and} \quad 2\alpha_5 + 2\alpha_6 = 0.$$

On the other hand, differentiating (10) with respect to x yields

$$2\alpha_1 + \alpha_2 + 2\alpha_5 = 0 \quad \text{and} \quad \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_6 = 0.$$

Solving all these relations simultaneously leads to

$$\alpha_1 = -\alpha_2 = -\alpha_3 = \alpha_4 = -2\alpha_5 = 2\alpha_6,$$

which gives (8). Since (8) is symmetric in x and y, neither differentiation with respect to y nor linearizing x and setting one of the new variables equal to y will give any new relations on the α_i's. Thus (8) is irreducible by Theorem 2.

Finally, let us consider irreducible identities of type [2, 1, 1]. By Lemma 1, the terms $((yz)x)x$, $(yz)x^2$, and $(yx)(zx)$ cannot occur. Then, setting $s_1 = ((yx)z)x$, $s_2 = ((yx)x)z$, $s_3 = (x^2y)z$, $s_4 = (x^2z)y$, $s_5 = ((zx)x)y$, and $s_6 = ((zx)y)x$, such an identity must be of the form

$$\sum_{i=1}^{6} \alpha_i s_i = 0. \tag{11}$$

Lemma 1 also tells us that $\alpha_6 = -\alpha_1$, $\alpha_5 = -\alpha_2$, and $\alpha_4 = -\alpha_3$, while setting $z = x$ in (11) gives $\alpha_1 + \alpha_2 = 0$, $\alpha_3 + \alpha_6 = 0$, and $\alpha_4 + \alpha_5 = 0$. Also, differentiating (11) with respect to x gives $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0$, $\alpha_1 + \alpha_6 = 0$, and $2\alpha_4 + 2\alpha_5 + \alpha_6 = 0$. Solving these relations simultaneously, we readily see that all the α_i's are zero. Thus, no identity of type [2, 1, 1] is irreducible, and the theorem is proved.

3. In this final section we find all identities of degree five which are irreducible relative to the commutative law. Algebras satisfying identities from the first two of the five families that turn up are investigated in **(4)** and **(2)** respectively.

THEOREM 5. *Over a field of characteristic not* 2, 3, *or* 5, *an identity of degree* 5 *is irreducible relative to the commutative law if and only if it belongs to one of the following families of identities:*

$$2((x^2x)x)x - 3(x^2x^2)x + (x^2x)x^2 = 0, \tag{12}$$

$$\begin{cases} \beta_1[y(x^3\cdot x) - 4(yx^3)x + 6((yx^2)x)x - 3((yx\cdot x)x)x] \\ \quad + \beta_2[-y(x^2\cdot x^2) + 5(yx^3)x - 9((yx^2)x)x + 4((yx\cdot x)x)x \\ \quad + ((yx)x^2)x + (yx^2)x^2 - (yx)x^3] \\ \quad + \beta_3[((yx^2)x)x - ((yx\cdot x)x)x - (yx^2)x^2 + (yx\cdot x)x^2] = 0, \end{cases} \tag{13}$$

$$\begin{cases} \gamma_1[((yx\cdot x)y)x - ((yx\cdot x)x)y - ((yx^2y)x + ((yx^2)x)y] \\ \quad + \gamma_2[-((y^2x)x)x + ((yx\cdot y)x)x + ((yx\cdot x)x)y \\ \quad - ((yx^2)x)y + (y^2x)x^2 - (yx\cdot y)x^2 - (yx\cdot x)(yx) + (yx^2)(yx)] \\ \quad + \gamma_3[4((y^2x)x)x - 6((yx\cdot y)x)x \\ \quad - 2((yx\cdot x)x)y + 2((yx^2y)x + 4((yx^2)x)y - 2(yx^3)y \\ \quad - 4(y^2\cdot x^2)x + 4(yx\cdot yx)x + (y^2x)x^2 - 2(yx\cdot x)(yx) \\ \quad + y^2x^3] = 0, \end{cases} \tag{14}$$

https://doi.org/10.4153/CJM-1965-008-3 Published online by Cambridge University Press

$$(15)\quad \begin{cases} \delta_1[((yx\cdot x)z)x - ((zx\cdot x)y)x - ((yx\cdot x)x)z \\ \quad + ((zx\cdot x)x)y - ((yx^2)z)x + ((zx^2)y)x + ((yx^2)x)z - ((zx^2)x)y] \\ \quad + \delta_2[((yx\cdot z)x)x - ((zx\cdot y)x)x \\ \quad - ((yx\cdot x)x)z + ((zx\cdot x)x)y + ((yx^2)x)z - ((zx^2x)y \\ \quad - (yx\cdot z)x^2 + (zx\cdot y)x^2 + (yx\cdot x)(zx) - (zx\cdot x)(yx) \\ \quad - (yx^2)(zx) + (zx^2)(yx)] = 0, \end{cases}$$

$$(16)\quad \begin{cases} \epsilon_1[((zy\cdot y)x)x + ((zx\cdot x)y)y - ((y^2z)x)x + ((yx\cdot z)y)x \\ \quad + ((yx\cdot z)x)y - ((x^2z)y)y - (zy\cdot y)x^2 + (zy\cdot x)(yx) \\ \quad + (zx\cdot y)(yx) - (zx\cdot x)y^2 + (y^2z)x^2 - 2(yx\cdot z)(yx) \\ \quad + (x^2z)y^2 - (y^2x)(zx) + (yx\cdot y)(zx) + (yx\cdot x)(zy) - (x^2y)(zy)] \\ \quad + \epsilon_2[((zy\cdot x)y)x + ((zx\cdot y)x)y - ((y^2x)z)x \\ \quad + ((yx\cdot y)z)x + ((yx\cdot x)z)y - ((x^2y)z)y] \\ \quad - (\epsilon_1+\epsilon_2)[((zy\cdot x)x)y + ((zx\cdot y)y)x - ((y^2x)x)z + ((yx\cdot y)x)z \\ \quad + ((yx\cdot x)y)z - ((x^2y)y)z] = 0. \end{cases}$$

Observing that (6) and (7) imply (12) and that (8) implies an identity of the family (13), we see that Theorem 5 has the following

COROLLARY. *Let A be a commutative algebra with unity element of characteristic not* 2, 3, *or* 5, *and let A satisfy an identity of degree $\leqslant 5$ not implied by the commutative law. Then A satisfies at least one of the identities* (12)–(16).

To prove Theorem 5, we need to consider the identity types [5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], and [2, 1, 1, 1]. The remaining type of degree 5, [1, 1, 1, 1, 1], cannot have any irreducible identities by Lemma 1. The general equation of type [5] is

$$(17)\qquad \alpha_1((x^2x)x)x + \alpha_2(x^2x^2)x + \alpha_3(x^2x)x^2 = 0.$$

Setting the derivative of this equation identically equal to zero gives the relations $5\alpha_1 + 4\alpha_2 + 2\alpha_3 = 0$ and $\alpha_2 + 3\alpha_3 = 0$, or $\alpha_2 = -3\alpha_3$ and $\alpha_1 = 2\alpha_3$, so that (17) reduces to (12). Conversely, the vanishing of its derivative is sufficient to make an identity in one variable irreducible by Theorem 2.

The possible terms occurring in an identity of type [4, 1] are $u_1 = y(x^2\cdot x^2)$, $u_2 = y(x^3\cdot x)$, $u_3 = (yx^3)x$, $u_4 = ((yx^2)x)x$, $u_5 = ((yx\cdot x)x)x$, $u_6 = ((yx)x^2x$, $u_7 = (yx^2)x^2$, $u_8 = (yx\cdot x)x^2$, and $u_9 = (yx)x^3$. The general identity of this type is then

$$(18)\qquad \sum_{i=1}^{9} \alpha_i u_i = 0.$$

Setting $y = x$ in (18) gives the relations

$$\alpha_1 + \alpha_6 = 0, \qquad \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0, \qquad \alpha_7 + \alpha_8 + \alpha_9 = 0,$$

while setting $y = 1$ in (18) gives

$$\alpha_1 + \alpha_7 + \alpha_8 = 0 \quad\text{and}\quad \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_9 = 0.$$

https://doi.org/10.4153/CJM-1965-008-3 Published online by Cambridge University Press

These equations are easily seen to be equivalent to

$$\alpha_1 = -\alpha_6 = \alpha_9 = -\alpha_7 - \alpha_8 \quad \text{and} \quad \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0. \tag{19}$$

On the other hand, differentiating (18) with respect to x yields the relations

$$\begin{aligned} 4\alpha_1 + 4\alpha_2 + \alpha_3 + \alpha_9 &= 0, & 3\alpha_3 + 2\alpha_4 + \alpha_6 + 2\alpha_7 &= 0, \\ 2\alpha_4 + 4\alpha_5 + 2\alpha_6 + 2\alpha_8 &= 0, & \alpha_6 + 2\alpha_7 + 2\alpha_8 + 3\alpha_9 &= 0, \end{aligned}$$

which reduce to

$$4\alpha_2 + \alpha_3 - 5\alpha_7 - 5\alpha_8 = 0 \quad \text{and} \quad 3\alpha_3 + 2\alpha_4 + 3\alpha_7 + \alpha_8 = 0, \tag{20}$$

after using (19) to eliminate α_1, α_5, α_6, and α_9. We may equivalently replace (20) and the second equation of (19) by

$$\begin{cases} \alpha_3 = -4\alpha_2 + 5\alpha_7 + 5\alpha_8, \qquad \alpha_4 = 6\alpha_2 - 9\alpha_7 - 8\alpha_8, \\ \alpha_5 = -3\alpha_2 + 4\alpha_7 + 3\alpha_8. \end{cases} \tag{21}$$

Thus, α_2, α_7, α_8 may be determined arbitrarily, and the remaining α_i's determined from them. Setting $\beta_1 = \alpha_2$, $\beta_2 = \alpha_7$, $\beta_3 = \alpha_8$ in (18) and using (21) and the first part of (19) to express the other coefficients in terms of the β_i's, we obtain (13). Since we have been careful not to lose any relations in solving for the α_i's, we may conclude from Theorem 2 that any way of choosing values for the β_i's gives an irreducible identity.

Turning next to identities of type [3, 2], the possible terms are $v_1 = ((y^2x)x)x$, $v_2 = ((yx\cdot y)x)x$, $v_3 = ((yx\cdot x)y)x$, $v_4 = ((yx\cdot x)x)y$, $v_5 = ((yx^2)y)x$, $v_6 = ((yx^2)x)y$, $v_7 = (yx^3)y$, $v_8 = (y^2\cdot x^2)x$, $v_9 = (yx\cdot yx)x$, $v_{10} = ((yx)x^2)y$, $v_{11} = (y^2x)x^2$, $v_{12} = (yx\cdot y)x^2$, $v_{13} = (yx\cdot x)(yx)$, $v_{14} = (yx^2)(yx)$, and $v_{15} = y^2x^3$, and the general equation is

$$\sum_{i=1}^{15} \alpha_i v_i = 0. \tag{22}$$

Linearizing y in (22) and setting one of these variables equal to x, we get the following relations:

$$\begin{cases} 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0, \qquad \alpha_2 + \alpha_5 + \alpha_6 = 0, \\ \alpha_3 + \alpha_5 + \alpha_7 = 0, \qquad \alpha_4 + \alpha_6 + \alpha_7 = 0, \qquad 2\alpha_8 + 2\alpha_9 + \alpha_{10} = 0, \\ \alpha_{10} = 0, \qquad 2\alpha_{11} + \alpha_{12} + \alpha_{13} = 0, \qquad \alpha_{12} + \alpha_{14} = 0, \\ \alpha_{13} + \alpha_{14} + 2\alpha_{15} = 0; \end{cases} \tag{23}$$

and differentiating (22) with respect to y leads to the relations

$$\begin{aligned} &2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_9 + \alpha_{13} = 0, \\ &\alpha_3 + 2\alpha_5 + \alpha_6 + 2\alpha_8 + \alpha_{14} = 0, \qquad \alpha_4 + \alpha_6 + 2\alpha_7 + \alpha_{10} + 2\alpha_{15} = 0, \\ &\alpha_{10} + 2\alpha_{11} + 2\alpha_{12} + \alpha_{13} + \alpha_{14} = 0, \end{aligned}$$

which reduce to

$$\alpha_2 + 2\alpha_9 + \alpha_{13} = 0, \qquad \alpha_7 + 2\alpha_{15} = 0, \tag{24}$$

using the relations (23). We may now easily solve (23) and (24) for 11 of the α_i's in terms of the remaining 4 to obtain a set of relations equivalent to (23) and (24):

$$(25)\quad \begin{cases} \alpha_{13} = -\alpha_{14} - 2\alpha_{15}, \quad \alpha_{12} = -\alpha_{14}, \quad \alpha_{11} = \alpha_{14} + \alpha_{15}, \\ \alpha_{10} = 0, \quad \alpha_8 = -\alpha_9, \quad \alpha_7 = -2\alpha_{15}, \quad \alpha_6 = \alpha_3 + 2\alpha_9 - \alpha_{14} - 4\alpha_{15}, \\ \alpha_5 = -\alpha_3 + 2\alpha_{15}, \quad \alpha_4 = -\alpha_3 - 2\alpha_9 + \alpha_{14} + 6\alpha_{15}, \\ \alpha_2 = -2\alpha_9 + \alpha_{14} + 2\alpha_{15}, \quad \alpha_1 = 2\alpha_9 - \alpha_{14} - 4\alpha_{15}. \end{cases}$$

On the other hand, differentiating (22) with respect to x gives

$$\begin{aligned} &3\alpha_1 + \alpha_2 + 2\alpha_8 + 2\alpha_{11} = 0, \qquad 2\alpha_2 + 2\alpha_3 + 2\alpha_5 + 2\alpha_9 + 2\alpha_{12} = 0, \\ &\alpha_3 + 3\alpha_4 + 2\alpha_6 + 2\alpha_{10} + \alpha_{13} = 0, \\ &\alpha_5 + \alpha_6 + 3\alpha_7 + \alpha_{10} + \alpha_{14} = 0, \qquad \alpha_8 + \alpha_{11} + \alpha_{12} + 3\alpha_{15} = 0, \\ &\alpha_9 + 2\alpha_{13} + 2\alpha_{14} = 0, \end{aligned}$$

and using (25) to eliminate all the a_i's except for α_3, α_9, α_{14}, α_{15} yields the relation $\alpha_9 - 4\alpha_{15} = 0$ six times. Replacing α_9 by $4\alpha_{15}$ in (25) and making the substitutions (25) in (22), we obtain (14) with $\gamma_1 = \alpha_3$, $\gamma_2 = \alpha_{14}$, and $\gamma_3 = \alpha_{15}$. Again Theorem 2 guarantees that all of the identities of this family are irreducible.

Let us consider next identities of type [3, 1, 1]. By Lemma 1, an irreducible identity of this type cannot contain terms which are carried into themselves when the two linear variables are switched, and terms which are carried into each other when the two linear variables are switched have coefficients which add to zero. Thus, the irreducible identities of this type must be of the form

$$(26)\qquad \sum_{i=1}^{10} \alpha_i t_i = 0,$$

where

$$\begin{aligned} t_1 &= ((yx\cdot z)x)x - ((zx\cdot y)x)x, & t_2 &= ((yx\cdot x)z)x - ((zx\cdot x)y)x, \\ t_3 &= ((yx\cdot x)x)z - ((zx\cdot x)x)y, & t_4 &= ((yx^2)z)x - ((zx^2)y)x, \\ t_5 &= ((yx^2)x)z - ((zx^2)x)y, & t_6 &= (yx^3)z - (zx^3)y, \\ t_7 &= ((yx)x^2)z - ((zx)x^2)y, & t_8 &= (yx\cdot z)x^2 - (zx\cdot y)x^2 \\ t_9 &= (yx\cdot x)(zx) - (zx\cdot x)(yx), & t_{10} &= (yx^2)(zx) - (zx^2)(yx). \end{aligned}$$

Setting $z = x$ in (26) gives the relations

$$(27)\quad \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0, \quad -\alpha_1 + \alpha_4 + \alpha_5 = 0, \quad -\alpha_2 - \alpha_4 + \alpha_6 = 0, \\ -\alpha_3 - \alpha_5 - \alpha_6 = 0, \quad \alpha_7 = 0, \quad \alpha_8 + \alpha_9 = 0, \quad -\alpha_8 + \alpha_{10} = 0, \\ -\alpha_9 - \alpha_{10} = 0, \end{cases}$$

while setting $z = 1$ in (26) yields

$$(28)\quad \begin{cases} \alpha_2 + \alpha_3 + \alpha_9 = 0, \quad -\alpha_2 + \alpha_5 + \alpha_{10} = 0, \quad -\alpha_3 - \alpha_5 - \alpha_7 = 0, \\ \alpha_7 - \alpha_9 - \alpha_{10} = 0. \end{cases}$$

https://doi.org/10.4153/CJM-1965-008-3 Published online by Cambridge University Press

Choosing α_2 and α_9 as parameters, we may now solve for the remaining α_i's using (27) and (28) to obtain

$$(29)\qquad \begin{cases} \alpha_{10} = -\alpha_9, & \alpha_8 = -\alpha_9, \quad \alpha_7 = 0, \alpha_6 = 0, \\ \alpha_5 = \alpha_2 + \alpha_9, & \alpha_4 = -\alpha_2, \quad \alpha_3 = -\alpha_2 - \alpha_9, \quad \alpha_1 = \alpha_9. \end{cases}$$

Substituting these values in (26) gives (15) with $\delta_1 = \alpha_2$ and $\delta_2 = \alpha_9$. In order to use Theorem 2 to show that every identity of this family is irreducible, we must show that differentiating (26) with respect to x imposes no relations on the α_i's not implied by (29). But this differentiation yields

$$2\alpha_1 + 2\alpha_2 + 2\alpha_4 + 2\alpha_8 = 0, \qquad \alpha_2 + 3\alpha_3 + 2\alpha_5 + 2\alpha_7 + \alpha_9 = 0,$$
$$\alpha_4 + \alpha_5 + 3\alpha_6 + \alpha_7 + \alpha_{15} = 0,$$

which all follow from (29).

The argument for identities of type [2, 1, 1, 1] is very similar to the one just completed. Again using Lemma 1, an irreducible identity of this type must have the form

$$(30)\qquad \sum_{i=1}^{4} \alpha_i q_i,$$

where

$$q_1 = ((x^2y)z)w + ((x^2z)w)y + ((x^2w)y)z - ((x^2y)w)z - ((x^2w)z)y - ((x^2z)y)w,$$
$$q_2 = ((xy\cdot x)z)w + ((xz\cdot x)w)y + ((xw\cdot x)y)z - ((xy\cdot x)w)z - ((xw\cdot x)z)y - ((xz\cdot x)y)w,$$
$$q_3 = ((xy\cdot z)x)w + ((xz\cdot w)x)y + ((xw\cdot y)x)z - ((xy\cdot w)x)z - ((xw\cdot zx)y - ((xz\cdot y)x)w,$$
$$q_4 = ((xy\cdot z)w)x + ((xz\cdot w)y)x + ((xw\cdot y)z)x - ((xy\cdot w)z)x - ((xw\cdot z)y)x - ((xz\cdot y)x)w.$$

But, letting $w = x$ in (30), the coefficient of $((x^2y)z)x$ is $\alpha_1 + \alpha_4 = 0$, while differentiating (30) with respect to x yields

$$2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 0, \qquad \alpha_3 = 0, \qquad \text{and } \alpha_4 = 0.$$

Hence, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, and there are no irreducible identities of this type.

There remain only the identities of type [2, 2, 1] to be considered. This time the possible terms are:

$$\begin{aligned} p_1 &= ((zy\cdot y)x)x, & p_2 &= ((zy\cdot x)y)x, & p_3 &= ((zy\cdot x)x)y, \\ p_4 &= ((zx\cdot y)y)x, & p_5 &= ((zx\cdot y)x)y, & p_6 &= ((zx\cdot x)y)y, \\ p_7 &= ((y^2z)x)x, & p_8 &= ((yx\cdot z)y)x, & p_9 &= ((yx\cdot z)x)y, \\ p_{10} &= ((x^2z)y)y, & p_{11} &= ((y^2x)z)x, & p_{12} &= ((yx\cdot y)z)x, \\ p_{13} &= ((yx\cdot x)z)y, & p_{14} &= ((x^2y)z)y, & p_{15} &= ((y^2x)x)z, \\ p_{16} &= ((yx\cdot y)x)z, & p_{17} &= ((yx\cdot x)y)z, & p_{18} &= ((x^2y)y)z, \end{aligned}$$

$$
\begin{array}{lll}
p_{19} = (y^2 \cdot x^2)z, & p_{20} = (yx \cdot yx)z, & p_{21} = (zy \cdot yx)x, \\
p_{22} = ((zy)x^2)y, & p_{23} = ((zx)y^2)x, & p_{24} = (zx \cdot yx)y, \\
p_{25} = (zy \cdot y)x^2, & p_{26} = (zy \cdot x)(yx), & p_{27} = (zx \cdot y)(yx), \\
p_{28} = (zx \cdot x)y^2, & p_{29} = (y^2 z)x^2, & p_{30} = (yx \cdot z)(yx), \\
p_{31} = (x^2 z)y^2, & p_{32} = (y^2 x)(zx), & p_{33} = (yx \cdot y)(zx), \\
p_{34} = (yx \cdot x)(zy), & p_{35} = (x^2 y)(zy), &
\end{array}
$$

and the general identity of this type is

$$\sum_{i=1}^{35} \alpha_i p_i = 0. \tag{31}$$

Linearizing y in (31) and setting one of the new variables equal to x, the relations involving the first six terms are

$$
\begin{array}{ll}
\alpha_1 + \alpha_2 + \alpha_3 = 0, & \alpha_1 + \alpha_4 + \alpha_5 = 0, \\
\alpha_2 + \alpha_4 + \alpha_6 = 0, & \alpha_3 + \alpha_5 + \alpha_6 = 0,
\end{array}
$$

which are equivalent to

$$\alpha_6 = \alpha_1, \qquad \alpha_5 = \alpha_2, \qquad \alpha_4 = \alpha_3 = -\alpha_1 - \alpha_2. \tag{32}$$

The next four terms yield the relations $2\alpha_7 + \alpha_8 + \alpha_9 = 0$, $\alpha_8 + \alpha_{10} = 0$, and $\alpha_9 + \alpha_{10} = 0$, which lead to

$$\alpha_{10} = -\alpha_9 = -\alpha_8 = \alpha_7. \tag{33}$$

In each of these two cases we have treated together all those terms which differ from each other only in that x's and y's have been interchanged. The difference is that in the first case none of the four positions involved may be interchanged using the commutative law (so that the resulting relations are exactly as in the proof of Theorem 3), while in the second case exactly one pair of positions may be interchanged using the commutative law (the first two positions). Since the relations that arise from a set of terms differing only in the order of their x's and y's depend only on which positions may be interchanged using the commutative law, the other five cases where exactly one pair of positions may be interchanged will come out exactly like (33), giving

$$
\begin{array}{ll}
\alpha_{14} = -\alpha_{13} = -\alpha_{12} = \alpha_{11}, & \alpha_{18} = -\alpha_{17} = -\alpha_{16} = \alpha_{15}, \\
\alpha_{24} = -\alpha_{23} = -\alpha_{22} = \alpha_{21}, & \alpha_{28} = -\alpha_{27} = -\alpha_{26} = \alpha_{25}, \\
\alpha_{35} = -\alpha_{34} = -\alpha_{33} = \alpha_{32}. &
\end{array} \tag{34}
$$

The two remaining cases have more symmetries, and easily yield

$$\alpha_{20} = -\alpha_{19}, \qquad \alpha_{31} = -\tfrac{1}{2}\alpha_{30} = \alpha_{29}. \tag{35}$$

Using (32)–(35) we have found 25 of the α_i's in terms of the remaining 10. To get some relations between these 10, we now set $z = x$ in (31). Some of the relations obtained in this way are

$$
\begin{array}{lll}
\alpha_6 + \alpha_{10} = 0, & \alpha_2 + \alpha_{11} = 0, & \alpha_3 + \alpha_{15} = 0, \\
\alpha_{20} + \alpha_{21} = 0, & \alpha_{28} + \alpha_{31} = 0, & \text{and } \alpha_{25} + \alpha_{33} = 0,
\end{array}
$$

https://doi.org/10.4153/CJM-1965-008-3 Published online by Cambridge University Press

which may be reduced using (32)–(35) to

$$\alpha_7 = -\alpha_1, \quad \alpha_{11} = -\alpha_2, \quad \alpha_{15} = -\alpha_1 - \alpha_2, \quad \alpha_{21} = \alpha_{19}, \quad \alpha_{32} = -\alpha_{29} = \alpha_{25}. \tag{36}$$

Combining this with (32)–(35) allows all the α_i's to be expressed in terms of α_1, α_2, α_{19}, and α_{25}.

If we now differentiate (31) with respect to y, two of the relations we get are $2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_7 + \alpha_{21} + \alpha_{26} = 0$ and $\alpha_2 + 2\alpha_4 + \alpha_5 + \alpha_8 + 2\alpha_{23} + \alpha_{27} = 0$, which may be reduced to $-\alpha_1 + \alpha_{19} - \alpha_{25} = 0$ and $-\alpha_1 - 2\alpha_{19} - \alpha_{25} = 0$ using (31)–(36). But the last two equations imply

$$\alpha_{19} = 0, \quad \alpha_{25} = -\alpha_1. \tag{37}$$

Using (32)–(37) to express the α_i's in (31) in terms of α_1 and α_2, we obtain (16) with $\epsilon_1 = \alpha_1$ and $\epsilon_2 = \alpha_2$. It may be verified that (16) is symmetric with respect to x and y, and that (16) vanishes either when we set $z = x$ or when it is differentiated with respect to y. Hence every identity of this family is irreducible.

References

1. F. Kosier and J. M. Osborn, *Nonassociative algebras of degree 3*, Trans. Amer. Math. Soc., *110* (1964), 484–492.
2. N. Losey, *Simple commutative non-associative algebras satisfying a polynomial identity of degree five*, Ph.D. thesis, Univ. of Wisconsin (1963).
3. J. M. Osborn, *An identity of degree four*, to appear in Proc. Amer. Math. Soc.
4. ——— *A generalization of power-associativity*, to appear in Pacific J. Math.

University of Wisconsin