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Reciprocity sheaves

Bruno Kahn, Shuji Saito, Takao Yamazaki and Kay Rülling

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Bruno Kahn, Shuji Saito and Takao Yamazaki

With two appendices by Kay Rülling

Abstract

We start developing a notion of reciprocity sheaves, generalizing Voevodsky's homotopy invariant presheaves with transfers which were used in the construction of his triangulated categories of motives. We hope that reciprocity sheaves will eventually lead to the definition of larger triangulated categories of motivic nature, encompassing non-homotopy invariant phenomena.

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Introduction

In this paper, we start developing a notion of reciprocity sheaves modelled on Voevodsky's theory of presheaves with transfers. Reciprocity is a weaker condition than homotopy invariance, used by Voevodsky as the main building block for constructing his triangulated categories of motives in [Voe00b]; as in [Voe00b, p. 195], we hope that reciprocity sheaves will eventually lead to the definition of larger triangulated categories of motivic nature, encompassing non-homotopy invariant phenomena which emerged in [Kah91, IR12, Rus13, KS11, BS14].

In the whole paper we fix a base field k. Our reciprocity sheaves form a full abelian subcategory **Rec** of **PST**, the abelian category of presheaves with transfers on the category **Sm** of smooth schemes over k (see § 1.1.3 for the definition). It contains the subcategory **HI** of objects $F \in \mathbf{PST}$ which are homotopy invariant (i.e. $F(X) \simeq F(\mathbf{A}^1_X)$ for $X \in \mathbf{Sm}$). It also contains the object represented by a smooth commutative algebraic group G over k (recall that such an object is in **HI** if and only if G^0 is a semi-abelian variety). Typical examples of objects of **Rec** not contained in **HI** are the additive group $\mathbf{G}_a$ and the modules of absolute Kähler differentials Ω^i.

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As predecessors of reciprocity sheaves, there were notions of reciprocity functors studied in [Kah91, IR12]. All definitions are inspired by the following theorem of Rosenlicht–Serre [Ser59, ch. III].

Theorem 1. *Assume that k is algebraically closed. Let G be a smooth connected commutative algebraic group over k and $\alpha : C \to G$ be a morphism, where C is a smooth irreducible curve over k. Let $C \hookrightarrow \overline{C}$ be its smooth compactification. Then there is an effective divisor D on $\overline{C}$ supported in $\overline{C} - C$ such that α has modulus D in the sense of Rosenlicht–Serre, which means that*

$$\sum_{x \in C(k)} v_x(f)\alpha(x) = 0 \quad \text{in } G(k) \text{ for any } f \in G(\overline{C}, D),$$

where $v_x : k(C)^\times \to \mathbb{Z}$ is the normalized valuation at x and

$$G(\overline{C}, D) := \bigcap_{x \in D} \mathrm{Ker}(\mathcal{O}^\times_{\overline{C},x} \to \mathcal{O}^\times_{D,x}). \tag{1}$$

A distinguished feature of our reciprocity sheaves is their relation to relative Chow groups of zero cycles with moduli studied in [KS11]. Let X be a smooth variety over k and choose a compactification $X \hookrightarrow \overline{X}$ with $\overline{X}$ integral and proper over k. For a (not necessarily reduced) closed subscheme $Y \subset \overline{X}$ with $X = \overline{X} - Y$, the relative Chow group $\mathrm{CH}_0(\overline{X}, Y)$ of zero cycles is the quotient of the group of zero cycles $Z_0(X)$ on X by 'rational equivalence with modulus Y'. More precisely, one defines in [KS11]

$$\mathrm{CH}_0(\overline{X}, Y) = \mathrm{Coker}(\Phi(\overline{X}, Y)(\mathrm{Spec}\, k) \xrightarrow{\partial} Z_0(X)),$$

where

$$\Phi(\overline{X}, Y)(\mathrm{Spec}\, k) = \bigoplus_{\varphi : \overline{C} \to \overline{X}} G(\overline{C}, \varphi^* Y), \tag{2}$$

the direct sum being over all finite morphisms $\varphi : \overline{C} \to \overline{X}$, where $\overline{C}$ is a normal proper curve over k such that $\varphi(\overline{C}) \not\subset Y$, $G(\overline{C}, \varphi^* Y)$ is defined as (1) with $\varphi^* Y = \overline{C} \times_{\overline{X}} Y$ and ∂ is induced by the divisor map on $\overline{C}$ and the pushforward of zero cycles by φ.

Now the key idea to define reciprocity sheaves is to enhance the abelian group $\mathrm{CH}_0(\overline{X}, Y)$ into an object $h(\overline{X}, Y)$ in **PST**. It is the cokernel of a map in **PST**:

$$\Phi(\overline{X}, Y) \to \mathbb{Z}_{\mathrm{tr}}(X),$$

where $\mathbb{Z}_{\mathrm{tr}}(X)$ is the object of **PST** represented by X and $\Phi(\overline{X}, Y) \in \mathbf{PST}$ is defined by an analogue of (2). We have

$$h(\overline{X}, Y)(\mathrm{Spec}\, k) = \mathrm{CH}_0(\overline{X}, Y). \tag{3}$$

To give more details, we introduce a terminology: a pair $(\overline{X}, Y)$ is called a *modulus pair* if $\overline{X}$ is integral and proper over k, $Y \subset \overline{X}$ is a (possibly non-reduced) closed subscheme and $X := \overline{X} - Y$ is smooth and quasi-affine. Then, for a modulus pair $(\overline{X}, Y)$ and for a section $a \in F(X)$ of a presheaf with transfers F, we define the notion of a having modulus Y, or Y being a modulus for a (see Definition 2.1.2), as a generalization of the modulus in the sense of Rosenlicht–Serre (see Theorem 1).

Then $F \in \mathbf{PST}$ is defined to have reciprocity (or to be a reciprocity presheaf) if for any $a \in F(X)$ with X quasi-affine smooth over k and for any dense open immersion $X \hookrightarrow \overline{X}$ with $\overline{X}$ integral proper over k, there exists a closed subscheme $Y \subset \overline{X}$ such that $X = \overline{X} - Y$ and that Y is a modulus for a. The first main result is the following.

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Theorem 2 (see Theorem 2.1.5). *Let $(\overline{X}, Y)$ be a modulus pair with $X = \overline{X} - Y$. Then the functor*

$$\mathbf{PST} \to \mathbf{Ab}, \quad F \mapsto \{a \in F(X) \mid a \text{ has modulus } Y\}$$

is represented by an object $h(\overline{X}, Y) \in \mathbf{PST}$. If moreover Y is a Cartier divisor on $\overline{X}$, then $h(\overline{X}, Y)$ has reciprocity.

It will become clear from its construction that $h(\overline{X}, Y)$ satisfies Formula (3). When $\overline{X}$ is smooth of dimension one, $h(\overline{X}, Y)$ recovers Rosenlicht's generalized Jacobian of [Ser59, ch. V]; see Proposition 9.4.1.

It is easy to see that reciprocity sheaves form a full abelian subcategory **Rec** of **PST**. Theorems 3–5 show that **Rec** contains reasonably many interesting objects of **PST**.

Theorem 3 (see Theorem 3.1.1). *If $F \in \mathbf{PST}$ is homotopy invariant, F has reciprocity.*

Recall from [SS03, BK16] that a presheaf represented by a commutative algebraic group has the structure of a presheaf with transfers.

Theorem 4 (see Theorem 4.1.1). *If $F \in \mathbf{PST}$ is represented by a smooth commutative algebraic group, F has reciprocity.*

In the appendices, Kay Rülling proves the following.

Theorem 5 (see Theorems A.6.2 and B.2.2). (a) *The presheaf with transfers $X \mapsto H^0(X, \Omega^i_X)$ has reciprocity, where Ω^i_X denotes the sheaf of absolute Kähler differentials. If k is perfect, the same is true with Ω^i_X replaced by relative differentials $\Omega^i_{X/k}$.*

(b) *If k is perfect of positive characteristic, the de Rham–Witt presheaves $X \mapsto H^0(X, W_n\Omega^i_X)$ have structures of presheaves with transfers and have reciprocity.*

An open problem is the following.

Question 1. Is **Rec** closed under extensions in **PST**?

The next results extend part of Voevodsky's main theorems for homotopy invariant presheaves with transfers to reciprocity sheaves: cf. [MVW06, Proposition 11.1] for Theorem 6 and [MVW06, Theorems 22.1, 22.2 and 22.15] for Theorem 7.

Theorem 6 (See Theorem 7.1.1 and Corollary 7.1.3). *Let $F \in \mathbf{PST}$ be a reciprocity presheaf.*

(1) *Let X be a smooth semi-local k-scheme and $V \subset X$ an open dense subset. Then the map $F(X) \to F(V)$ is injective.*

(2) *For an open dense immersion $U \subset X$ in $\mathbf{Sm}$, $F_{\mathrm{Zar}}(X) \to F_{\mathrm{Zar}}(U)$ is injective, where F_{Zar} is the Zariski sheafification of F as a presheaf.*

(3) *If $F(E) = 0$ for any field E, then $F_{\mathrm{Zar}} = 0$.*

We note that in this theorem, (2) and (3) are easy consequences of (1) (cf. [MVW06, Lemma 22.8 and Corollary 11.2]).

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THEOREM 7 (see Theorems 9.1.2, 9.2.2 and 9.3.2). *Let $F \in \mathbf{PST}$ be a reciprocity presheaf.*

(1) *F_{Zar} has a unique structure of a presheaf with transfers such that $F \to F_{\mathrm{Zar}}$ is a morphism in* **PST**.

(2) *If k is perfect, F_{Zar} has reciprocity.*

(3) *We have $F_{\mathrm{Zar}} = F_{\mathrm{Nis}}$, where F_{Nis} is the Nisnevich sheafification of F as a presheaf.*

Theorems 6 and 7 may be viewed as the degree-0 part of the following conjecture.

CONJECTURE 1. Suppose that k is perfect. Let F be a reciprocity presheaf which is a sheaf for the Nisnevich topology.

(1) (Gersten's conjecture.) For any smooth semi-local k-scheme X essentially of finite type, the Cousin resolution

$$0 \to F(X) \to \bigoplus_{x \in X^{(0)}} F(x) \to \bigoplus_{x \in X^{(1)}} H^1_x(X, F) \to \cdots \to \bigoplus_{x \in X^{(n)}} H^n_x(X, F) \to \cdots$$

is universally exact in the sense of [Gra85].

(2) The presheaves $H^i_{\mathrm{Zar}}(-, F)$ and $H^i_{\mathrm{Nis}}(-, F)$ coincide, and have reciprocity.

Note that the conjecture is known if F is homotopy invariant [Voe00a, Theorems 4.37, 5.6 and 5.7].

As remarked before, there are objects of **Rec** which are not homotopy invariant. But we show the following result in § 6.

THEOREM 8 (see Theorem 6.1.1). *Let F be a reciprocity presheaf which is separated for the Zariski topology. Then F is $\mathbf{P}^1$-invariant, namely for any $X \in \mathbf{Sm}$, the projection $p_X : \mathbf{P}^1_X \to X$ induces $p_X^* : F(X) \xrightarrow{\sim} F(\mathbf{P}^1_X)$.*

This is related to the approach to Gersten's conjecture in [CHK97, § 5.4] (where the perfectness of k is not required). Note that $\mathbf{P}^1$-invariance is weaker than reciprocity (Remark 6.1.2).

1. Notation

In the whole paper we fix a base field k. Let **Sm** be the category of separated smooth schemes of finite type over k. For $S \in \mathbf{Sm}$, let $\mathbf{Sch}/S$ be the category of schemes of finite type over S. We put $\mathbf{Sch} := \mathbf{Sch}/\operatorname{Spec} k$.

1.1.1. For an integral scheme $\overline{C}$ and a closed subscheme $D \subset \overline{C}$, we put

$$\begin{aligned} G(\overline{C}, D) &= \bigcap_{x \in D} \operatorname{Ker}(\mathcal{O}^{\times}_{\overline{C},x} \to \mathcal{O}^{\times}_{D,x}) \\ &= \varinjlim_{D \subset U \subset \overline{C}} \Gamma(U, \operatorname{Ker}(\mathcal{O}^*_U \to \mathcal{O}^*_D)), \end{aligned} \tag{1.1}$$

where U ranges over open subsets of $\overline{C}$ containing D.

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1.1.2. Let $S \in \mathbf{Sm}$. For $X \in \mathbf{Sch}/S$, $c(X/S)$ denotes the free abelian group on the set of closed integral subschemes of X which are finite over S and surjective over a component of S: this group is denoted by $C_0(X/S)$ in [SV96, § 3] and by $c_{\mathrm{equi}}(X/S, 0)$ in [Voe00a, § 2].

For any morphism $f : T \to S$ in $\mathbf{Sm}$, there is a homomorphism

$$f^* : c(X/S) \to c(X \times_S T/T)$$

called pullback of cycles (see [SV00, § 3.5] and [Voe00a, p. 90], where f^* is denoted by $cycl(f)$; see also [MVW06, Example 1A.12]).

For a morphism $f : X \to Y$ in $\mathbf{Sch}/S$, a pushforward

$$f_* : c(X/S) \to c(Y/S)$$

is defined as follows (see [Voe00a, paragraph before Proposition 2.1]): let $Z \subset X$ be a closed integral subscheme which belongs to $c(X/S)$. Since it is finite over S, its image $f(Z)$ in Y defines an element of $c(Y/S)$, and $f|_Z : Z \to f(Z)$ is finite and surjective. We then define $f_*(Z) = \deg(f|_Z) f(Z)$.

1.1.3. Recall the category $\mathbf{Cor}$ of finite correspondences and the category of presheaves with transfers $\mathbf{PST}$ [Voe00b, §§ 2.1 and 3.1]: the objects of $\mathbf{Cor}$ are those of $\mathbf{Sm}$ and, for $X, Y \in \mathbf{Sm}$, the group of morphisms $\mathbf{Cor}(X, Y)$ is $c(X \times Y/X)$. The category $\mathbf{PST}$ is the category of contravariant functors $\mathbf{Cor} \to \mathbf{Ab}$.

Let $\widetilde{\mathbf{Sm}}$ be the category of k-schemes X which are written as limits $X = \varprojlim_{i \in I} X_i$ over a filtered set I, where $X_i \in \mathbf{Sm}$ and all transition maps are open immersions. We frequently allow $F \in \mathbf{PST}$ to take values on objects of $\widetilde{\mathbf{Sm}}$ by $F(X) := \varinjlim_{i \in I} F(X_i)$.

2. Reciprocity sheaves and representability

In this section, we introduce reciprocity sheaves, our fundamental objects of study in this paper, and prove Theorem 2.

2.1 Definition of reciprocity sheaves

Definition 2.1.1. A modulus pair is a pair $(\overline{X}, Y)$, where $\overline{X} \in \mathbf{Sch}/k$ is integral and proper over k and $Y \subset X$ is a closed subscheme such that $X = \overline{X} - |Y|$ is quasi-affine and smooth over k, where $|Y|$ is the support of Y.

Let $(\overline{X}, Y)$ be a modulus pair with $X = \overline{X} - |Y|$. For $S \in \mathbf{Sm}$, we consider commutative diagrams (which we denote by $(\varphi : \overline{C} \to \overline{X} \times S)$)

$$\begin{array}{ccc} & & \overline{X} \\ & \overset{\gamma_\varphi}{\nearrow} & \uparrow \\ \overline{C} & \xrightarrow{\varphi} & \overline{X} \times S \\ & \underset{p_\varphi}{\searrow} & \downarrow \\ & & S \end{array} \tag{2.1}$$

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where:

(A) $\overline{C} \in \mathbf{Sch}$ is integral normal and φ is a finite morphism;
(B) for some generic point η of S, $\dim \overline{C} \times_S \eta = 1$;
(C) the image of γ_φ is not contained in Y.

These conditions imply that p_φ is proper and surjective over a connected component of S.

Let $G(\overline{C}, \gamma_\varphi^* Y)$ be as in (1.1) for $D = \gamma_\varphi^* Y = \varphi^*(Y \times S)$. We will see in § 2.5 that the divisor map on $\overline{C}$ induces

$$\operatorname{div}_{\overline{C}} : G(\overline{C}, \gamma_\varphi^* Y) \to c(C/S), \tag{2.2}$$

where $C = \overline{C} - |\gamma_\varphi^* Y|$ and $c(C/S)$ is defined in § 1.1.2.

Definition 2.1.2. Let F be a presheaf with transfers, $(\overline{X}, Y)$ a modulus pair with $X = \overline{X} - |Y|$ and $a \in F(X)$. We say that Y is a modulus for a (or a has modulus Y) if, for any diagram (2.1), we have

$$(\varphi_* \operatorname{div}_{\overline{C}}(f))^*(a) = 0 \text{ in } F(S) \quad \text{for any } f \in G(\overline{C}, \gamma_\varphi^* Y), \tag{2.3}$$

where $\varphi_* : c(C/S) \to c(X \times S/S) = \mathbf{Cor}(S, X)$ (see § 1.1.2).

Definition 2.1.3. We say that $F \in \mathbf{PST}$ has *reciprocity* (or F is a *reciprocity presheaf*) if for any quasi-affine $X \in \mathbf{Sm}$, any $a \in F(X)$ and any open immersion $X \hookrightarrow \overline{X}$ with $\overline{X}$ integral proper over k, a has modulus Y for some closed subscheme $Y \subset \overline{X}$ such that $X = \overline{X} - |Y|$. Let $\mathbf{Rec}$ be the full subcategory of $\mathbf{PST}$ consisting of reciprocity presheaves.

Remark 2.1.4. It is evident that $\mathbf{Rec}$ is closed in $\mathbf{PST}$ under taking sub and quotient objects, so that $\mathbf{Rec}$ is an abelian subcategory of $\mathbf{PST}$.

We now state the main result of this section.

Theorem 2.1.5. *Let $(\overline{X}, Y)$ be a modulus pair with $X = \overline{X} - |Y|$.*

(1) *The functor*

$$\mathbf{PST} \to \mathbf{Ab}, \quad F \mapsto \{a \in F(X) \mid a \text{ has modulus } Y\}$$

is represented by a presheaf with transfers $h(\overline{X}, Y)$.

(2) *Suppose that Y is a Cartier divisor on $\overline{X}$. Then $h(\overline{X}, Y)$ has reciprocity.*

This theorem is proven as follows. The object $h(\overline{X}, Y)$ is constructed as the cokernel of a map $\tau : \Phi(\overline{X}, Y) \to \mathbb{Z}_{\mathrm{tr}}(X)$ in $\mathbf{PST}$, which we describe in § 2.2 (see Proposition 2.2.2). After proving auxiliary lemmas in § 2.3, we construct $\Phi(\overline{X}, Y)$ and τ in § 2.4 and in § 2.5, respectively. Then (1) becomes obvious from its construction. The proof of (2) occupies §§ 2.6–2.9. We introduce a notion of *admissible correspondences* in § 2.6. Using a preliminary result proven in § 2.7, we show in § 2.8 a functoriality of $\Phi(\overline{X}, Y)$ with respect to admissible correspondences, from which (2) is deduced in § 2.9.

Remark 2.1.6. Let $(\overline{X}, Y)$ and X be as in Theorem 2.1.5(1). By definition, there is a surjection $\pi : \mathbb{Z}_{\mathrm{tr}}(X) \to h(\overline{X}, Y)$ with the following universal property: let $F \in \mathbf{PST}$ and $a \in F(X)$, which we regard as a morphism $a : \mathbb{Z}_{\mathrm{tr}}(X) \to F$ by Yoneda's lemma. Then a factors through π if and only if Y is a modulus for a.

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Corollary 2.1.7. *Let $F \in \mathbf{PST}$. Then F has reciprocity if for any quasi-affine $X \in \mathbf{Sm}$ and any $a \in F(X)$, there is a modulus pair $(\overline{X}, Y)$ with $X = \overline{X} - |Y|$ such that Y is a Cartier divisor on $\overline{X}$ and Y is a modulus for a.*

Proof. Let $X \in \mathbf{Sm}$ be quasi-affine and $a \in F(X)$. Let $(\overline{X}, Y)$ be a modulus pair given by the hypothesis. Take any open immersion $X \hookrightarrow \overline{X}'$ with $\overline{X}'$ integral proper over k. We need to find a closed subscheme $Y' \subset \overline{X}'$ such that $|Y'| = \overline{X}' - X$ and that Y' is a modulus for a. Let $a : \mathbb{Z}_{\mathrm{tr}}(X) \to F$ be the map in $\mathbf{PST}$ corresponding to $a \in F(X)$. By the assumption, it factors as

$$\mathbb{Z}_{\mathrm{tr}}(X) \xrightarrow{\pi} h(\overline{X}, Y) \xrightarrow{\theta} F.$$

Noting that Y is a Cartier divisor on $\overline{X}$, $h(\overline{X}, Y)$ has reciprocity by Theorem 2.1.5(2). Hence, there exists a closed subscheme $Y' \subset \overline{X}'$ such that $|Y'| = \overline{X}' - X$ and that Y' is a modulus for $\pi \in h(\overline{X}, Y)(X)$. This implies that π factors through $\mathbb{Z}_{\mathrm{tr}}(X) \to h(\overline{X}', Y')$ and hence so does a, which means that Y' is a modulus for a. □

Remark 2.1.8. For any given $X \in \mathbf{Sm}$, we can use Nagata's compactification theorem, then blowup and normalization to find a modulus pair $(\overline{X}, Y)$ such that $\overline{X}$ is normal and Y is a Cartier divisor. If $\mathrm{ch}(k) = 0$, then $\overline{X}$ can be taken to be smooth and $\overline{X} - X$ to be the support of a normal crossing divisor on $\overline{X}$.

2.2 Reformulation of Theorem 2.1.5(1)

Definition 2.2.1. Let $(\overline{X}, Y)$ be a modulus pair. For $S \in \mathbf{Sm}$, let $\mathcal{C}_{(\overline{X},Y)}(S)$ be the class of all morphisms $\varphi : \overline{C} \to \overline{X} \times S$ satisfying Conditions (A), (B), (C) of § 2.1, and put (see (2.1))

$$\Phi(\overline{X}, Y)(S) = \bigoplus_{(\varphi : \overline{C} \to \overline{X} \times S) \in \mathcal{C}_{(\overline{X},Y)}(S)} G(\overline{C}, \gamma_\varphi^* Y).$$

For $(\overline{C} \xrightarrow{\varphi} \overline{X} \times S) \in \mathcal{C}_{(\overline{X},Y)}(S)$ and $f \in G(\overline{C}, \gamma_\varphi^* Y)$, we prove that $\varphi_* \mathrm{div}_{\overline{C}}(f)$ belongs to $\mathbf{Cor}(S, X)$ in § 2.5 below, thereby obtaining a map

$$G(\overline{C}, \gamma_\varphi^* Y) \to \mathbf{Cor}(S, X). \tag{2.4}$$

Collecting these maps, we get

$$\tau_S : \Phi(\overline{X}, Y)(S) \to \mathbf{Cor}(S, X). \tag{2.5}$$

Theorem 2.1.5(1) follows immediately from the following proposition.

Proposition 2.2.2. *The assignment $S \mapsto \Phi(\overline{X}, Y)(S)$ gives a presheaf with transfers, and the τ_S define a morphism*

$$\tau : \Phi(\overline{X}, Y) \to \mathbb{Z}_{\mathrm{tr}}(X) \quad \textit{in } \mathbf{PST}.$$

Moreover,

$$h(\overline{X}, Y) := \mathrm{Coker}(\Phi(\overline{X}, Y) \to \mathbb{Z}_{\mathrm{tr}}(X)) \in \mathbf{PST}$$

represents the functor in Theorem 2.1.5(1).

Remark 2.2.3. Formula (3) is obvious from the definition.

The proof of this proposition is given in §§ 2.4–2.5. Before this, we need some preparation, which is the object of the next subsection. Until the end of § 2.5, we fix a modulus pair $(\overline{X}, Y)$ and put $X = \overline{X} - Y$.

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2.3 Auxiliary lemmas

Let $(\varphi : \overline{C} \to \overline{X} \times S) \in \mathcal{C}_{(\overline{X},Y)}(S)$ with S connected. Put $C := \overline{C} - |\gamma_\varphi^* Y|$. We consider a condition for a Cartier divisor α on $\overline{C}$:

$$\text{the support } |\alpha| \text{ of } \alpha \text{ is contained in } C. \tag{2.6}$$

Lemma 2.3.1. (1) *If α satisfies (2.6), then any irreducible component D of $|\alpha|$ is finite and surjective over S.*

(2) *For any $g \in G(\overline{C}, \gamma_\varphi^* Y)$, $\alpha = \operatorname{div}(g)$ satisfies (2.6).*

Proof. (1) Since X is quasi-affine, C is quasi-affine over S and hence D is quasi-affine over S. Since D is closed in $\overline{C}$, it is proper over S and hence finite by Lemma 2.3.2 below. Since D is pure of codimension one in $\overline{C}$, $\dim(D) = \dim(S)$ and $D \to S$ must be surjective, hence (1).

Part (2) is obvious from the definition. □

Lemma 2.3.2. *Let $X, S \in \mathbf{Sch}$. Then a morphism $f : X \to S$ which is quasi-affine and proper is finite.*

Proof. Factor f as $X \xrightarrow{j} X' \xrightarrow{f'} S$, where j is a dense open immersion and f' is affine. Since f is proper, j is proper and hence is an isomorphism. Therefore, f is affine and the lemma is well known. □

Let $Z \in \mathbf{Sch}$ be integral with generic point η and let $f : Z \to S$ be a morphism in $\mathbf{Sch}$. We denote by $\tilde{\Lambda}(\varphi, Z)$ the set of all irreducible components of $\overline{C} \times_S Z$ and by $\beta : \overline{C} \times_S Z \to \overline{C}$ the projection map. Define

$$\Lambda(\varphi, Z) := \{T \in \tilde{\Lambda}(\varphi, Z) \mid \dim T_\eta = 1, T \not\subset |\beta^*(\gamma_\varphi^* Y)|\}, \tag{2.7}$$

where $T_\eta := T \times_Z \eta$.

Lemma 2.3.3. *Let $T \in \tilde{\Lambda}(\varphi, Z)$ and let α be a Cartier divisor on $\overline{C}$ satisfying (2.6). (Note that $\beta^*(\alpha)$ is well defined by Lemma 2.3.1(1).)*

(1) *Let $s \in Z$ and let Σ be an irreducible component of $T_s := T \times_Z s$.*

  (a) *We have $\dim \Sigma \geqslant 1$ and $\Sigma \not\subset |\beta^*(\alpha)|$.*

  (b) *If $\Sigma \cap |\beta^*(\alpha)| \neq \emptyset$, then we have $\dim \Sigma = 1$.*

(2) *If $T \cap |\beta^*(\alpha)| \neq \emptyset$, then $T \in \Lambda(\varphi, Z)$.*

Proof. (1a) The first assertion follows from Chevalley's theorem applied at $f(s)$ [EGAIV, Lemme 13.1.1]. As all components of $\beta^*(\alpha) \times_S s$ are finite over s by Lemma 2.3.1(1), the second assertion follows from the first.

(1b) On the one hand, we have $\dim \Sigma \geqslant 1$ by (1a). On the other hand, $\Sigma \cap |\beta^*(\alpha)|$ is of codimension $\leqslant 1$ in Σ since it is the support of a Cartier divisor on Σ and hence we get $\dim \Sigma \leqslant 1$, noting that $\dim \Sigma \cap |\beta^*(\alpha)| = 0$ by Lemma 2.3.1(1). This proves (1b).

(2) The assumption that $T \cap |\beta^*(\alpha)| \neq \emptyset$ and (2.6) imply that $T \not\subset |\beta^*(\gamma_\varphi^* Y)|$. It remains to show that $\dim T_\eta = 1$. Let $W \subset Z$ be the closure of the image of $T \to Z$ and $\xi \in W$ be the generic point. We separate two cases.

(i) Assume that $W = Z$ (so that $\xi = \eta$). Then $T_\eta \cap |\beta^*(\alpha)| \neq \emptyset$ by Lemma 2.3.1(1). Thus, we may apply (1b) with $s = \eta$ to conclude that $\dim T_\eta = 1$.

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(ii) Assume that $W \neq Z$. We show that $T \cap |\beta^*(\alpha)| = \emptyset$. By Lemma 2.3.4 below, we have $\dim T \geqslant \dim Z + 1$ and hence $\dim T_\xi = \dim T - \dim W > 1$ since $W \neq Z$. By Chevalley's theorem, this implies that for any $s \in W$ and for any irreducible component Σ of T_s, we have $\dim \Sigma > 1$. Hence, by (1b), we conclude that $\Sigma \cap |\beta^*(\alpha)| = \emptyset$. This proves that $T \cap |\beta^*(\alpha)| = \emptyset$. □

Lemma 2.3.4. *Let $S, Z, C \in \mathbf{Sch}$ and let $p : C \to S$, $f : Z \to S$ be two morphisms. Then, for any irreducible component T of $C \times_S Z$, one has*

$$\dim T \geqslant \dim C + \dim Z - \dim S.$$

Proof.[1] Viewing $C \times_S Z$ as the inverse image in $C \times Z$ of the diagonal $\Delta_S \subset S \times S$ via the projection $C \times Z \to S \times S$, we reduce to showing that

$$\operatorname{codim}_{C \times Z} T \leqslant \operatorname{codim}_{S \times S} \Delta_S,$$

which easily follows from [Bou85, ch. VIII, p. 34, Proof of Corollary 4]. □

2.4 $\Phi(\overline{X}, Y)$ is a presheaf with transfers

For $S, S' \in \mathbf{Sm}$ and $Z \in \mathbf{Cor}(S', S)$, we define a homomorphism

$$Z^* : \Phi(\overline{X}, Y)(S) \to \Phi(\overline{X}, Y)(S') \tag{2.8}$$

(see Definition 2.2.1). We may assume that S, S', Z are integral; the definition is then extended linearly to the general case.

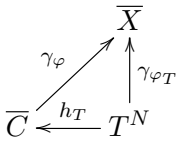
Take $(\varphi : \overline{C} \to \overline{X} \times S) \in \mathcal{C}_{(\overline{X},Y)}(S)$ and $T \in \Lambda(\varphi, Z)$. Let $T^N \to T$ be the normalization and consider the composite map

$$\varphi_T : T^N \to \overline{C} \times_S Z \xrightarrow{\varphi \times_S Z} \overline{X} \times Z \to \overline{X} \times S'. \tag{2.9}$$

Noting that φ and $Z \to S'$ are finite, so is φ_T and we have

$$(\varphi_T : T^N \to \overline{X} \times S') \in \mathcal{C}_{(\overline{X},Y)}(S'). \tag{2.10}$$

We have a commutative diagram



We define a map

$$Z_T^* : G(\overline{C}, \gamma_\varphi^* Y) \xrightarrow{h_T^*} G(T^N, \gamma_{\varphi_T}^* Y) \hookrightarrow \Phi(\overline{X}, Y)(S'), \tag{2.11}$$

where the second term comes from (2.10). To explain h_T^*, take $g \in G(\overline{C}, \gamma_\varphi^* Y)$ and put $\Sigma = |\operatorname{div}_{\overline{C}}(g)|$. Note that $g \in \Gamma(\overline{C} \backslash \Sigma, \mathcal{O}_{\overline{C}}^\times)$ since $\overline{C}$ is normal. By Lemma 2.3.3(1a), $T^N - h_T^{-1}(\Sigma)$ is a dense open subset of T^N and we get $h_T^*(g) \in \Gamma(T^N - h_T^{-1}(\Sigma), \mathcal{O}_{T^N}^\times)$. As $\gamma_{\varphi_T}^*(Y) = h_T^*(\gamma_\varphi(Y))$, we find that $h_T^*(g) \in G(T^N, \gamma_{\varphi_T}^*(Y))$.

[1] We thank J. Œsterlé for his help with this proof.

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Letting m_T be the multiplicity of T in $\overline{C} \times_S Z$, we then define

$$Z^* = \sum_{T \in \Lambda(\varphi, Z)} m_T \cdot Z_T^* : G(\overline{C}, \gamma_\varphi^* Y) \to \Phi(\overline{X}, Y)(S'),$$

which induces the desired map (2.8).

Let $S, S', S'' \in \mathbf{Sm}$ and $Z \in \mathbf{Cor}(S', S)$, $Z' \in \mathbf{Cor}(S'', S')$. We need to show that

$$Z'^* Z^* = (ZZ')^* : \Phi(\overline{X}, Y)(S) \to \Phi(\overline{X}, Y)(S''), \tag{2.12}$$

so that $\Phi(\overline{X}, Y)$ is an object of **PST**. To verify (2.12), one may suppose that S, S', Z and Z' are integral. Let $W_1, \dots, W_r$ be the irreducible components of $Z \times_{S'} Z' \subset S \times S''$ and n_j the multiplicity of W_j. Then we have $ZZ' = \sum_j n_j W_j$ (see [Dég07, Proof of Lemma 4.1.15]). Consider the following two subclasses of $\mathcal{C}_{(\overline{X}, Y)}(S'')$ (where $\Lambda(-,-)$ is as in (2.7) and $\varphi_{T'}$ is as in (2.9)):

$$\Xi_1 = \{\varphi_{T'} \mid T' \in \Lambda(\varphi_T, Z') \text{ for some } T \in \Lambda(\varphi, Z)\},$$
$$\Xi_2 = \bigcup_{1 \leqslant j \leqslant r} \{\varphi_{T'} \mid T' \in \Lambda(\varphi, W_j)\}.$$

Note that in the definition of Ξ_2, the sum is disjoint since the morphism $T' \to W_j$ for $T' \in \Lambda(\varphi, W_j)$ is surjective. For $T \in \Lambda(\varphi, Z)$ and $T' \in \Lambda(\varphi_T, Z')$ (so that $\varphi_{T'} \in \Xi_1$), we have a commutative diagram

$$\begin{array}{ccccccc} T' & \xrightarrow{\hookrightarrow} & T^N \times_{S'} Z' & & & & \\ \downarrow & & \downarrow & & & & \\ \tilde{T}' & \xrightarrow{\hookrightarrow} & T \times_{S'} Z' & \xrightarrow{\hookrightarrow} & \overline{C} \times_S Z \times_{S'} Z' & \xleftarrow{\hookleftarrow} & \overline{C} \times_S W_j \end{array} \tag{2.13}$$

where $\tilde{T}'$ is the image of T' in $T \times_{S'} Z'$. It suffices to prove the following.

(i) There is an inclusion $\Xi_2 \hookrightarrow \Xi_1$.

(ii) For $\varphi_{T'} \in \Xi_1$, if there is a Cartier divisor α on $\overline{C}$ such that $|\alpha| \subset C$ and that $\tilde{T}' \cap (|\alpha| \times_S Z \times_{S'} Z') \neq \emptyset$, then $\varphi_{T'} \in \Xi_2$.

(iii) Let $T \in \Lambda(\varphi, Z)$, $T' \in \Lambda(\varphi_T, Z')$, $T'' \in \Lambda(\varphi, W_j)$, $1 \leqslant j \leqslant r$ and suppose that $\varphi_{T'} \in \Xi_1$ corresponds to $\varphi_{T''} \in \Xi_2$ by (i). Then we have $m_T m_{T'} = n_j m_{T''}$.

We show (i). Take $\varphi_{T''}$ from Ξ_2. The canonical map $T''^N \to \overline{C}$ must factor through the normalization T^N of some irreducible component T of $\overline{C} \times_S Z$. We claim that $T \in \Lambda(\varphi, Z)$ (see (2.7)), which yields a desired correspondence. It is obvious that $T \not\subset |\beta^*(\gamma_\varphi^* Y)|$. If $\dim T_\eta \neq 1$, Lemma 2.3.3(1a) shows that $\dim T_\eta > 1$ and hence we get $\dim T''_\eta > 1$ by Chevalley's theorem [EGAIV, 13.1.1], which contradicts $T'' \in \Lambda(\varphi, W_j)$. Thus, we get $\dim T_\eta = 1$, and (i) is proved.

Next we show (ii). In view of (2.13), there are $j \in [1, r]$ and an irreducible component of T'' of $\overline{C} \times_S W_j$ such that the map $T' \to \overline{C} \times_S Z \times_{S'} Z'$ factors through T''. By the assumption, we have $T'' \cap (|\alpha| \times_S W_j) \neq \emptyset$. By Lemma 2.3.3(2), this implies that $T'' \in \Lambda(\varphi, W_j)$.

Finally, (iii) can be seen by a computation:

$$\begin{aligned} m_T m_{T'} &= l(\mathcal{O}_{T \times_{S'} Z', \eta_{T'}}) l(\mathcal{O}_{\overline{C} \times_S Z, \eta_T}) \\ &= l(\mathcal{O}_{Z \times_{S'} Z', \eta_{W_j}}) l(\mathcal{O}_{T \times_Z W_j, \eta_{T''}}) l(\mathcal{O}_{\overline{C} \times_S Z, \eta_T}) \\ &= l(\mathcal{O}_{Z \times_{S'} Z', \eta_{W_j}}) l(\mathcal{O}_{\overline{C} \times_S W_j, \eta_{T''}}) = n_j m_{T''}, \end{aligned}$$

where we denote by $l(R)$ the length of an Artin local ring R and by η_V the generic point of an integral scheme V.

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Remark 2.4.1. Here is an example where the inclusion $\Xi_2 \hookrightarrow \Xi_1$ in (i) is not surjective.[2] Let $\overline{X} = \mathbf{P}^1 \times \mathbf{P}^1$ and $Y := \infty \times \mathbf{P}^1 \cup \mathbf{P}^1 \times \infty$. Let B be the blow-up of $S := \mathbf{A}^2$ at 0, regarded as a closed subscheme of $\mathbf{P}^1 \times S$. Denote by $i : B \hookrightarrow \mathbf{P}^1 \times S$ the inclusion map. Put $\overline{C} := \mathbf{P}^1 \times B$ and define $\varphi = \mathrm{id}_{\mathbf{P}^1} \times i : \overline{C} \to \overline{X} \times S$. Then φ defines an element of $\mathcal{C}_{(\overline{X},Y)}(S)$. Set $S = \mathbf{A}^2 \supset Z = S' = 0 \times \mathbf{A}^1 \supset Z' = S'' = 0 \times 0$. Let $E \subset B$ be the exceptional curve, $L \subset B$ the strict transform of S' and $p := L \cap E$. Then $\Xi_2 = \emptyset$, because $\overline{C} \times_S S'' = \mathbf{P}^1 \times E$ does not belong to $\Lambda(\varphi, Z')$ (2.7). On the other hand, we have $\Xi_1 = \{\varphi_{\mathbf{P}^1 \times p}\}$ because $\Lambda(\varphi, Z) = \{\mathbf{P}^1 \times L\}$ and $\Lambda(\varphi_{\mathbf{P}^1 \times L}, Z') = \{\mathbf{P}^1 \times p\}$.

2.5 Construction of τ

Let $S \in \mathbf{Sm}$, $(\varphi : \overline{C} \to \overline{X} \times S) \in \mathcal{C}_{(\overline{X},Y)}(S)$ and put $C = \varphi^{-1}(X \times S)$. Lemma 2.3.1 shows that $\mathrm{div}_{\overline{C}}(g)$ satisfies (2.6) for any $g \in G(\overline{C}, \gamma_\varphi^*(Y))$. Hence, the divisor map on $\overline{C}$ induces (2.2). We obtain the map τ_S in (2.5) by composing (2.2) with

$$\varphi_* : c(C/S) \to \mathbf{Cor}(S, X) = c(X \times S/S).$$

To show that τ is a morphism in $\mathbf{PST}$, we need to show that the following diagram commutes for any $Z \in \mathbf{Cor}(S', S)$, $S, S' \in \mathbf{Sm}$:

$$\begin{array}{ccc} \Phi(\overline{X}, Y)(S) & \xrightarrow{\tau_S} & \mathbf{Cor}(S, X) \\ \downarrow{\scriptstyle Z^*} & & \downarrow{\scriptstyle Z^*} \\ \Phi(\overline{X}, Y)(S') & \xrightarrow{\tau_{S'}} & \mathbf{Cor}(S', X) \end{array} \tag{2.14}$$

As $\mathbf{Cor}(S', X) \to \mathbf{Cor}(S' \times k', X \times k')$ is injective for any extension k'/k, we may suppose that k is perfect. Also note that $\mathbf{Cor}(S', X) \to \mathbf{Cor}(V, X)$ is injective for any open dense immersion $V \to S'$. After such a base change, we may assume that Z is regular and hence $Z \in \mathbf{Sm}$ by the assumption that k is perfect. By (2.12), we may assume that Z is either (i) the transpose of a finite surjective morphism $S = Z \to S'$ or (ii) the graph of a morphism $f : S' = Z \to S$. In the case of (i), we have $c(C/S) = c(C/S')$ and the statement becomes trivial. We consider the case (ii). By shrinking Z further, $f : Z \to S$ can be written as the composite of a flat map and regular immersions of codimension one in S. Again by (2.12), we may assume that f is one such morphism. Here we present a proof of the commutativity of (2.14) assuming that f is a regular immersion of codimension one in S. We omit the proof for the case f is flat, as it can be shown in a similar (and much easier) way.

Let $(\varphi : \overline{C} \to \overline{X} \times S) \in \mathcal{C}_{(\overline{X},Y)}(S)$. Then the desired assertion will follow from the commutativity of the diagram

$$\begin{array}{ccccc} G(\overline{C}, \gamma_\varphi^* Y) & \xrightarrow{\mathrm{div}_{\overline{C}}} & c(C/S) & \xrightarrow{\varphi_*} & c(X \times S/S) \\ \downarrow{\scriptstyle \sum h_T^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \bigoplus_{T \in \Lambda(\varphi, Z)} G(T^N, \gamma_{\varphi_T}^* Y) & & c(C_Z/Z) & \xrightarrow{\varphi_*} & c(X \times Z/Z) \\ & \searrow^{\oplus \mathrm{div}_{T^N}} & \uparrow{\scriptstyle \sum m_T (i_T)_*} & & \\ & & \bigoplus_{T \in \Lambda(\varphi, Z)} c(T^0/Z) & & \end{array}$$

[2] This example was communicated to us by R. Sugiyama.

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where $C = \overline{C} - |\gamma_\varphi^* Y|$, $T^0 := T^N - |\gamma_{\varphi_T}^* Y|$, $C_Z = C \times_S Z$ and $i_T : T^0 \to C_Z$ is the natural map. (See (2.11) for h_T^* and § 1.1.2 for f^*). The commutativity of the right-hand squares is obvious. To prove that of the left-hand pentagon, we use the following facts.

(1) For a Cartier divisor $\iota : D \hookrightarrow C$, we have an identity

$$D \cdot [C_Z] = C_Z \cdot [D] \quad \text{as cycles on } D \cap C_Z,$$

where $[C_Z]$ (respectively $[D]$) is the cycle on C associated to the Cartier divisor C_Z (respectively D) and $D \cdot -$ (respectively $C_Z \cdot -$) is the intersection product (see [Ful98, Theorem 2.4]).

(2) If $D = \mathrm{div}_C(g)$ for $g \in k(C)^\times$ and $W \subset C$ is integral closed such that $W \not\subset |D|$, we have

$$D \cdot [W] = \mathrm{div}_W(g_{|W}) = \pi_* \mathrm{div}_{W^N}(g_{|W^N}),$$

where $\pi : W^N \to W$ is the normalization of W.

Take $g \in G(\overline{C}, \gamma_\varphi^* Y)$. We have $\mathrm{div}_{\overline{C}}(g) = \mathrm{div}_C(g)$ and

$$f^* \mathrm{div}_{\overline{C}}(g) = C_Z \cdot [\mathrm{div}_{\overline{C}}(g)] \overset{(1)}{=} \mathrm{div}_{\overline{C}}(g) \cdot [C_Z].$$

For each $T \in \Lambda(\varphi, Z)$, we have

$$\mathrm{div}_{\overline{C}}(g) \cdot [T^0] \overset{(2)}{=} \mathrm{div}_{T^0}(g|_{T^0}) = \mathrm{div}_{T^N}(h_T^*(g)).$$

Noting that $C_Z = \sum_{T \in \Lambda(\varphi, Z)} m_T T^0$ as a divisor on C, we get the desired commutativity.

The last claim of Proposition 2.2.2 (representability) is now tautologically true. This completes the proof of Proposition 2.2.2. □

2.6 Admissible correspondences

In order to prove Theorem 2.1.5(2), we need to show a functoriality of $\Phi(\overline{X}, Y)$ with respect to modulus pairs $(\overline{X}, Y)$: this will be done in Proposition 2.8.1. In this subsection, we introduce a notion of admissible correspondences, and prove their existence in suitable cases.

For two closed subschemes Y_1, Y_2 in a scheme $\overline{X}$, we write $Y_1 \geqslant Y_2$ if the inclusion $Y_2 \to \overline{X}$ factors through $Y_1 \to \overline{X}$ (equivalently, if for all $x \in \overline{X}$ one has $\mathcal{I}_{Y_1,x} \subset \mathcal{I}_{Y_2,x}$, where $\mathcal{I}_{Y_i}$ is the ideal sheaf of Y_i).

Definition 2.6.1. Let $M_i = (\overline{X}_i, Y_i)$ $(i = 1, 2)$ be modulus pairs and put $X_i = \overline{X}_i \backslash |Y_i|$. Let $Z \in \mathbf{Cor}(X_1, X_2)$ be an integral finite correspondence. We write $\bar{Z}^N$ for the normalization of the closure of Z in $\overline{X}_1 \times \overline{X}_2$ and $p_i : \bar{Z}^N \to \overline{X}_i$ for the canonical morphisms for $i = 1, 2$. We say that Z is *admissible* for (M_1, M_2) if $p_1^* Y_1 \geqslant p_2^* Y_2$. An element of $\mathbf{Cor}(X_1, X_2)$ is called admissible if all of its irreducible components are admissible.

Lemma 2.6.2. *Let $Z \in \mathbf{Cor}(X_1, X_2)$, where $X_1, X_2 \in \mathbf{Sm}$ are quasi-affine. For $i = 1, 2$, let $\overline{X}_i \in \mathbf{Sch}$ be a normal proper k-scheme which contains X_i as an open dense subset. Let $Y_2 \subset \overline{X}_2$ be a closed subscheme supported on $\overline{X}_2 - X_2$. Then there exists a closed subscheme $Y_1 \subset \overline{X}_1$ supported on $\overline{X}_1 - X_1$ such that Z is admissible for $((\overline{X}_1, Y_1), (\overline{X}_2, Y_2))$.*

Proof. We may assume that $Z \subset X_1 \times X_2$ is integral and finite surjective over X_1. Let $\overline{Z} \subset \overline{X}_1 \times \overline{X}_2$ be the closure of Z and $p_i : \overline{Z} \to \overline{X}_i$ $(i = 1, 2)$ be the natural maps. We remark that

$$p_1^{-1}(\overline{X}_1 - X_1)_{\mathrm{red}} = \overline{Z} - Z \quad \text{and} \quad p_2^{-1}(Y_2)_{\mathrm{red}} \subset \overline{Z} - Z. \tag{2.15}$$

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The first assertion follows from Lemma 2.6.3 below, noting that $Z \to X_1$ is finite. The second assertion is obvious since p_2 induces $Z \to X_2$. Hence, one can find a closed subscheme $Y_1 \subset \overline{X}_1$ supported on $\overline{X}_1 - X_1$ such that $p_1^* Y_1 \geqslant p_2^* Y_2$ on $\overline{Z}$[3] and therefore on $\overline{Z}^N$. □

Lemma 2.6.3. *Let*

$$\begin{array}{ccc} X & \xrightarrow{j} & \bar{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\ Y & \xrightarrow{j'} & \bar{Y} \end{array}$$

be a commutative diagram of schemes, where j, j' are dense open immersions and p is proper. Then $\bar{p}^{-1}(Y) = X$.

Proof. Let $Z = \bar{p}^{-1}(Y)$. The dense open immersion $X \hookrightarrow Z$ is proper and hence an isomorphism. □

2.7 An invariance property for $G(\overline{C}, D)$

Proposition 2.7.1. *Let $f : \overline{C}' \to \overline{C}$ be a proper surjective morphism of k-varieties and $D \subset \overline{C}$ be an effective Cartier divisor. Assume that $f_* \mathcal{O}_{\overline{C}'} = \mathcal{O}_{\overline{C}}$. Then, for $D' = \overline{C}' \times_{\overline{C}} D$, we have*

$$f^* : G(\overline{C}, D) \xrightarrow{\sim} G(\overline{C}', D').$$

For the proof, we need a lemma.

Lemma 2.7.2. *Let $f : \overline{C}' \to \overline{C}$ be a proper surjective morphism of schemes. Let $D \subset \overline{C}$ be a closed subscheme. Then the system*

$$\{f^{-1}(U) \mid U \supset D, U \text{ open}\}$$

is cofinal among the open neighbourhoods of $f^{-1}(D)$ in $\overline{C}'$.

Proof. Let U' be an open neighbourhood of $f^{-1}(D)$ and $Z = \overline{C}' - U'$. Then $f(Z)$ is closed and $f^{-1}(D) \subset f^{-1}(U) \subseteq U'$ with $U = \overline{C} - f(Z)$. □

Proof of Proposition 2.7.1. By Lemma 2.7.2, it suffices to show that

$$\mathcal{O}^*_{U|D} \xrightarrow{\sim} f_* \mathcal{O}^*_{U'|D'}$$

for U running through the open neighbourhoods of D in $\overline{C}$ and $U' = f^{-1}(U)$. In the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}^*_{U|D} & \longrightarrow & \mathcal{O}^*_{U} & \longrightarrow & \mathcal{O}^*_{D} \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ 0 & \longrightarrow & f_*\mathcal{O}^*_{U'|D'} & \longrightarrow & f_*\mathcal{O}^*_{U'} & \longrightarrow & f_*\mathcal{O}^*_{D'} \end{array}$$

[3] Note that if $F \hookrightarrow W$ is a closed immersion in **Sch** defined by the ideal sheaf $\mathcal{I}$, there exists $n > 0$ such that $\mathcal{I} \supseteq \mathcal{I}_0^n$, where $\mathcal{I}_0$ is the ideal sheaf of F_{red}.

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b is an isomorphism by the assumption that $f_*\mathcal{O}_{\overline{C}'} = \mathcal{O}_{\overline{C}}$. Hence, it suffices to show that c is injective. For this, it suffices to show that c' is injective in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_D & \longrightarrow & \mathcal{O}_U & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle b'}{\scriptstyle \simeq} & & \downarrow{\scriptstyle c'} & & \\ 0 & \longrightarrow & f_*\mathcal{I}_{D'} & \longrightarrow & f_*\mathcal{O}_{U'} & \longrightarrow & f_*\mathcal{O}_{D'} & & \end{array}$$

which will follow from the surjectivity of a'. To see this, note that

$$f^*\mathcal{I}_D = \mathcal{I}_D \otimes_{\mathcal{O}_U} \mathcal{O}_{U'} \xrightarrow{\sim} \mathcal{I}_D\mathcal{O}_{U'} = \mathcal{I}_{D'}$$

by the assumption that D is a Cartier divisor. Thus, we get

$$f_*\mathcal{O}_{U'} \otimes \mathcal{I}_D \simeq f_*(f^*\mathcal{I}_D) \xrightarrow{\sim} f_*\mathcal{I}_{D'}$$

by the projection formula and hence the claim since $\mathcal{O}_U \xrightarrow{\sim} f_*\mathcal{O}_{U'}$. □

2.8 Functoriality of $\Phi(\overline{X}, Y)$

Proposition 2.8.1. *Let $M_i = (\overline{X}_i, Y_i)$ $(i = 1, 2)$ be modulus pairs with $X_i = \overline{X}_i \backslash |Y_i|$, and let $Z \in \mathbf{Cor}(X_1, X_2)$ be admissible for (M_1, M_2). We assume that Y_2 is a Cartier divisor. Then there is a morphism*

$$Z_* : \Phi(\overline{X}_1, Y_1) \to \Phi(\overline{X}_2, Y_2) \quad \textit{in } \mathbf{PST}$$

which fits into a commutative diagram

$$\begin{array}{ccc} \Phi(\overline{X}_1, Y_1) & \xrightarrow{\tau} & \mathbb{Z}_{\mathrm{tr}}(X_1) \\ \downarrow{\scriptstyle Z_*} & & \downarrow{\scriptstyle Z_*} \\ \Phi(\overline{X}_2, Y_2) & \xrightarrow{\tau} & \mathbb{Z}_{\mathrm{tr}}(X_2) \end{array} \tag{2.16}$$

so that Z_* induces a map $h(\overline{X}_1, Y_1) \to h(\overline{X}_2, Y_2)$.

Proof. We may assume that $Z \subset X_1 \times X_2$ is integral and finite surjective over X_1. Let $\overline{Z} \subset \overline{X}_1 \times \overline{X}_2$ be the closure of Z and $p_i : \overline{Z} \to \overline{X}_i$ $(i = 1, 2)$ be the natural maps. Take $S \in \mathbf{Sm}$ connected and

$$(\varphi : \overline{C} \to \overline{X}_1 \times S) \in \mathcal{C}_{(\overline{X}_1, Y_1)}(S) \quad \text{with } C = \varphi^{-1}(X_1 \times S).$$

Let $C_{Z,i}$ $(i \in I)$ be the irreducible components of $C \times_{X_1} Z$, $\overline{C}_{Z,i}$ be the closure of $C_{Z,i}$ in $\overline{C} \times_{\overline{X}_1} \overline{Z}$ and $\overline{C}^N_{Z,i}$ be its normalization. We have the composite map

$$\psi'_i : \overline{C}^N_{Z,i} \to \overline{C} \times_{\overline{X}_1} \overline{Z} \xrightarrow{\varphi \times_{\overline{X}_1} \overline{Z}} \overline{Z} \times S \to \overline{X}_2 \times S.$$

By construction, ψ'_i is proper. Let

$$\overline{C}^N_{Z,i} \xrightarrow{p_{2,i}} \overline{C}_{X_2,i} \xrightarrow{\psi_i} \overline{X}_2 \times S \tag{2.17}$$

be its Stein factorization. Note that $\overline{C}_{X_2,i}$ is normal and ψ_i is finite. Let $J \subset I$ be the subset of those $i \in I$ such that $\dim(\overline{C}_{X_2,i} \times_S \eta) = 1$, where η is the generic point of S.

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For $i \in J$, we have

$$(\psi_i : \overline{C}_{X_2,i} \to \overline{X}_2 \times S) \in \mathcal{C}_{(\overline{X}_2,Y_2)}(S).$$

Indeed, it suffices to check that the image of ψ_i' is not contained in $Y_2 \times S$. The normalization $C_{Z,i}^N$ of $C_{Z,i}$ is dense open in $\overline{C}_{Z,i}^N$ and ψ_i' induces a morphism

$$C_{Z,i}^N \to C \times_{X_1} Z \xrightarrow{\varphi \times_{\overline{X}_1} \overline{Z}} Z \times S \to X_2 \times S,$$

which implies the desired assertion. We obtain a commutative diagram

$$\begin{array}{ccccc}
\overline{C} & \xleftarrow{p_{1,i}} & \overline{C}_{Z,i}^N & \xrightarrow{p_{2,i}} & \overline{C}_{X_2,i} \\
\downarrow{\scriptstyle \gamma_\varphi} & & \downarrow{\scriptstyle \gamma_{Z,i}} & & \downarrow{\scriptstyle \gamma_{X_2,i}} \\
\overline{X}_1 & \xleftarrow{p_1} & \overline{Z} & \xrightarrow{p_2} & \overline{X}_2
\end{array}$$

From this, we get a composition

$$\begin{aligned} Z_*^i : G(\overline{C}, \gamma_\varphi^* Y_1) \to G(\overline{C}_{Z,i}^N, p_{1,i}^* \gamma_\varphi^* Y_1) &= G(\overline{C}_{Z,i}^N, \gamma_{Z,i}^* p_1^* Y_1) \hookrightarrow G(\overline{C}_{Z,i}^N, \gamma_{Z,i}^* p_2^* Y_2) \\ &\simeq G(\overline{C}_{X_2,i}, \gamma_{X_2,i}^* Y_2) \subset \Phi(\overline{X}_2, Y_2)(S), \end{aligned}$$

where the first map is the pullback by $p_{1,i}$, the middle inclusion comes from the assumption that $p_1^* Y_1 \geqslant p_2^* Y_2$ and the isomorphism is the inverse of the pullback along $p_{2,i}$, which is an isomorphism by Proposition 2.7.1 (here we use the assumption that Y_2 is a Cartier divisor; also, $p_{2,i}$ satisfies the assumption of Proposition 2.7.1 by the construction (2.17)). Letting m_i be the multiplicity of $C_{Z,i}$ in $C \times_{X_1} Z$, we then define

$$Z_* = \sum_{i \in J} m_i Z_*^i : G(\overline{C}, \gamma_\varphi^* Y_1) \to \Phi(\overline{X}_2, Y_2)(S),$$

which induces $Z_* : \Phi(\overline{X}_1, X_1)(S) \to \Phi(\overline{X}_2, Y_2)(S)$. It is easy to check (essentially as a special case of (2.12)) that this induces a map $Z_* : \Phi(\overline{X}_1, Y_1) \to \Phi(\overline{X}_2, Y_2)$ in **PST**.

To prove the commutativity of (2.16), we may assume that $S = \operatorname{Spec} K$ for a field K by the same reason as for the commutation of (2.14). Then $\overline{C}$ and $\overline{C}_{Z,i}^N$ are regular of dimension one, so that $p_{1,i}$ is flat. We are reduced to showing the commutativity of the diagram

$$\begin{array}{ccccc}
G(\overline{C}, \gamma_\varphi^* Y_1) & \xrightarrow{\operatorname{div}_{\overline{C}}} & c(C/K) & \xrightarrow{\varphi_*} & c(X_1 \times K/K) \\
\downarrow{\scriptstyle \oplus m_i p_{2,i*} p_{1,i}^*} & & \downarrow{\scriptstyle \oplus m_i p_{2,i*} p_{1,i}^*} & & \downarrow{\scriptstyle Z_*} \\
\bigoplus_{i \in J} G(\overline{C}_{X_2,i}, \gamma_{X_2,i}^* Y_2) & \xrightarrow{\oplus \operatorname{div}_{\overline{C}_{X_2,i}}} & \bigoplus_{i \in J} c(C_{X_2,i}/K) & \xrightarrow{\oplus \psi_{i*}} & c(X_2 \times K/K)
\end{array}$$

where $p_{1,i}^*$ is the flat pullback of cycles. Each square of the diagram is easily seen to be commutative. This completes the proof of Proposition 2.8.1. □

2.9 Proof of Theorem 2.1.5(2)

Let $(\overline{X}, Y)$ be as in the theorem. Assume that we are given a connected quasi-affine $V \in \mathbf{Sm}$ and $a \in h(\overline{X}, Y)(V)$. Let $\overline{V}$ be a proper normal scheme over k containing V as a dense open subset.

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Let $\tilde{a} \in \mathbb{Z}_{\mathrm{tr}}(X)(V) = \mathbf{Cor}(V, X)$ be a lift of a under the canonical surjection $\mathbb{Z}_{\mathrm{tr}}(X)(V) \to h(\overline{X}, Y)(V)$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_{\mathrm{tr}}(V) & \xrightarrow{\tilde{a}_*} & \mathbb{Z}_{\mathrm{tr}}(X) \\ & \searrow^{a_*} & \downarrow \\ & & h(\overline{X}, Y) \end{array}$$

where a_* and $\tilde{a}_*$ are respectively induced by a and $\tilde{a}$ via the Yoneda embedding. By Lemma 2.6.2 and Proposition 2.8.1, there is a closed subscheme $W \subset \overline{V}$ supported on $\overline{V} - V$ such that a_* factors through $\mathbb{Z}_{\mathrm{tr}}(V) \to h(\overline{V}, W)$, which implies that a has modulus W. □

3. Homotopy invariance implies reciprocity

3.1 Introduction

In this section, we prove Theorem 3. Actually, we prove the following stronger result.

Theorem 3.1.1. *Let $F \in \mathbf{PST}$. We consider the following condition.*

($\diamondsuit$) *If $X \in \mathbf{Sm}$ and $a \in F(X)$, then Y is a modulus for a for any modulus pair $(\overline{X}, Y)$ with $X = \overline{X} - Y$. (Equivalently: for any modulus pair $(\overline{X}, Y)$ with Y reduced, Y is a modulus for a.)*

Then we have the following.

(1) *If F is homotopy invariant, then ($\diamondsuit$) holds.*

(2) *If F is separated for the Zariski topology and satisfies ($\diamondsuit$), then F is homotopy invariant.*

For the proof, we need some preliminary results on relative Picard groups and relative Suslin homology, which will occupy §§ 3.2 and 3.3. The proofs of (1) and (2) will be given in §§ 3.4 and 3.5, respectively.

3.2 Relative Picard group

We recall [SV96, §§ 2–3] with some modifications. Let $\overline{X}$ be an integral scheme and Y a closed subscheme of $\overline{X}$. We denote by $\mathrm{Pic}(\overline{X}, Y)$ the group of all isomorphism classes of pairs (L, σ) of an invertible sheaf L and an isomorphism $\sigma : L|_Y \cong \mathcal{O}_Y$. Note that $\mathrm{Pic}(\overline{X}) = \mathrm{Pic}(\overline{X}, \emptyset)$. We have an exact sequence

$$\Gamma(\overline{X}, \mathcal{O}_{\overline{X}}^{\times}) \to \Gamma(Y, \mathcal{O}_Y^{\times}) \to \mathrm{Pic}(\overline{X}, Y) \to \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

and an isomorphism

$$\mathrm{Pic}(\overline{X}, Y) \cong H^1_{\mathrm{Zar}}(\overline{X}, \mathbf{G}_{\overline{X},Y}),$$

where $\mathbf{G}_{\overline{X},Y} := \ker(\mathcal{O}_{\overline{X}}^{\times} \to \mathcal{O}_Y^{\times})$. An element (L, σ) of $\mathrm{Pic}(\overline{X}, Y)$ is called liftable if there exists a pair $(U, \tilde{\sigma})$ consisting of an open subset $U \subset \overline{X}$ and an isomorphism $\tilde{\sigma} : L|_U \cong \mathcal{O}_U$ such that $Y \subset U$ and $\tilde{\sigma}|_Y = \sigma$. We define $\widetilde{\mathrm{Pic}}(\overline{X}, Y)$ to be the subgroup of $\mathrm{Pic}(\overline{X}, Y)$ consisting of liftable elements. Let $\mathrm{Div}(\overline{X}, Y)$ be the group of Cartier divisors on $\overline{X}$ whose support does not intersect with Y.

Lemma 3.2.1. *Let $\overline{X}$ and Y be as above.*

(1) *We have an exact sequence*

$$0 \to \Gamma(\overline{X}, \mathbf{G}_{\overline{X},Y}) \to G(\overline{X}, Y) \xrightarrow{\mathrm{div}^{\overline{X}}} \mathrm{Div}(\overline{X}, Y) \to \widetilde{\mathrm{Pic}}(\overline{X}, Y) \to 0.$$

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(2) *If $\overline{X}$ is normal and Y is reduced, the pullback by the projection $p : X \times \mathbf{A}^1 \to X$ induces isomorphisms*

$$p^* : \mathrm{Pic}(\overline{X}, Y) \simeq \mathrm{Pic}(\overline{X} \times \mathbf{A}^1, Y \times \mathbf{A}^1), \tag{3.1}$$

$$p^* : \widetilde{\mathrm{Pic}}(\overline{X}, Y) \simeq \widetilde{\mathrm{Pic}}(\overline{X} \times \mathbf{A}^1, Y \times \mathbf{A}^1). \tag{3.2}$$

(3) *If Y has an affine open neighbourhood in $\overline{X}$, then $\widetilde{\mathrm{Pic}}\,(\overline{X}, Y) = \mathrm{Pic}(\overline{X}, Y)$.*

Proof. The assertions are shown in [SV96, 2.3 and 2.5] except for (3.2), which we deduce from (3.1). Let

$$i^* : \mathrm{Pic}(\overline{X} \times \mathbf{A}^1, Y \times \mathbf{A}^1) \to \mathrm{Pic}(\overline{X}, Y)$$

be the pullback along the zero section $i : \mathrm{Spec}\, k \to \mathbf{A}^1$. By definition,

$$i^*(\widetilde{\mathrm{Pic}}(\overline{X} \times \mathbf{A}^1, Y \times \mathbf{A}^1)) \subset \widetilde{\mathrm{Pic}}(\overline{X}, Y)$$

and i^*p^* is the identity. Hence, (3.2) follows from the fact that $\mathrm{Ker}(i^*) = 0$, which follows from (3.1). □

3.3 Relative Suslin homology

We take a connected $S \in \mathbf{Sm}$ and $\overline{X} \in \mathbf{Sch}/S$. Let $Y \subset \overline{X}$ be a closed subscheme. Suppose that $\overline{X}$ is normal and set $X := \overline{X} - Y$. Let $c(X/S)$ be as in § 1.1.2. We define for each $n \in \mathbb{Z}_{\geqslant 0}$

$$C_n(X/S) := c(X \times \Delta^n / S \times \Delta^n),$$

where $\Delta^n = \mathrm{Spec}\, k[t_0, \ldots, t_n]/(\sum t_i - 1)$ is the standard cosimplicial scheme over k. The complex $C_*(X/S)$ of abelian groups thus obtained is called the *relative Suslin complex*. Its homology group is denoted by $H^S_*(X/S)$ and called the *relative Suslin homology*.

Now suppose that the generic fiber of $X \to S$ is one dimensional and that X is quasi-affine over S. By definition, the components of $Z \in C_n(X/S)$ are closed in $\overline{X} \times \Delta^n$, so that Z is a Weil divisor on $\overline{X} \times \Delta^n$. Let $\widetilde{C}_n(X/S) \subset C_n(X/S)$ be the subgroup of all $Z \in C_n(X/S)$ which are Cartier divisors on $\overline{X} \times \Delta^n$. We have

$$\widetilde{C}_n(X/S) = \mathrm{Div}(\overline{X} \times \Delta^n, Y \times \Delta^n). \tag{3.3}$$

As a pullback of a Cartier divisor is Cartier, $\widetilde{C}_*(X/S)$ is a subcomplex of $C_*(X/S)$. Its homology groups are denoted by $\widetilde{H}^S_*(X/S)$.

Theorem 3.3.1. *Assume that Y reduced. Then we have*

$$\widetilde{H}^S_n(X/S) \cong \begin{cases} \widetilde{\mathrm{Pic}}(\overline{X}, Y) & (n = 0), \\ 0 & (n > 0). \end{cases}$$

Proof. By Lemma 3.2.1(1) and (3.3), we have an exact sequence of complexes of abelian groups

$$0 \to \Gamma(\overline{X} \times \Delta^*, \mathbf{G}_{\overline{X} \times \Delta^*, Y \times \Delta^*}) \to G(\overline{X} \times \Delta^*, Y \times \Delta^*)$$
$$\to \widetilde{C}_n(X/S) \to \widetilde{\mathrm{Pic}}(\overline{X} \times \Delta^*, Y \times \Delta^*) \to 0.$$

The assumption that X is quasi-affine over S is preserved under any base change [EGAII, (5.1.10)(iii)] and hence remains true after passing to the geometric fibers. This shows that Y meets every component of these fibers, which implies as in [SV96, Proof of Theorem 3.1] that the first term vanishes: $\Gamma(\overline{X} \times \Delta^*, \mathbf{G}_{\overline{X} \times \Delta^*, Y \times \Delta^*}) = 0$. It is also proved in loc. cit. that $G(\overline{X} \times \Delta^*, Y \times \Delta^*)$ is acyclic. Now the theorem follows from Lemma 3.2.1(2). □

Corollary 3.3.2. *Under the same assumption as in the previous theorem, the following composition is zero:*

$$G(\overline{X}, Y) \to \widetilde{C}_0(X/S) \to \widetilde{H}^S_0(X/S).$$

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3.4 Proof of Theorem 3.1.1(1)

Suppose that $F \in \mathbf{PST}$ is homotopy invariant. Let $(\overline{X}, Y)$ and $a \in F(X)$ be as in $(\diamondsuit)$. We show that Y is a modulus for a. Let $S \in \mathbf{Sm}$ and $(\varphi : \overline{C} \to \overline{X}_S) \in \mathcal{C}_{(\overline{X},Y)}(S)$. Put $Y_{\overline{C}} = \varphi^*(Y_S)$, $C = \overline{C} - Y_{\overline{C}}$. We need to show that the composition

$$G(\overline{C}, Y_{\overline{C}}) \to c(C/S) \xrightarrow{\varphi_*} c(X_S/S) = \mathbb{Z}_{\mathrm{tr}}(X)(S) \xrightarrow{a} F(S) \qquad (*)$$

is zero. By [MVW06, Lemma 7.5], the homotopy invariance of F implies that the last map factors as

$$c(X_S/S) \to H^S_0(X_S/S) \to F(S).$$

Therefore, the map $(*)$ factors as

$$G(\overline{C}, Y_{\overline{C}}) \subset G(\overline{C}, (Y_{\overline{C}})_{\mathrm{red}}) \to \widetilde{C}_0(C/S) \to \widetilde{H}^S_0(C_S/S) \to H^S_0(C_S/S) \to H^S_0(X_S/S) \to F(S),$$

which is zero by Corollary 3.3.2. □

3.5 Proof of Theorem 3.1.1(2)

This proof is adapted from [Voe00a, Proposition 3.11]. Suppose that $F \in \mathbf{PST}$ is separated for Zariski topology and satisfies $(\diamondsuit)$, and take $S \in \mathbf{Sm}$. Let $p : S \times \mathbf{A}^1 \to S$ be the projection and $i : \operatorname{Spec} k \hookrightarrow \mathbf{A}^1$ be the 0-section. We must show that $p^* : F(S) \to F(S \times \mathbf{A}^1)$ is an isomorphism, which will follow from the injectivity of $(\mathrm{id}_S \times i)^* : F(S \times \mathbf{A}^1) \to F(S)$ (see Proof of Lemma 3.2.1). Since $F \in \mathbf{PST}$ is separated for Zariski topology, we may assume that S is affine. Consider the commutative diagram

$$\begin{array}{ccccc} S \times \mathbf{A}^1 & \xrightarrow{\mathrm{id}_S \times \delta} & (S \times \mathbf{A}^1) \times \mathbf{A}^1 & \xrightarrow{p \times \mathrm{id}_{\mathbf{A}^1}} & S \times \mathbf{A}^1 \\ & & \uparrow{\scriptstyle \mathrm{id}_{S \times \mathbf{A}^1} \times i} & & \uparrow{\scriptstyle \mathrm{id}_S \times i} \\ & & S \times \mathbf{A}^1 & \xrightarrow{\quad p \quad} & S \end{array}$$

Since $(p \times \mathrm{id}_{\mathbf{A}^1})(\mathrm{id}_S \times \delta) = \mathrm{id}_{S \times \mathbf{A}^1}$, it suffices to show that

$$(\mathrm{id}_S \times \delta)^* = (\mathrm{id}_{S \times \mathbf{A}^1} \times i)^* : F((S \times \mathbf{A}^1) \times \mathbf{A}^1) \to F(S \times \mathbf{A}^1). \qquad (*)$$

Take a proper integral variety $\overline{S}$ which contains S as an open dense subscheme. Put $X := (S \times \mathbf{A}^1) \times \mathbf{A}^1$, $\overline{X} := (\overline{S} \times \mathbf{P}^1) \times \mathbf{P}^1$ and $Y := (\overline{X} - X)_{\mathrm{red}}$, so that $(\overline{X}, Y)$ is a modulus pair (here we used the assumption that S is affine). Let $\overline{C} := (S \times \mathbf{A}^1) \times \mathbf{P}^1$ and let $\varphi : \overline{C} \to \overline{X} \times (S \times \mathbf{A}^1)$ be the morphism defined by the natural inclusion $\overline{C} \hookrightarrow \overline{X}$ and the projection $\overline{C} \to S \times \mathbf{A}^1$. Then φ is an element of $\mathcal{C}_{(\overline{X},Y)}(S \times \mathbf{A}^1)$ (cf. Definition 2.2.1). Since Y is a modulus for any $a \in F(X)$ by $(\diamondsuit)$, we get $(\varphi_* \operatorname{div}_{\overline{C}}(f))^*(a) = 0$ for any $f \in G(\overline{C}, Y|_{\overline{C}})$, where $Y|_{\overline{C}} = \varphi^*(Y \times (S \times \mathbf{A}^1)) = (S \times \mathbf{A}^1) \times \infty$. If we write x, y for the coordinates of $\mathbf{P}^1 \times \mathbf{P}^1$, then the function $f = 1 - (x/y)$ belongs to $G(\overline{C}, Y|_{\overline{C}})$, and $\varphi_* \operatorname{div}_{\overline{C}}(f) \in \mathbf{Cor}(S \times \mathbf{A}^1, (S \times \mathbf{A}^1) \times \mathbf{A}^1)$ agrees with the difference of the graphs of $\mathrm{id}_S \times \delta$ and $\mathrm{id}_{S \times \mathbf{A}^1} \times i$. This proves $(*)$ and hence completes the proof. □

Remark 3.5.1. Let $(\overline{X}, Y)$ be a modulus pair with Y reduced. Put $X = \overline{X} - |Y|$. Let $h_0(X)$ be the presheaf with transfers introduced in [Voe00b, p. 207], which is characterized as the maximal homotopy invariant quotient of $\mathbb{Z}_{\mathrm{tr}}(X)$. By Theorem 3, we get a (surjective) map $h(\overline{X}, Y) \to h_0(X)$. It is an interesting problem to know when this is an isomorphism: it is true (after Zariski sheafification) if $\dim X = 1$ by Theorem 9.4.1 below.

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4. Algebraic groups have reciprocity

4.1 Introduction

In this section, we show the following result by adapting an argument of [Ser59].

Theorem 4.1.1. *Any smooth commutative group scheme G locally of finite type over k, regarded as a presheaf with transfers, has reciprocity.*

The proof goes as follows. In § 4.2, we reduce the proof to the case k is algebraically closed and G connected. We then prove Theorem 4.1.1 for characteristic zero in § 4.3, and for characteristic $p > 0$ in § 4.4.

In what follows we identify G with the object of **PST** represented by G (cf. [SS03, Proof of Lemma 3.2] and [BK16, Lemma 1.4.4]). Before going to the proof of Theorem 4.1.1, we make a simple remark, which will be used frequently in this paper.

Remark 4.1.2. Assume that k is perfect. Let $(\overline{X}, Y)$ be a modulus pair with $X = \overline{X} - Y$, $F \in \mathbf{PST}$ and $a \in F(X)$. Assume that F has global injectivity, which means that for any dense open immersion $j : U \hookrightarrow X$ in **Sm**, $j^* : F(X) \to F(U)$ is injective. Then, in order to show that Y is a modulus for a, it suffices to verify the following condition (cf. (2.3)).

Let $K = k(S)$ be the function field of a connected $S \in \mathbf{Sm}$, $\overline{C}$ a normal integral proper curve over K and $\varphi : \overline{C} \to \overline{X} \times K$ a finite morphism such that $\varphi(\overline{C}) \not\subset Y \times K$. Put $C = \varphi^{-1}(X \times K)$ and let $\psi : C \to X$ be the induced map. Since C is regular and k is perfect, we have $C \in \widetilde{\mathbf{Sm}}$, whence $\psi^* : F(X) \to F(C)$ (cf. § 1.1.3). Then

$$(\mathrm{div}_{\overline{C}}(g))^*(\psi^*(a)) = 0 \quad \text{in } F(K) \text{ for any } g \in G(\overline{C}, D),$$

where $D = \varphi^*(Y \times K)$ and $\mathrm{div}_{\overline{C}}(g) \in c(C/K)$ is viewed as an element of $\mathbf{Cor}(K, C)$ by the map $C \to C \times K$ induced by the identity on C and the projection $C \to \mathrm{Spec}\, K$.

4.2 Reduction to algebraically closed and connected cases

Let $\bar{k}$ be an algebraic closure of k. We assume that $G_{\bar{k}}$ has reciprocity (over $\bar{k}$), and prove that G has reciprocity (over k). Take $X \in \mathbf{Sm}$ connected quasi-affine, $a \in G(X)$ and $\overline{X}$ a compactification of X: we must find a closed subset $Y \subset \overline{X}$ with support $\overline{X} - X$ which is a modulus for a. By hypothesis, there is such a $Y \subset \overline{X}_{\bar{k}}$ for the image $a_{\bar{k}}$ of a in $G(X_{\bar{k}})$ (more accurately, this is true component by component in case X is not geometrically connected). We may choose Y such that $\mathcal{I}_Y = \mathcal{I}^n_{(\overline{X}_{\bar{k}} - X_{\bar{k}})_{\mathrm{red}}}$ for some $n > 0$, which implies that Y is defined over k. We claim that Y is the desired modulus for a. Indeed, let $S \in \mathbf{Sm}$ and $\overline{C} \in \mathcal{C}_{(\overline{X},Y)}(S)$: with the notation of Definition 2.1.2, we obviously have for $f \in G(\overline{C}, \gamma_\varphi^* Y)$

$$((\varphi_* \mathrm{div}_{\overline{C}}(f))^*(a))_{\bar{k}} = ((\varphi_{\bar{k}})_* \mathrm{div}_{\overline{C}_{\bar{k}}}(f_{\bar{k}}))^*(a_{\bar{k}}) = 0 \in G(S_{\bar{k}})$$

and hence the claim since $G(S) \to G(S_{\bar{k}})$ is injective.

Now assume that k is algebraically closed. We reduce the proof of Theorem 4.1.1 to the case G is connected. Indeed, G then splits as a direct product $G^0 \times D$, where G^0 is the connected component of 0 and D is discrete; in particular, the choice of such a splitting yields a translation isomorphism $\tau_\delta : G^\delta \xrightarrow{\sim} G^0$ for any $\delta \in D$. By construction of the action of finite correspondences, a finite correspondence of degree d in $\mathbf{Cor}(S, X)$ maps $G^\delta(X)$ to $G^{d\delta}(S)$ for any $\delta \in D$; in particular, $(\varphi_* \mathrm{div}_{\overline{C}}(f))^*(a)$ maps $G^\delta(X)$ to $G^0(S)$. Since the action of $(\varphi_* \mathrm{div}_{\overline{C}}(f))^*(a)$ clearly commutes with τ_δ, reciprocity for G^0 implies reciprocity for G.

From now on until the end of this section, we assume that k is algebraically closed and G connected.

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4.3 Proof of Theorem 4.1.1 in characteristic zero

We take an integral quasi-affine $X \in \mathbf{Sm}$ and $a \in G(X) = \mathrm{Mor}_k(X, G)$. Let $\overline{X}$ be a proper smooth integral variety that contains X as an open dense subset and $W := (\overline{X} - X)_{\mathrm{red}}$ be a normal crossing divisor (Remark 2.1.8). In view of Corollary 2.1.7, it suffices to find $n \in \mathbb{Z}_{>0}$ such that $(\overline{X}, nW)$ is a modulus for a.

Let $\{\omega_1, \dots, \omega_d\}$ be a basis of the space of invariant differential forms on G. Noting that $\varinjlim_n \Gamma(\overline{X}, \Omega^1_{\overline{X}/k} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}}(nW)) = \Gamma(X, \Omega^1_{X/k})$, take $n > 0$ such that

$$a^*\omega_i \in \Gamma(\overline{X}, \Omega^1_{\overline{X}/k} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}}(nW)) \quad \text{for all } i = 1, \dots, d.$$

We claim that $Y := nW$ is a modulus for a. For this, it suffices to verify the condition in Remark 4.1.2, but it follows from the following result from [Ser59]. □

Proposition 4.3.1 [Ser59, III, Proposition 10]. *Let G be a commutative algebraic group over a field K of characteristic zero, and let $\{\omega_1, \dots, \omega_d\}$ be a basis of the space of invariant differential forms on G. Let $\overline{C}$ be a proper normal curve over K, C an open dense subscheme of $\overline{C}$ and $a : C \to G$ a morphism. Let D be an effective divisor on $\overline{C}$ such that $|D| = \overline{C} - C$ and that*

$$a^*\omega_i \in \Gamma(\overline{C}, \Omega^1_{\overline{C}/K} \otimes_{\mathcal{O}_{\overline{C}}} \mathcal{O}_{\overline{C}}(D)) \quad \textit{for all } i = 1, \dots, d.$$

Then we have $(\mathrm{div}_{\overline{C}}(g))^*(a) = 0 \in G(K)$ for any $g \in G(\overline{C}, D)$, where a is viewed as a section over C of the object of $\mathbf{PST}$ represented by G.

4.4 Proof of Theorem 4.1.1 in positive characteristic

In this case, the following lemma is proved in [Ser59, III, §§ 7 and 9].

Lemma 4.4.1. *There exist a semi-abelian variety G_1, a unipotent group U and a homomorphism $\theta : G \to G_1 \times U$ with finite kernel.*

Here we include a sketch of the proof taken from [Ser59]. Let A be the maximal abelian quotient of G, and put $L = \ker(G \to A)$. Write $L = L_u \times L_m$, where L_u is a unipotent group and L_m is a torus. Put $G_1 := G/L_u$, which is semi-abelian. Take an $N \in \mathbb{Z}_{>0}$ such that $p^N L_u = 0$ and put $U := (G/L_m)/p^N$, which is unipotent. Define θ to be the product of the canonical surjective maps $G \to G_1$ and $G \to U$. One checks that $\ker(\theta)$ is finite. □

Proposition 4.4.2 (cf. [Ser59, III, Proposition 14]). *Let $\theta : G \to G'$ be a morphism of smooth connected commutative algebraic groups such that $\mathrm{Ker}(\theta)$ is finite. Let $(\overline{X}, Y)$ be a modulus pair such that Y is a Cartier divisor on $\overline{X}$. Put $X = \overline{X} - Y$ and let $a \in G(X)$. If $(\overline{X}, Y)$ is a modulus for $\theta(a) \in G'(X)$, then $(\overline{X}, Y)$ is a modulus for $a \in G(X)$.*

Proof. Since G satisfies global injectivity, it suffices to check the condition in Remark 4.1.2. We use the notation therein.

Let $g \in G(\overline{C}, D)$ be non-constant and $g : \overline{C} \to \mathbf{P}^1_K$ be the corresponding morphism. Let $U_g = \mathbf{P}^1_K - g(\overline{C} - C) \subset \mathbf{P}^1_K$ and $C_g = g^{-1}(U') \subset C$. Then C_g is finite over U_g and we let $[g] \in \mathbf{Cor}(U_g, C)$ denote the finite correspondence given by C_g, which induces $[g]_* : G(C) \to G(U_g)$. It is proved in [Ser59, III, Proposition 9] that, for any effective divisor D on $\overline{C}$ such that $|D| = \overline{C} - C$ and $b \in G(C)$, the following conditions are equivalent:

(1) $(\mathrm{div}_{\overline{C}}(g))^*(b) = 0 \in G(K)$ for any $g \in G(\overline{C}, D)$;

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(2) for any non-constant $g \in G(\overline{C}, D)$, $[g]_*(b) \in G(U_g)$, viewed as a morphism $U_g \to G$, is constant.

Thus, we are reduced to showing (2) for $D = \varphi^*(Y \times K)$ and $b = \varphi^*(a)$ with $\varphi : C \to X \times K$. Let us take a non-constant $g \in G(\overline{C}, D)$. Since $\theta(a) \in G'(X)$ has modulus $(\overline{X}, Y)$ by assumption, (1) holds for $\theta(b) \in G'(C)$ instead of $b \in G(C)$. Hence, the morphism $[g]_*(\theta(b)) : U_g \to G'$ is constant. Note that $[g]_*(\theta(b))$ factors as

$$U_g \xrightarrow{[g]_*(b)} G \xrightarrow{\theta} G'.$$

Since $\ker(\theta)$ is finite and U_g is connected, $[g]_*(b)$ must be constant too. □

Proposition 4.4.3. *If G is unipotent, it has reciprocity.*

Proof. We take integral quasi-affine $X \in \mathbf{Sm}$ and $a \in G(X)$. We also take a modulus pair $(\overline{X}, W)$ such that $\overline{X}$ is normal, that W is an effective Cartier divisor on $\overline{X}$ and that $X = \overline{X} - |W|$ (Remark 2.1.8). In view of Corollary 2.1.7, it suffices to find $n \in \mathbb{Z}_{>0}$ such that $(\overline{X}, nW)$ is a modulus for a.

We fix an embedding $G \to \mathrm{GL}_r$. Then our section $a \in G(X)$ can be represented as a matrix $(a_{ij})_{i,j=1}^r \in \mathrm{GL}_r(X)$, where $a_{ij} \in \mathcal{O}(X)$ are regular functions on X. Take such $n > 0$ that

$$a_{ij} \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(nW)) \quad \text{for all } i, j = 1, \dots, r.$$

We claim that $Y = nW$ is a modulus for a. For this, it suffices to verify the condition in Remark 4.1.2, but it follows from the following result from [Ser59]. □

Proposition 4.4.4 [Ser59, III, Proposition 15]. *Let G be a connected unipotent commutative algebraic subgroup of GL_r. Let $\overline{C}$ be a proper normal curve over K, C an open dense subscheme of $\overline{C}$ and $a : C \to G$ a morphism. Write $a_{ij} \in \mathcal{O}(C)$ for the (i,j)th-entry of $a \in G(C) \subset \mathrm{GL}_r(C)$. Let D be an effective divisor on $\overline{C}$ such that $|D| = \overline{C} - C$ and that*

$$a_{ij} \in \Gamma(\overline{C}, \mathcal{O}_{\overline{C}}(D)) \quad \text{for all } i, j = 1, \dots, r.$$

Then we have $(\mathrm{div}_{\overline{C}}(g))^*(a) = 0 \in G(K)$ for any $g \in G(\overline{C}, D)$, where a is viewed as a section over C of the object of $\mathbf{PST}$ represented by G.

Proof of Theorem 4.1.1 in positive characteristic. Lemma 4.4.1 and Proposition 4.4.2 reduce us to the cases of a semi-abelian variety and of a unipotent group. We have proved the theorem for unipotent groups in Proposition 4.4.3. A semi-abelian variety is homotopy invariant and hence has reciprocity by Theorem 3. □

Remark 4.4.5. Let G be a commutative algebraic group over an arbitrary field k. Let $\overline{C}$ be a smooth projective curve over k, D an effective divisor on $\overline{C}$, $C := \overline{C} - D$ and $a \in G(C)$. If D is a modulus for a in the sense of Rosenlicht–Serre (cf. Theorem 1), then D is a modulus for a in our sense. Indeed, by Remark 4.1.2 it suffices to show that, for any field $K \in \widetilde{\mathbf{Sm}}$, the image a_K of a in $G(C \times K)$ has modulus $D \times K$. But this follows from the proof in this section.

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5. Weak reciprocity

In this section, we introduce the notion of 'weak reciprocity'. It is weaker than reciprocity defined in Definition 2.1.3 but strong enough to imply the injectivity property stated in Theorem 6. An advantage of reciprocity in Definition 2.1.3 over weak reciprocity is that the analogue of Theorem 2 fails for weak reciprocity (Remark 5.1.8). Weak reciprocity can be formulated for pretheories, a more general notion than presheaves with transfers (see Definition 5.1.3). Theorem 6 follows from Theorem 7.1.1.

5.1 Definition of weak reciprocity

Definition 5.1.1. For $S \in \mathbf{Sm}$, $rel\mathcal{C}(S)$ is the class of the morphisms $p : X \to S$ in $\mathbf{Sm}$ which are quasi-affine and equidimensional of relative dimension one. We sometimes write X/S for $p : X \to S$.

5.1.2. For $S \in \mathbf{Sm}$ and $(p : X \to S) \in rel\mathcal{C}(S)$, we have a map

$$\psi_{X/S} : c(X/S) \to \mathbf{Cor}(S, X) = c(X \times S/S) \tag{5.1}$$

induced by $X \overset{\mathrm{id}_X \times p}{\longrightarrow} X \times S$. For $F \in \mathbf{PST}$, we define a pairing

$$\begin{aligned} \langle , \rangle_{X/S} &: c(X/S) \times F(X) \to F(S), \\ \langle Z, a \rangle_{X/S} &= \psi_{X/S}(Z)^*(a) \quad \text{for } Z \in c(X/S),\ a \in F(X). \end{aligned} \tag{5.2}$$

It satisfies the following conditions.

(i) If $i : S \to X$ is a section of $X \to S$, then $\langle i(S), a \rangle_{X/S} = i^*a$ for $a \in F(X)$.

(ii) If $f : S' \to S$ is a morphism, then $f^*\langle Z, a \rangle_{X/S} = \langle f^*Z, g^*a \rangle_{X'/S'}$ for $Z \in c(X/S)$ and any $a \in F(X)$, where $X' = S' \times_S X$ and $g : X' \to X$ is the second projection (see § 1.1.2 for f^*Z).

(iii) For $(p_i : X_i \to S) \in rel\mathcal{C}(S)$ with $i = 1, 2$ and for an S-morphism $f : X_1 \to X_2$, we have $\langle f_*(Z), \alpha \rangle_{X_2/S} = \langle Z, f^*(a) \rangle_{X_1/S}$ for $Z \in c(X/S)$ and $a \in F(X_2)$.

5.1.3. Following Voevodsky [Voe00a, Definition 3.1], a *pretheory over* k is defined as a presheaf $F : \mathbf{Sm}^{\mathrm{op}} \to \mathbf{Ab}$, commuting with coproducts and provided with bilinear pairings (5.2) for all $S \in \mathbf{Sm}$ and $X/S \in rel\mathcal{C}(S)$ subject to the conditions (i) and (ii). If it satisfies additionally (iii), F is called of homological type. Pretheories form an abelian category containing $\mathbf{PST}$ as a (non-full) subcategory via (5.1).

Definition 5.1.4. Let $S \in \mathbf{Sm}$ and $X/S \in rel\mathcal{C}(S)$. A *good compactification* of X/S is a dense open immersion $j : X \hookrightarrow \overline{X}$ of S-schemes such that:

(1) $\overline{X}$ is normal and $\overline{p} : \overline{X} \to S$ is proper and equidimensional of dimension one;

(2) $\overline{X} - X$ has an affine neighbourhood in $\overline{X}$.

A *good compactification* of X/S with modulus is a pair $(j : X \hookrightarrow \overline{X}, Y)$ of a good compactification of X/S and a closed subscheme $Y \subset \overline{X}$ with $X = \overline{X} - Y$. We sometimes write $(\overline{X}/S, Y)$ for $(j : X \hookrightarrow \overline{X}, Y)$ for simplicity. Note that this is not a modulus pair in the sense of Definition 2.1.1 unless S is proper [EGAII, 5.4.3(ii)].

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5.1.5. Let $(\overline{X}/S, Y)$ be as in Definition 5.1.4 with $X = \overline{X} - Y$. Let $\mathrm{Pic}(\overline{X}, Y)$ be the relative Picard group (see § 3.2). As in the proof of [SV96, Theorem 3.1], it follows from [SV96, Lemma 2.3] that we have a short exact sequence

$$0 \to G(\overline{X}, Y) \xrightarrow{\mathrm{div}_{\overline{X}}} c(X/S) \xrightarrow{\tau} \mathrm{Pic}(\overline{X}, Y) \to 0 \tag{5.3}$$

(compare (3.3)). The map τ can be described as follows. The components of $Z \in c(X/S)$ are closed and of codimension one in $\overline{X}$, so that Z is a Cartier divisor on $\overline{X}$ whose support is disjoint from Y and that there is a natural isomorphism $\sigma : \mathcal{O}_{\overline{X}}(Z)_{|Y} \simeq \mathcal{O}_Y$. Then τ sends Z to the class of the pair $(\mathcal{O}_{\overline{X}}(Z), \sigma)$.

Definition 5.1.6. Let F be a pretheory.

(1) Given $S \in \mathbf{Sm}$ and $X/S \in rel\mathcal{C}(S)$ and a good compactification $(\overline{X}/S, Y)$ of X/S with modulus and $a \in F(X)$, we say that Y is a *weak modulus* for $a \in F(X)$ (or a has *weak modulus* Y) if

$$\langle \mathrm{div}_{\overline{X}}(g), a \rangle_{X/S} = 0 \quad \text{for any } g \in G(\overline{X}, Y);$$

equivalently, the morphism

$$\langle -, a \rangle_{X/S} : c(X/S) \to F(S)$$

factors through τ in (5.3).

(2) We say that F *has weak reciprocity* if, for any affine $S \in \mathbf{Sm}$ and $X/S \in rel\mathcal{C}(S)$ with a good compactification $X \hookrightarrow \overline{X}$, any section $a \in F(X)$ has a weak modulus.

Lemma 5.1.7. *If $F \in \mathbf{PST}$ has reciprocity, it has weak reciprocity.*

Proof. Let $S \in \mathbf{Sm}$ be affine and take $C/S \in rel\mathcal{C}(S)$ with a good compactification $C \hookrightarrow \overline{C}$. (Note that C is quasi-affine because S is affine and $X/S \in rel\mathcal{C}(S)$.) Take $a \in F(C)$. We must show that a has a weak modulus. Take an integral proper k-scheme $\overline{X}$ which contains $\overline{C}$ as a dense open subset. By the assumption on F, a has a modulus $Y \subset \overline{X}$ in the sense of Definition 2.1.2 such that $|Y| = \overline{X} - C$. Let φ be the composite map

$$\overline{C} \xrightarrow{\gamma_{\bar{p}}} \overline{C} \times S \hookrightarrow \overline{X} \times S$$

with $\bar{p} : \overline{C} \to S$ the projection and $\gamma_{\bar{p}}$ its graph. Then $(\varphi : \overline{C} \to \overline{X} \times S) \in \mathcal{C}_{(\overline{X},Y)}(S)$ and $\overline{C} \times_{\overline{X} \times S} (Y \times S)$ is a weak modulus for a (cf. Definition 2.2.1). □

Remark 5.1.8. The analogue of Theorem 2.1.5(1) still holds for weak reciprocity, but the analogue of Theorem 2.1.5(2) does not hold. More precisely, for $(\overline{X}/S, Y)$ as in Definition 5.1.4 with $X = \overline{X} - Y$, the functor

$$\mathbf{PST} \to \mathbf{Ab}, \quad F \mapsto \{a \in F(X) \mid a \text{ has weak modulus } Y\}$$

is representable. But the representing object does not have weak reciprocity, even if $S = \mathrm{Spec}\, k$.

5.2 Local symbols

In this subsection, we show the equivalence between weak reciprocity for $F \in \mathbf{PST}$ in the sense of Definition 5.1.6 for curves over a function field and the existence of local symbols satisfying 'Weil reciprocity', in analogy with [Ser59, ch. III].

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Let $E = k(S)$ be the function field of a connected $S \in \mathbf{Sm}$. Let X be a normal projective irreducible curve over E with function field $K = E(X)$. Let $X_{(0)}$ be the set of closed points of X. For each $x \in X_{(0)}$, let $v_x : K^\times \to \mathbb{Z}$ be the normalized valuation at x.

We fix $F \in \mathbf{PST}$. Set, for $a \in F(\mathcal{O}_{X,x})$,

$$a(x) = \langle x, \tilde{a} \rangle_{U/E},$$

where U is an open neighbourhood of x such that a is the image of some $\tilde{a} \in F(U)$. Note that $a(x)$ is independent of choices of U and $\tilde{a}$. In fact, we have $a(x) = f_{x*} i_x^* a$, where $i_x : x \to \operatorname{Spec} \mathcal{O}_{X,x}$ is the closed immersion and f_x is the finite morphism $x \to \operatorname{Spec} E$. (By Proposition 6.1.5 and Lemma 6.3.3 below, this formula holds more generally for a pretheory which is $\mathbf{P}^1$-rigid in the sense of Definition 6.1.3.)

Proposition 5.2.1. *The following conditions are equivalent.*

(1) *For any dense open affine $U \subset X$ and $a \in F(U)$, there exists an effective divisor $Z \subset X$ such that $U = X \backslash |Z|$ and that*

$$\langle \operatorname{div}_X(f), a \rangle_{U/E} = 0 \in F(E) \quad \text{for all } f \in G(X, Z). \tag{♣}$$

(2) *There exists a family of bilinear pairings*

$$\{(-\,,-)_x : K^\times \times F(K) \to F(E)\}_{x \in X_{(0)}}$$

which satisfies the following conditions:

(a) *for any $x \in X_{(0)}$, $a \in F(\mathcal{O}_{X,x})$ and $g \in K^\times$, we have $(g, a|_K)_x = v_x(g) a(x)$;*

(b) *for any $x \in X_{(0)}$ and $a \in F(K)$, we have $(U_x^{(m)}, a)_x = 0$ for sufficiently large $m > 0$, where*

$$U_x^{(m)} = \{u \in K^\times \mid v_x(u - 1) \geqslant m\};$$

(c) *$\sum_{x \in X_{(0)}} (g, a)_x = 0$ for any $a \in F(K)$ and $g \in K^\times$.*

Moreover, if (2) holds, (♣) for $a \in F(U)$ is equivalent to the condition

$$(U_x^{(m_x)}, a)_x = 0 \quad \text{for all } x \in X - U, \tag{♡}$$

where m_x is the multiplicity of x in Z.

Proof. Assume (1). By (5.3), we have an isomorphism

$$\operatorname{Coker}(G(X, Z) \to c(U/E)) \simeq \operatorname{Pic}(X, Z).$$

Hence, passing to the limit over all $U \subset X$ and Z, the pairings $\langle -\,, - \rangle_{U/E}$ induce

$$\langle -\,, - \rangle_{K/E} : \varprojlim_Z \operatorname{Pic}(X, Z) \times F(K) \to F(E).$$

By weak approximation, we have an isomorphism

$$\operatorname{Pic}(X, Z) \simeq \operatorname{Coker}\left(K^\times \to \bigoplus_{x \in X_{(0)} - |Z|} \mathbb{Z} \oplus \bigoplus_{x \in |Z|} K^\times / U_x^{(m_x)} \right),$$

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where m_x is the multiplicity of x in Z. Whence a natural map

$$\pi_x : K^\times \to \varprojlim_Z \mathrm{Pic}(X, Z) \quad \text{for } x \in X_{(0)}$$

is induced by the projections $K^\times \to K^\times/U_x^{(m_x)}$. Define

$$(-,-)_x : K^\times \times F(K) \to F(E)$$

by $(f,a)_x = \langle \pi_x(f), a\rangle_{K/E}$ for $f \in K^\times$ and $a \in F(K)$. It is easy to verify (a)–(c). Hence, (1) $\Rightarrow$ (2).

Now suppose (2) and take $a \in F(U), U \subset X$ as in (1). For each $x \in X\backslash U$, there exists an integer $m_x \geqslant 1$ such that $(U_x^{(m_x)}, a)_x = 0$ by (b). Define $Z = \sum_{x\in X\backslash U} m_x x$. For $g \in G(\bar{X}, Z)$, we get

$$\langle \mathrm{div}_X(g), a\rangle_{U/E} \overset{(a)}{=} \sum_{u\in U_{(0)}} (g,a)_u \overset{(!)}{=} \sum_{x\in X_{(0)}} (g,a)_x \overset{(c)}{=} 0.$$

Here (!) holds since $g \in G(\bar{X}, Z)$ implies that $g \in U_x^{(m_x)}$ for $x \in |Z|$. This proves (2) $\Rightarrow$ (1) and the implication (♡) $\Rightarrow$ (♣).

It remains to show the implication (♣) $\Rightarrow$ (♡). Fix $x \in X - U$ and let m_x be the multiplicity of x in Z. Take $f \in U_x^{(m_x)}$. By (2)(b), for each $y \in X - U$ there exists $n_y \geqslant m_y$ such that $(U_y^{(n_y)}, a)_y = 0$. By the approximation theorem, we find $g \in K^\times$ such that $g/f \in U_x^{(n_x)}$ and $g \in U_y^{(n_y)}$ for all $y \in X - U$. This implies that $g \in G(X, Z)$ and hence (♣) and (2)(a) imply that

$$0 = \langle \mathrm{div}_X(g), a\rangle_{U/E} = \sum_{y\in U_{(0)}} v_y(g)a(y).$$

Then, using properties in (2), we compute

$$(f,a)_x = (g,a)_x \overset{(c)}{=} -\sum_{y\in X_{(0)}-\{x\}} (g,a)_y = -\sum_{y\in U_{(0)}} (g,a)_y \overset{(a)}{=} -\sum_{y\in U_{(0)}} v_y(g)a(y) = 0.$$

The proposition is proved. □

6. $\mathbf{P}^1$-invariance

In this section, we discuss $\mathbf{P}^1$-invariance and $\mathbf{P}^1$-rigidity for a pretheory with weak reciprocity, and draw a few consequences.

6.1 $\mathbf{P}^1$-invariance and $\mathbf{P}^1$-rigidity

Theorem 6.1.1. *Let F be a pretheory which is separated for the Zariski topology. If F has weak reciprocity, then it is $\mathbf{P}^1$-invariant, namely $\pi^* : F(S) \xrightarrow{\sim} F(\mathbf{P}^1_S)$ for any $S \in \mathbf{Sm}$, where $\pi : \mathbf{P}^1_S \to S$ is the projection.*

Remark 6.1.2. The property of $\mathbf{P}^1$-invariance is clearly stable under arbitrary products in $\mathbf{PST}$. On the other hand, one can easily see that the Zariski separated presheaf with transfers $\mathbf{G}_a^{\mathbb{N}}$ does not have (weak) reciprocity, by checking that the evaluation in the proof of Proposition 4.4.3 is optimal.

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Theorem 6.1.1 will be deduced from Proposition 6.1.5 below, for which we need to introduce the notion of $\mathbf{P}^1$-rigidity. For $t \in \mathbf{P}^1(k)$, let $i_t : S \to \mathbf{P}^1_S$ be the corresponding section of π at t.

DEFINITION 6.1.3. A presheaf $F : \mathbf{Sm}^{\mathrm{op}} \to \mathbf{Ab}$ of abelian groups is $\mathbf{P}^1$-rigid if $i_0^* = i_\infty^* : F(\mathbf{P}^1_S) \to F(S)$ for any $S \in \mathbf{Sm}$.

PROPOSITION 6.1.4. *For a presheaf F of abelian groups, $\mathbf{P}^1$-invariance implies $\mathbf{P}^1$-rigidity. The converse is true if F is separated for the Zariski topology.*

Proof. It is obvious that $\mathbf{P}^1$-invariance implies $\mathbf{P}^1$-rigidity. We prove the converse: it suffices to show that $\mathrm{Ker}(F(\mathbf{P}^1_S) \xrightarrow{i_0^*} F(S)) = 0$ for any $S \in \mathbf{Sm}$. We apply $\mathbf{P}^1$-rigidity to $X = \mathbf{A}^1_S$ to get

$$\tilde{i}_0^* = \tilde{i}_\infty^* : F(\mathbf{P}^1_X) \to F(X), \tag{6.1}$$

where (for clarity) $\tilde{i}_t$ is relative to the base X. Consider the morphisms

$$\varphi_1, \varphi_2 : \mathbf{P}^1_X = \mathbf{P}^1 \times \mathbf{A}^1_S \to \mathbf{P}^1_S$$

given by

$$\varphi_1((u_0 : u_1), t) = (u_0 : u_1 + tu_0), \quad \varphi_2((u_0 : u_1), t) = (u_1 + tu_0 : u_0).$$

Letting $p : \mathbf{A}^1_S \to S$ be the projection, we compute

$$\begin{array}{ll} \varphi_1 \circ \tilde{i}_0 = j_\infty : \mathbf{A}^1_S \to \mathbf{P}^1_S, & t \mapsto (1 : t), \\ \varphi_1 \circ \tilde{i}_1 = i_1 \circ p : \mathbf{A}^1_S \to \mathbf{P}^1_S, & t \mapsto 1, \\ \varphi_2 \circ \tilde{i}_0 = j_0 : \mathbf{A}^1_S \to \mathbf{P}^1_S, & t \mapsto (t : 1), \\ \varphi_2 \circ \tilde{i}_1 = i_0 \circ p : \mathbf{A}^1_S \to \mathbf{P}^1_S, & t \mapsto 0. \end{array} \tag{6.2}$$

Take $a \in \mathrm{Ker}(F(\mathbf{P}^1_S) \xrightarrow{i_0^*} F(S))$. Then $a \in \mathrm{Ker}(F(\mathbf{P}^1_S) \xrightarrow{i_\infty^*} F(S))$ by rigidity. By (6.1),

$$\tilde{i}_0^* \varphi_\nu^*(a) = \tilde{i}_\infty^* \varphi_\nu^*(a) \quad (\nu = 1, 2).$$

By (6.2), this implies that

$$j_\infty^* a = p^* i_\infty^* a = 0, \quad j_0^* a = p^* i_0^* a = 0.$$

Noting that F is separated for the Zariski topology, this implies that $a = 0$ since $\mathbf{P}^1_S = j_0(\mathbf{A}^1_S) \cup j_1(\mathbf{A}^1_S)$ is an open covering of $\mathbf{P}^1_S$. □

Theorem 6.1.1 now follows from the next proposition.

PROPOSITION 6.1.5. *A pretheory F having weak reciprocity is $\mathbf{P}^1$-rigid.*

Proof. Let $S \in \mathbf{Sm}$ and $a \in F(\mathbf{P}^1_S)$. Let t be the standard coordinate on $\mathbf{A}^1_k = \mathbf{P}^1_k - \{\infty\}$. Let $j_0 : \mathbf{A}^1_S \hookrightarrow \mathbf{P}^1_S$ be the natural inclusion and j_∞ be the composite of j_0 and the morphism $\mathbf{P}^1_S \to \mathbf{P}^1_S; t \to t^{-1}$. These define two good compactifications of $\mathbf{A}^1_S$. Since F has weak reciprocity, there exists $n_0 > 0$ such that for all $n \geqslant n_0$, we have

$$\langle \mathrm{div}_{\mathbf{A}^1_S}(1 + t^{-n}), j_0^* a \rangle_{\mathbf{A}^1_S/S} = \langle \mathrm{div}_{\mathbf{A}^1_S}(1 + t^n), j_\infty^* a \rangle_{\mathbf{A}^1_S/S} = 0 \in F(S).$$

Noting that

$$\frac{1 + t^n}{1 + t^{-n}} = t^n,$$

we deduce that

$$n(i_0^* a - i_\infty^* a) = 0.$$

Replacing n by $n + 1$, we get the desired rigidity. □

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6.2 Application to 0-cycles

Corollary 6.2.1. *Let $F \in \mathbf{PST}$ be such that the associated pretheory has weak reciprocity, and let X be a smooth proper k-variety. Then the natural pairing $\langle , \rangle_X : Z_0(X) \times F(X) \to F(k)$ factors through a pairing*

$$\mathrm{CH}_0(X) \otimes F(X) \to F(k).$$

Proof. Let $C \subset X$ be an irreducible curve and let $f \in k(C)^*$: we must show that the map $F(X) \to F(k)$ induced by pairing with $\mathrm{div}(f)$ is 0. Let $\tilde{C}$ be the normalization of C: we have a diagram

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \\ \mathbf{P}^1 & & \end{array}$$

Let $a \in F(X)$. We have

$$\langle \mathrm{div}(f), a\rangle_X = \langle \mathrm{div}(f), i^*a\rangle_{\tilde{C}} = \langle 0 - \infty, f_* i^* a\rangle_{\mathbf{P}^1} = 0,$$

where the last equality follows from Proposition 6.1.5. □

6.3 Functoriality for pretheories with weak reciprocity

We apply the above to show that weak reciprocity is sufficient to yield some important properties to a pretheory, which are automatic for presheaves with transfers. This generalizes [Voe00a, Propositions 3.12, 3.14 and 3.15] (the homotopy invariant case).

Proposition 6.3.1. *Let F be a $\mathbf{P}^1$-rigid pretheory. Let $S \in \mathbf{Sm}$ and $X/S \in rel\mathcal{C}(S)$ and $j : U \hookrightarrow X$ be an open embedding. Then, for any $(Z, a) \in c(U/S) \times F(X)$, one has*

$$\langle j_* Z, a\rangle_{X/S} = \langle Z, j^* a\rangle_{U/S}.$$

Proof. This is proven by adapting the proof of [Voe00a, Proposition 3.12]. Let $\Xi = X - U$ and $W = X \times \mathbf{P}^1 \backslash \Xi \times \{0\}$, so that we have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{q} & X \\ \downarrow & & \downarrow \\ S \times \mathbf{P}^1 & \xrightarrow{\pi} & S \end{array}$$

where q is induced by the projection $X \times \mathbf{P}^1 \to X$. For the sections i_0 (at 0) and i_1 (at 1) of π, we have $i_0^* W = U$ and $i_1^* W = X$. Let $\tilde{Z} \in c(W/S \times \mathbf{P}^1)$ be the unique element whose image in $c(X \times \mathbf{P}^1/S \times \mathbf{P}^1)$ equals $\pi^* Z = Z \times \mathbf{P}^1$. Then we have $i_0^* \tilde{Z} = Z$ and $i_1^* \tilde{Z} = j_* Z$. Put $\varphi = \langle \tilde{Z}, q^* a\rangle_{W/S \times \mathbf{P}^1} \in F(X \times \mathbf{P}^1)$. By condition (ii) in § 5.1.2, we get

$$i_0^* \varphi = \langle Z, j^* a\rangle_{U/S}, \quad i_1^* \varphi = \langle j_* Z, a\rangle_{X/S}.$$

By $\mathbf{P}^1$-rigidity, we have $i_0^* \varphi = i_1^* \varphi$ and hence the claim. □

Let E be a finite separable extension of k and $f : \mathrm{Spec}\, E \to \mathrm{Spec}\, k$ the projection. Let F be a pretheory. In [Voe00b, p. 101], Voevodsky defined a trace map $f_* : F(E) \to F(k)$ by

$$f_* a = \langle \{0\}_E, p^* a\rangle_{\mathbf{A}^1_E/k},$$

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where $p : \mathbf{A}^1_E \to \operatorname{Spec} E$ is the structural map. Taking $T \in \mathbf{Sm}$, we apply the above construction to the pretheory $S \to F(T \times S)$ and get a map $f_* : F(T_E) \to F(T)$ with $T_E = T \times \operatorname{Spec} E$. By construction, f_* satisfies the obvious functoriality with respect to $T \in \mathbf{Sm}$.

Proposition 6.3.2. *If F has weak reciprocity, one has the identity*

$$f_* f^* = \times [E_2 : E_1],$$

where $f^* : F(T) \to F(T_E)$ is the pullback by f.

Proof. We adapt that of [Voe00b, Corollary 3.15]. We give details, especially as one line of that proof has been omitted from the published edition. We may assume that $T = \operatorname{Spec} k$. Let $x_0,\ x_\infty \in \mathbf{P}^1$ be the points at zero and infinity. Since E/k is separable, the choice of a primitive element α yields a point $x \in \mathbf{P}^1$ with residue field E. Let P be the minimal polynomial of α and put $d := [E : k]$. For $n \in \mathbb{Z}_{>0}$, the function $t^{-dn} P^n$ belongs to $G(\mathbf{P}^1, ndx_\infty)$ and has divisor $nx - ndx_0$. Thus, by weak reciprocity, for any $a \in F(\mathbf{P}^1_k)$ there exists $n_0 \in \mathbb{Z}_{>0}$ such that $\langle nx - ndx_0, a\rangle_{\mathbf{P}^1/k} = 0$ for all $n \geqslant n_0$. Applying this to $n = n_0$ and $n_0 + 1$, we get

$$\langle x - dx_0, a\rangle_{\mathbf{P}^1/k} = 0.$$

Let $\pi : \mathbf{P}^1 \to \operatorname{Spec} k$ be the structure map and $i_y : y \to \mathbf{P}^1$ the natural embedding for $y = x, x_0$. Taking $a = \pi^* a_0$ for $a_0 \in F(k)$, we get from the lemma below that

$$0 = f_* i_x^* \pi^* a_0 - d i_{x_0}^* \pi^* a_0 = f_* f^* a_0 - d a_0,$$

as requested. □

Lemma 6.3.3. *Let C be a smooth curve over k and let x be a closed point of C with separable residue field. Then, for any $\mathbf{P}^1$-rigid pretheory F and any $a \in F(C)$, one has*

$$f_{x*} i_x^* a = \langle x, a\rangle_{C/k},$$

where $f_x : x \to \operatorname{Spec} k$ is the structural map and $i_x : x \hookrightarrow C$ is the closed immersion.

Proof. This is proven by adapting the argument of [Voe00a, Proposition 3.13] in the same manner as Proposition 6.3.1. The details are left to the reader. □

7. Injectivity

7.1 Statement of the results

The purpose of this section is to prove Theorem 6 of the introduction. It results from the following stronger theorem.

Theorem 7.1.1. *Theorem 6 is valid for any pretheory F which has weak reciprocity.*

Theorem 7.1.1 is in turn a direct consequence of the following, whose proof will be given in § 7.2.

Theorem 7.1.2. *Let F be a pretheory with weak reciprocity. Let $X \in \mathbf{Sm}$, $V \subset X$ be an open dense subset and $x_1, \dots, x_n \in X$ a finite collection of points. Then there exists an open neighbourhood U of $\{x_1, \dots, x_n\}$ such that we have a injection*

$$\operatorname{Ker}(F(X) \to F(V)) \subset \operatorname{Ker}(F(X) \to F(U)).$$

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This strengthens [MVW06, Theorem 11.3], where the same assertion is proven assuming that F is homotopy invariant. By the argument of [MVW06, Lemma 22.8] and [MVW06, Corollary 11.2], the above theorem implies the following.

Corollary 7.1.3. *Let F be as in Theorem 7.1.1. Let F_{Zar} be the Zariski sheafification of F as a presheaf.*

(1) *For an open dense immersion $U \hookrightarrow X$ in $\mathbf{Sm}$, $F_{\mathrm{Zar}}(X) \to F_{\mathrm{Zar}}(U)$ is injective.*
(2) *If $F(E) = 0$ for any field E, then $F_{\mathrm{Zar}} = 0$.* □

Remark 7.1.4. We prove Theorem 7.1.2 following Voevodsky's proof of [MVW06, Theorem 11.3]. Under the additional assumption that F is a Nisnevich sheaf, one could alternatively deduce Theorem 7.1.2 from $\mathbf{P}^1$-invariance (Theorem 6.1.1) by the method of [CHK97].

7.2 Proof of Theorem 7.1.2

We will use the notion of standard triples as in [Voe00a, Definition 4.1] and [MVW06, Definitions 11.5 and 11.11] and two results about them (Proposition 7.2.3 and Lemma 7.2.4).

Definition 7.2.1. Let $S \in \mathbf{Sm}$ be connected.

(1) A triple $(\bar{p} : \bar{X} \to S, X_\infty, Z)$ with $X := \bar{X} \backslash X_\infty$ is called a *standard triple* if the following conditions are satisfied:

(a) $Z, X_\infty \subset \overline{X}$ are closed reduced and $Z \cap X_\infty = \emptyset$;
(b) $X/S \in rel\mathcal{C}(S)$ with $X \to S$ smooth, and $\bar{p} : \overline{X} \to S$ is its good compactification;
(c) $Z \cup X_\infty$ has an affine open neighbourhood in $\overline{X}$.

(2) A standard triple $(\bar{p} : \bar{X} \to S, X_\infty, Z)$ is called *split* over an open subset $U \subset \bar{X} \backslash X_\infty$ if $L|_{U \times_S Z}$ is trivial, where L is the line bundle on $U \times_S \bar{X}$ corresponding to the graph of the diagonal map.

Remark 7.2.2. By [MVW06, Remark 11.6], Definition 7.2.1(1) implies the following:

(1) S is affine, and both Z and X_∞ are finite over S;
(2) $\overline{X}$ is a good compactification of both X and $X \backslash Z$.

Proposition 7.2.3 (Mark Walker, see [MVW06, Theorem 11.17]). *Assume that k is infinite. Let $W \in \mathbf{Sm}$ be connected and quasi-projective over k and $Y \subsetneq W$ a closed subset with points $y_1, \ldots, y_n \in Y$. Then there exist an affine open neighbourhood $X \subset W$ of $y_1, \ldots, y_n$ and a standard triple $(\bar{X} \to S, X_\infty, Z)$ such that $(X, X \cap Y) \simeq (\bar{X} \backslash X_\infty, Z)$.*

Lemma 7.2.4 [MVW06, Lemma 11.14]. *Let $T = (\bar{X} \to S, X_\infty, Z)$ be a standard triple, and $x_1, \ldots, x_n \in X := \bar{X} \backslash X_\infty$. Then there exists an open neighbourhood $U \subset X$ of $x_1, \ldots, x_n$ such that T is split over U.*

We need a lemma. Its second assertion will be used in § 8.

Lemma 7.2.5. *Let $(\overline{X}/S, X_\infty, Z)$ be a standard triple and $Y \subset \overline{X}$ a closed subscheme with $|Y| = X_\infty$. We have a commutative diagram*

$$\begin{array}{ccc} c(X - Z/S) & \xrightarrow{\tau} & \mathrm{Pic}(\overline{X}, Y \sqcup Z) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle f} \\ c(X/S) & \xrightarrow[\tau]{} & \mathrm{Pic}(\overline{X}, Y) \end{array}$$

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where the left vertical map is the pushforward along immersion $j : X - Z \hookrightarrow X$ and the right vertical map is defined by $f(\mathcal{L}, \sigma) = (\mathcal{L}, \sigma|_Y)$. Moreover, we have an exact sequence

$$\mathcal{O}^\times(Z) \to \mathrm{Pic}(\overline{X}, Y \sqcup Z) \xrightarrow{j_*} \mathrm{Pic}(\overline{X}).$$

Proof. The first part is proven in [Dég07, Lemma 3.1.5(2)]. To show the second part, we consider a commutative diagram with exact rows (see § 3.2)

$$\begin{array}{ccccccc}
\mathcal{O}^\times(\bar{X}) & \longrightarrow & \mathcal{O}^\times(Y \sqcup Z) & \longrightarrow & \mathrm{Pic}(\bar{X}, Y \sqcup Z) & \longrightarrow & \mathrm{Pic}(\bar{X}) \\
\| & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle j_*} & & \| \\
\mathcal{O}^\times(\bar{X}) & \longrightarrow & \mathcal{O}^\times(Y) & \longrightarrow & \mathrm{Pic}(\bar{X}, Y) & \longrightarrow & \mathrm{Pic}(\bar{X})
\end{array}$$

As the Chinese reminder theorem shows that $\ker(a) = \mathcal{O}^\times(Z)$, the lemma follows from the diagram. □

The following proposition is the key to the proof of Theorem 7.1.2.

Proposition 7.2.6. *Let F be as in Theorem 7.1.2. Let $T = (\bar{X} \xrightarrow{p} S, X_\infty, Z)$ be a standard triple. Let $U \subset X := \bar{X} \backslash X_\infty$ be an affine open subset such that T is split over U. Then, for any $a \in F(X)$, there is a homomorphism $\varphi_a : F(X \backslash Z) \to F(U)$ such that $\varphi_a(a|_{X \backslash Z}) = a|_U$.*

Proof (Compare [MVW06, Proposition 11.15]). For any S-scheme W, we write $W_U = W \times_S U$ and denote by pr_W the projection $W_U \to W$. We write $j : U \to X$ and $j' : X \backslash Z \to X$ for the inclusion maps. Put $j'_U = j' \times_S 1_U : (X \backslash Z)_U \to X_U$. We then obtain a commutative diagram

$$\begin{array}{ccccc}
F(X \backslash Z) & \xleftarrow{j'^*} & F(X) & \xrightarrow{j^*} & F(U) \\
\downarrow{\scriptstyle pr^*_{X-Z}} & & \downarrow{\scriptstyle pr^*_X} & & \| \\
F((X \backslash Z)_U) & \xleftarrow{j'^*_U} & F(X_U) & \xrightarrow{\gamma_j^*} & F(U)
\end{array}$$

where $\gamma_j : U \to X_U$ is the graph of j.

Let $a \in F(X)$. Since $X_U/U \in rel\mathcal{C}(U)$ admits the good compactification $X_U \hookrightarrow \overline{X}_U$, there exists a weak modulus $Y \subset \overline{X}_U$ for $pr^*_X(a) \in F(X_U)$ such that $|Y| = (X_\infty)_U$. We shall construct a homomorphism

$$\tilde{\varphi}_Y : F((X \backslash Z)_U) \to F(U)$$

such that $\gamma_j^*(\tilde{a}) = \tilde{\varphi}_Y(j'^*_U(\tilde{a}))$ for any $\tilde{a} \in F(X_U)$ having weak modulus Y. The proposition will then follow by setting $\varphi_a := \tilde{\varphi}_Y \circ pr^*_{X-Z}$.

Consider the standard triple $T_U = (\bar{X}_U \to U, (X_\infty)_U, Z_U)$. Let $\gamma_j(U) \in c(X_U/U)$ be the image of $U \in c(U/U)$ under $\gamma_{j*} : c(U/U) \to c(X_U/U)$ and $(L, \sigma) \in \mathrm{Pic}(\bar{X}_U, Y)$ be the image of $\gamma_j(U)$ under τ in (5.3), where $L \in \mathrm{Pic}(\bar{X}_U)$ and $\sigma : L|_Y \simeq \mathcal{O}_Y$ is a trivialization. To say that T is split over U means that there exists a trivialization $\tau : L|_{Z_U} \simeq \mathcal{O}_{Z_U}$. Since Z_U is disjoint from Y, σ and τ define a trivialization $\sigma \oplus \tau : L|_{Y \sqcup Z_U} \simeq \mathcal{O}_{Y \sqcup Z_U}$.

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By construction, the image of $(L, \sigma \oplus \tau)$ via the canonical map

$$\mathrm{Pic}(\bar{X}_U, Y \sqcup Z_U) \to \mathrm{Pic}(\bar{X}_U, Y)$$

(see Lemma 7.2.5) agrees with (L, σ). Choose a lift $\tilde{\delta}$ of $(L, \sigma \oplus \tau)$ in $c((X\backslash Z)_U/U)$ via τ in (5.3). Now we define

$$\tilde{\varphi}_Y := \langle \tilde{\delta}, - \rangle_{(X\backslash Z)_U/U} : F((X\backslash Z)_U) \to F(U).$$

If $\tilde{a} \in F(X_U)$, by Proposition 6.3.1, we have

$$\tilde{\varphi}_Y({j'_U}^* \tilde{a}) = \langle \tilde{\delta}, {j'_U}^* \tilde{a} \rangle_{(X\backslash Z)_U/U} = \langle {j'_U}_* \tilde{\delta}, \tilde{a} \rangle_{X_U/U}.$$

If, moreover, $\tilde{a}$ has weak modulus Y, then $\langle {j'_U}_* \tilde{\delta}, \tilde{a} \rangle_{X_U/U}$ only depends on the image of ${j'_U}_* \tilde{\delta}$ in $\mathrm{Pic}(\bar{X}_U, Y)$ and hence

$$\langle {j'_U}_* \tilde{\delta}, \tilde{a} \rangle_{X_U/U} = \langle \gamma_j(U), \tilde{a} \rangle_{X_U/U} = \gamma_j^* \tilde{a}$$

by Definition 5.1.2(i). This completes the proof of the proposition. □

Remark 7.2.7 (Comparison with [MVW06, Theorem 11.3]). If F is homotopy invariant, then Y_{red} will be a weak modulus for all $\tilde{a} \in F(X_U)$. Consequently, one can choose φ_a independently of a. Hence, there is a homomorphism $\varphi : F(X - Z) \to F(U)$ such that $\varphi(a|_{X-Z}) = a|_U$ for any $a \in F(X)$.

We are now ready to prove Theorem 7.1.2.

Proof of Theorem 7.1.2. First we assume that k is infinite. In this case, Theorem 7.1.2 results from the following claim.

Claim 7.2.8. *Assume that k is infinite, and keep the notation and assumptions of Theorem 7.1.2. Then there exists an open neighbourhood U of $x_1, \ldots, x_n$ satisfying the following property: for any $a \in F(X)$, there is a homomorphism $\varphi_a : F(V) \to F(U)$ such that $\varphi_a(a|_V) = a|_U$.*

Proof. By replacing V by $V\backslash\{x_1, \ldots, x_n\}$, we may assume that $x_1, \ldots, x_n \in X\backslash V$. By Proposition 7.2.3, we can replace X by an open neighbourhood of $x_1, \ldots, x_n$ in such a way that there exists a standard triple $T = (\bar{X} \to S, X_\infty, Z)$ with $(X, X\backslash V) \simeq (\bar{X}\backslash X_\infty, Z)$. By Lemma 7.2.4, T splits over some open neighbourhood $U \subset X$ of $x_1, \ldots, x_n$. Now apply Proposition 7.2.6. □

Next we treat the case where k is finite. Let E be a finite separable extension of k and $f : \mathrm{Spec}\, E \to \mathrm{Spec}\, k$ the projection. For $T \in \mathbf{Sm}$, the transfer structure on F induces a map $f_* : F(T_E) \to F(T)$ with $T_E = T \times \mathrm{Spec}\, E$. It satisfies the standard functoriality with respect to $T \in \mathbf{Sm}$ together with the identity $f_* f^* = \times[E : k]$, where $f^* : F(T) \to F(T_E)$ is the pullback by f. Now a standard norm argument using Proposition 6.3.2 reduces us to the case where k is infinite. □

8. MV-effaceablity

In this section, we define (Definition 8.1.5) a condition for a pretheory to be *MV-effaceable*. We then prove that a pretheory having weak reciprocity is MV-effaceable (Theorem 8.1.6). This is a key technical result for the proof (to be given in the next section) of Theorem 7 in the introduction.

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8.1 MV-effaceable pretheories

Definition 8.1.1. An *upper distinguished square* is a Cartesian diagram

$$\begin{array}{ccc} B & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{i} & X \end{array} \tag{8.1}$$

of objects in **Sm** such that (i) i is an open immersion, (ii) f is étale and (iii) f induces an isomorphism $Y\backslash B \simeq X\backslash A$.

We denote the square (8.1) by $Q(X, Y, A)$. Note that this induces for a pretheory F a complex

$$0 \to F(X) \xrightarrow{\binom{i^*}{f^*}} F(A) \oplus F(Y) \xrightarrow{(-f^*, i^*)} F(B) \to 0.$$

Definition 8.1.2. Let $T_X = (\bar{X} \to S, X_\infty, Z_X)$ and $T_Y = (\bar{Y} \to S, Y_\infty, Z_Y)$ be standard triples, with $X = \bar{X} - X_\infty$ and $Y = \bar{Y} - Y_\infty$. A *covering morphism* $f : T_Y \to T_X$ is a finite morphism $f : \bar{Y} \to \bar{X}$ such that: (i) $f^{-1}(X_\infty) \subset Y_\infty$; (ii) $f|_Y : Y \to X$ is étale; (iii) $Z_Y = f^{-1}(Z_X) \cap Y$ and (iv) f induces an isomorphism $Z_Y \simeq Z_X$.

Remark 8.1.3. If $f : T_Y \to T_X$ is a covering morphism, then the square $Q = Q(X, Y, X\backslash Z_X)$ is upper distinguished with $Y - Z_Y = (X - Z_X) \times_X Y$.

The following lemma is proved in [MVW06, Lemma 21.3].

Lemma 8.1.4. *Let $f : T_Y \to T_X$ be a covering morphism. If T_X is split over $U \subset X := \bar{X}\backslash X_\infty$, then T_Y is split over $f^{-1}(U) \cap Y$.*

Definition 8.1.5. A pretheory F is said to be *MV-effaceable* if the following condition is satisfied: let $T_X = (\bar{X} \to S, X_\infty, Z_X)$ and $T_Y = (\bar{Y} \to S, Y_\infty, Z_Y)$ be standard triples, and $f : T_Y \to T_X$ a covering morphism so that $Q(X, Y, X\backslash Z_X)$ is upper distinguished. Put $A = X\backslash Z_X$, $B = Y\backslash Z_Y$. Let $Q' = Q'(X', Y', A')$ be another upper distinguished square such that X' and Y' are affine. Put $B' = A' \times_{X'} Y'$. Let

$$j = \begin{pmatrix} j_B & j_Y \\ j_A & j_X \end{pmatrix} : Q' = \begin{pmatrix} B' & \xrightarrow{i'} & Y' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \xrightarrow{i'} & X' \end{pmatrix} \to Q = \begin{pmatrix} B & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{i} & X \end{pmatrix}$$

be a morphism of squares. We then get a morphism of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(X) & \xrightarrow{\binom{i^*}{f^*}} & F(A) \oplus F(Y) & \xrightarrow{(-f^*, i^*)} & F(B) & \longrightarrow & 0 \\ & & {\scriptstyle j_X^*}\downarrow & & {\scriptstyle \begin{pmatrix} j_A^* & 0 \\ 0 & j_Y^* \end{pmatrix}}\downarrow & & {\scriptstyle j_B^*}\downarrow & & \\ 0 & \longrightarrow & F(X') & \xrightarrow{\binom{i'^*}{f'^*}} & F(A') \oplus F(Y') & \xrightarrow{(-f'^*, i'^*)} & F(B') & \longrightarrow & 0 \end{array} \tag{8.2}$$

The condition for F to be MV-effaceable is that (8.2) induces the zero map on all cohomology groups if $j_X : X' \to X$ is an open immersion, and the triple T_X is split over X'.

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In [MVW06, Theorem 21.6], a homotopy invariant presheaf with transfers is shown to be MV-effaceable. The following generalizes this result.

Theorem 8.1.6. *A pretheory having weak reciprocity is MV-effaceable.*

The proof of this theorem will be completed in § 8.3. Before that, we need to prepare a few lemmas in § 8.2.

8.2 Functoriality of the relative Picard group

Lemma 8.2.1. *Let $S \in \mathbf{Sm}$ and $X/S \in rel\mathcal{C}(S)$. Let $(\overline{X}/S, Y)$ be its good compactification with modulus. For any morphism $f : S' \to S$ in $\mathbf{Sm}$, we have a commutative diagram*

$$\begin{array}{ccc} c(X/S) & \xrightarrow{\tau} & \mathrm{Pic}(\overline{X}, Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ c(X \times_S S'/S') & \xrightarrow{\tau} & \mathrm{Pic}(\overline{X} \times_S S', Y \times_S S') \end{array}$$

where the right vertical map is defined by $f^*(\mathcal{L}, \sigma) = (f^*\mathcal{L}, f^*\sigma)$.

Proof. See [Dég07, Lemma 3.1.5(1)]. □

Lemma 8.2.2. *Let $S \in \mathbf{Sm}$. We take X/S, $Y/S \in rel\mathcal{C}(S)$ and let $(\overline{X}/S, V), (\overline{Y}/S, W)$ be their good compactifications with moduli. Let $f : \overline{X} \to \overline{Y}$ be a finite surjective S-morphism. Suppose that $f^*W \leqslant V$. Then there is a unique homomorphism*

$$f_* : \mathrm{Pic}(\overline{X}, V) \to \mathrm{Pic}(\overline{Y}, W),$$

which fits into a commutative diagram

$$\begin{array}{ccc} c(X/S) & \xrightarrow{\tau} & \mathrm{Pic}(\overline{X}, V) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ c(Y/S) & \xrightarrow{\tau} & \mathrm{Pic}(\overline{Y}, W) \end{array}$$

Proof. We may suppose that X and Y are irreducible. Uniqueness is obvious from the surjectivity of τ. Thus, it suffices to show that $N_{k(X)/k(Y)}(G(\overline{X}, V)) \subset G(\overline{Y}, W)$, which follows from Lemma 8.2.3 below. □

Lemma 8.2.3. *Let $R \subset R'$ be an extension of domains, where R is normal and R' is finite over R, and let $I \subset R$ be an ideal. Let K and K' be the fraction fields of R and R', respectively. Write $(1+I)^* = (1+I) \cap R^*$ and $(1+IR')^* = (1+IR) \cap R'^*$. Then $N_{K'/K}(1+IR')^* \subset (1+I)^*$.*

Proof. The hypotheses imply that $N_{K'/K}(R'^*) \subseteq R^*$ and $\mathrm{Tr}_{K'/K}(R') \subseteq R$ and hence clearly $\mathrm{Tr}_{K'/K}(IR') \subseteq I$. So, it suffices to show that $N_{K'/K}(1+a) \equiv 1 + \mathrm{Tr}_{K'/K}(a) \bmod I^2$ for any $a \in IR'$. We have

$$N_{K'/K}(1+a) = P_a(1),$$

where $P_a(T) = T^n - \sigma_1(a)T^{n-1} + \cdots \pm \sigma_n(a)$ is the characteristic polynomial of a, with $n = [K' : K]$. But $\sigma_i : K' \to K$ is given by a homogeneous polynomial of degree i with coefficients in K, and $\sigma_i(R') \subseteq R$ by integrality. In particular, if $a = \sum \lambda_\alpha \mu_\alpha$ with $\lambda_\alpha \in I$, $\mu_\alpha \in R'$, then $\sigma_i(a)$ is a homogeneous polynomial of degree i in the λ_α, with coefficients in R. □

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8.3 Proof of Theorem 8.1.6

We follow the method of [MVW06, Theorem 21.6]. We give ourselves all the data in Definition 8.1.5, and prove that (8.2) induces the zero map on all cohomology groups. We take a closed subscheme $V = V_\infty \sqcup V_0$ on $\bar{X}$ (respectively $W = W_\infty \sqcup W_0$ on $\bar{Y}$) such that $|V_\infty| = X_\infty$, $|V_0| = Z_X$ (respectively $|W_\infty| = Y_\infty, |W_0| = Z_Y$). We suppose that $f^*V \leqslant W$ and f induces $W_0 \simeq V_0$.

In the sequel, we use the notation

$$(-)_{X'} = - \times_S X' \quad \text{etc.}$$

throughout. The main part of the proof is in the following proposition.

Proposition 8.3.1. *There exist $\lambda_A \in c(A_{X'}/X'), \lambda_B \in c(B_{Y'}/Y')$ and $\psi \in c(B_{A'}/A')$ satisfying the following conditions:*

$$i_*\lambda_A - \tilde{j}_X \in \mathrm{Ker}(c(X_{X'}/X') \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Pic}(\bar{X}_{X'}, V_{\infty X'})), \tag{1}$$

$$i_*\lambda_B - \tilde{j}_Y \in \mathrm{Ker}(c(Y_{Y'}/Y') \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Pic}(\bar{Y}_{Y'}, W_{\infty Y'})), \tag{2}$$

$$f'^*\lambda_A - f_*\lambda_B \in \mathrm{Ker}(c(A_{Y'}/Y') \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Pic}(\bar{X}_{Y'}, V_{Y'})), \tag{3}$$

$$f_*\psi - i'^*\lambda_A + \tilde{j}_A \in \mathrm{Ker}(c(A_{A'}/A') \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Pic}(\bar{X}_{A'}, V_{A'})), \tag{4}$$

$$f'^*\psi - i'^*\lambda_B + \tilde{j}_B \in \mathrm{Ker}(c(B_{B'}/B') \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Pic}(\bar{Y}_{B'}, W_{B'})), \tag{5}$$

$$i_*\psi \in \mathrm{Ker}(c(Y_{A'}/A') \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Pic}(\bar{Y}_{A'}, W_{\infty A'})). \tag{6}$$

Here $\tilde{j}_X : X' \to X_{X'}$ is the graph of j_X, and similarly for $\tilde{j}_Y, \tilde{j}_A, \tilde{j}_B$.

Given $\lambda_A \in c(A_{X'}/X'), \lambda_B \in c(B_{Y'}/Y')$ and $\psi \in c(B_{A'}/A')$, we define maps

$$s_1 : F(A) \oplus F(Y) \to F(X'), \quad s_2 : F(B) \to F(A') \oplus F(Y') \tag{8.3}$$

by $s_1(a,b) = \langle \lambda_A, a_{X'} \rangle_{A_{X'}/X'}, s_2(a) = (\langle \psi, a_{A'} \rangle_{B_{A'}/A'}, \langle \lambda_B, a_{Y'} \rangle_{B_{Y'}/Y'})$.

Remark 8.3.2. When F is homotopy invariant, Theorem 8.1.6 is proved in [MVW06, Theorem 21.6] as follows: let us take V and W to be reduced. By applying Proposition 8.3.1, one obtains λ_A, λ_B and ψ. Then the maps s_1, s_2 given by (8.3) define a chain homotopy from (8.2) to zero. (By homotopy invariance, the map $\lambda_A^* : F(A) \to F(X')$ depends only on the class of $\mathrm{Pic}(\bar{X}_{X'}, V_{X'})$, for instance. Hence, the conditions in Proposition 8.3.1 contain sufficient information to prove this assertion.)

When F is not homotopy invariant (but has reciprocity), we need to choose V and W depending on elements of $F(X), F(A), F(Y)$ and $F(B)$. Thus, we cannot construct a globally defined chain homotopy. However, the following corollary can be deduced.

Corollary 8.3.3. *Let s_1, s_2 be as in (8.3).*

(a) *Let $a \in F(X)$. If the image of a in $F(X_{X'})$ has modulus $V_{\infty X'}$, then we have*

$$j_X^* a = s_1 \begin{pmatrix} i^* \\ f^* \end{pmatrix} a.$$

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(b) *Let $(a,b)\in F(A)\times F(Y)$. Suppose that the image of a in $F(A_{X'})$ (respectively in $F(A_{Y'})$) has modulus $V_{A'}$ (respectively $V_{Y'}$), and suppose that the image of b in $F(Y_{A'})$ (respectively in $F(Y_{Y'})$) has modulus $W_{\infty A'}$ (respectively $W_{\infty Y'}$). Then we have*

$$(j_A^*a, j_Y^*b) = \binom{i'^*}{f'^*} s_1(a,b) + s_2(-f^*, i^*)(a,b).$$

(c) *Let $a\in F(B)$. If the image a in $F(B_{B'})$ has modulus $W_{B'}$, then we have*

$$j_B^*a = (-f'^*, i'^*)s_2a.$$

Proof. Proposition 8.3.1(1) implies (a). Parts (2), (3), (4) and (6) imply (b). Part (5) implies (c). (We used all the axioms of a pretheory, plus Proposition 6.3.1.) □

Theorem 8.1.6 follows from this corollary, because, for instance, for any $a\in F(X)$ by weak reciprocity one can always find a closed subscheme V_∞ on $\overline{X}/S$ with $|V_\infty| = X_\infty$ such that $V_{\infty X'}$ is a modulus for a.

It remains to prove Proposition 8.3.1. We divide it into three steps.

8.3.1 *Construction of λ_A and λ_B.* We consider the commutative diagram

$$\begin{array}{ccc}
c(B_{Y'}/Y') & \xrightarrow{i_*} & c(Y_{Y'}/Y') \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
c(A_{Y'}/Y') & \xrightarrow{i_*} & c(X_{Y'}/Y') \\
{\scriptstyle f'^*}\uparrow & & \uparrow{\scriptstyle f'^*} \\
c(A_{X'}/X') & \xrightarrow{i_*} & c(X_{X'}/X')
\end{array}$$

Observe that $\tilde{\jmath}_X\in c(X_{X'}/X')$ and $\tilde{\jmath}_Y\in c(Y_{Y'}/Y')$ have the same image in $c(X_{Y'},Y')$. By passing to the quotient, we obtain the right half of the following commutative diagram (see Lemmas 7.2.5, 8.2.1 and 8.2.2):

$$\begin{array}{ccccc}
\mathcal{O}^*(W_{0Y'}) & \longrightarrow & \mathrm{Pic}(\bar{Y}_{Y'}, W_{Y'}) & \longrightarrow & \mathrm{Pic}(\bar{Y}_{Y'}, W_{\infty Y'}) \\
{\scriptstyle (*)}\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}^*(V_{0Y'}) & \longrightarrow & \mathrm{Pic}(\bar{X}_{Y'}, V_{Y'}) & \longrightarrow & \mathrm{Pic}(\bar{X}_{Y'}, V_{\infty Y'}) \\
 & & \uparrow & & \uparrow \\
 & & \mathrm{Pic}(\bar{X}_{X'}, V_{X'}) & \longrightarrow & \mathrm{Pic}(\bar{X}_{X'}, V_{\infty X'})
\end{array}$$

The upper two rows are exact by Lemma 7.2.5. Note also that $(*)$ is surjective since $f|_{Z_Y}: Z_Y = |W_0| \xrightarrow{\sim} Z_X = |V_0|$ and $f^*V_0\simeq W_0$.

Since T_X is split over X', as in the proof of Proposition 7.2.6 we get $\lambda_A\in c(A_{X'}/X')$ such that $i_*\lambda_A$ and $\tilde{\jmath}_X$ agree in $\mathrm{Pic}(\bar{X}_{X'}, V_{\infty X'})$. Similarly, in view of Lemma 8.1.4, there exists $\lambda_B\in c(B_{Y'}/Y')$ such that $i_*\lambda_B$ and $\tilde{\jmath}_Y$ agree in $\mathrm{Pic}(\bar{Y}_{Y'}, W_{\infty Y'})$. Moreover, using the surjectivity of $(*)$, λ_B can be chosen so that its image in $\mathrm{Pic}(\bar{X}_{Y'}, V_{Y'})$ agrees with that of λ_A. We have proven (1)–(3).

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8.3.2 *Preliminary computation.* Before we construct ψ, we do some computations on λ_A and λ_B. Let $\mathcal{L}$ be the invertible sheaf on $\bar{X}_{X'}$ corresponding to (the graph of) $j_X : X' \hookrightarrow X$. The image of λ_A in $\mathrm{Pic}(\bar{X}_{X'}, V_{X'})$ defines a trivialization $\tau = \tau_\infty \sqcup \tau_0$, where $\tau_\infty : \mathcal{L}|_{V_{\infty X'}} \simeq \mathcal{O}_{V_{\infty X'}}$, $\tau_0 : \mathcal{L}|_{V_{0X'}} \simeq \mathcal{O}_{V_{0X'}}$. Since $\mathcal{L}$ is defined by an effective divisor, it has a canonical global section $\sigma \in \mathcal{L}(\bar{X}_{X'})$ (given by the image of $1 \in \mathcal{O}(\bar{X}_{X'})$). The image of $\tilde{j}_X$ in $\mathrm{Pic}(\bar{X}_{X'}, V_{\infty X'})$ is given by the class of $(\mathcal{L}, \sigma|_{V_{\infty X'}})$, which equals the image of $i_*\lambda_A$. This proves that $\tau_\infty = \sigma|_{V_{\infty X'}}$. As τ_0 is a trivialization, there exists $r \in \mathcal{O}(V_{0X'})$ such that $\sigma|_{V_{0X'}} = r\tau_0$. By pulling back along $i' : A' \hookrightarrow X'$, we find that

$$\text{the class of } -\lambda_{A'} + j_A \text{ is represented by } (\mathcal{O}_{\bar{X}_{A'}}, 1 \sqcup (r|_{V_{0A'}})) \tag{8.4}$$

in $\mathrm{Pic}(\bar{X}_{A'}, V_{A'})$. (Note that $\sigma|_{V_{0A'}}$ defines a trivialization $\mathcal{L}|_{V_{0A'}} \simeq \mathcal{O}_{V_{0A'}}$ because $A' \subset X' \backslash Z_X$. Thus, $r|_{V_{0A'}} \in \mathcal{O}(V_{0A'})$ is invertible.)

Similarly, let $\mathcal{L}'$ be the invertible sheaf on $\bar{Y}_{Y'}$ corresponding to $j_Y : Y' \to Y$. The image of $i_*\lambda_B$ in $\mathrm{Pic}(\bar{Y}_{Y'}, W_{Y'})$ defines a trivialization $\tau' = \tau'_\infty \sqcup \tau'_0$, where $\tau'_\infty : \mathcal{L}'|_{W_{\infty Y'}} \simeq \mathcal{O}_{W_{\infty Y'}}$, $\tau'_0 : \mathcal{L}'|_{W_{0Y'}} \simeq \mathcal{O}_{W_{0Y'}}$. We also have a canonical global section $\sigma' \in \mathcal{L}'(\bar{Y}_{Y'})$. We have $\tau'_\infty = \sigma'|_{W_{\infty Y'}}$. As τ'_0 is a trivialization, there exists $r' \in \mathcal{O}(W_{0Y'})$ such that $\sigma'|_{W_{0Y'}} = r'\tau'_0$. We find that

$$\text{the class of } -\lambda_{B'} + j_B \text{ is represented by } (\mathcal{O}_{\bar{Y}_{B'}}, 1 \sqcup (r'|_{W_{0B'}})) \tag{8.5}$$

in $\mathrm{Pic}(\bar{Y}_{B'}, W_{B'})$. (Note that $\sigma'|_{W_{0B'}}$ defines a trivialization $\mathcal{L}'|_{W_{0N'}} \simeq \mathcal{O}_{W_{0B'}}$ because $B' \subset Y' \backslash Z_Y$. Thus, we have $r'|_{W_{0B'}} \in \mathcal{O}^*(W_{0B'})$.)

8.3.3 *Construction of ψ (compare [MVW06, 21.9]).* We consider the commutative diagram

$$\begin{array}{ccccccc}
W_{0Y'} & \xrightarrow{\subset} & \bar{Y}_{Y'} & \xrightarrow{f'} & \bar{Y}_{X'} & \xleftarrow{\supset} & W_{0X'} \\
\wr\downarrow & & f\downarrow & & \downarrow f & & \downarrow\wr \\
V_{0Y'} & \xrightarrow{\subset} & \bar{X}_{Y'} & \xrightarrow{f'} & \bar{X}_{X'} & \xleftarrow{\supset} & V_{0X'}
\end{array} \tag{8.6}$$

Let $\tilde{r} \in \mathcal{O}(W_{0X'})$ be the pullback of $r \in \mathcal{O}(V_{0X'})$ along f. By the definition of standard triple, there is an affine open neighbourhood $U \subset \bar{Y}$ of $Y_\infty \sqcup Z_Y$. Thus, $U_{X'}$ is an affine open neighbourhood of $W_{X'}$. By the Chinese reminder theorem, we can find $h \in \mathcal{O}(U_{X'})$ which is mapped to 1 in $\mathcal{O}(W_{\infty X'})$ and to $\tilde{r}$ in $\mathcal{O}(W_{0X'})$. (Thus, h is a rational function on $\bar{Y}_{X'}$.) Now *we define ψ to be* $-\mathrm{div}(h)$ *considered as an element of* $c(B_{A'}/A')$. Note that the support of ψ is contained in $(\bar{Y}\backslash U)_{A'} \subset B_{A'}$, since $r|_{V_{0A'}} \in \mathcal{O}(V_{0A'})^*$.

By definition, the image of ψ in $\mathrm{Pic}(\bar{Y}_{A'}, W_{A'})$ is represented by $\alpha := (\mathcal{O}_{\bar{Y}_{A'}}, 1 \sqcup \tilde{r}^{-1})$. We consider the images of α by three maps. First,

$$\mathrm{Pic}(\bar{Y}_{A'}, W_{A'}) \to \mathrm{Pic}(\bar{Y}_{A'}, W_{\infty A'})$$

(see Lemma 7.2.5) sends α to the class of $(\mathcal{O}_{\bar{Y}_{A'}}, 1)$. This proves (6). Second,

$$f_* : \mathrm{Pic}(\bar{Y}_{A'}, W_{A'}) \to \mathrm{Pic}(\bar{X}_{A'}, V_{A'})$$

(see Lemma 8.2.2) sends α to the class of $(\mathcal{O}_{\bar{X}_{A'}}, 1 \sqcup N(h)|^{-1}_{V_{0A'}})$. On the other hand, it is proved in [MVW06, 21.10] that $N(h)|_{V_{0A'}} = r|_{V_{0A'}}$ in $\mathcal{O}(V_{0A'})$. (In [MVW06, 21.10], V is assumed to be reduced, but this assumption is not used in the proof) Part (4) follows from this and (8.4). Finally,

$$f'^* : \mathrm{Pic}(\bar{Y}_{A'}, W_{A'}) \to \mathrm{Pic}(\bar{Y}_{B'}, W_{B'})$$

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(see Lemma 8.2.1) sends α to the class of $(\mathcal{O}_{\bar{Y}_{B'}}, 1 \sqcup (f'^{*}\tilde{r}^{-1}))$. In view of (8.5), (5) is reduced to $f'^{*}\tilde{r}|_{W_{0B'}} = r'|_{W_{0B'}}$ in $\mathcal{O}(W_{0B'})$. By the diagram (8.6) and the definition of $\tilde{r}$, we have $f'^{*}\tilde{r}|_{W_{0B'}} = f^{*}f'^{*}r|_{W_{0B'}}$. Since f induces an isomorphism $W_0 \simeq V_0$, it suffices to show that $f'^{*}r|_{V_{0B'}} = f_{*}r'|_{V_{0B'}}$, but this follows from (3). This completes the proof of Proposition 8.3.1.

9. Sheafification preserves reciprocity

We prove Theorem 7 in the introduction. Parts (1), (2) and (3) are respectively shown in §§ 9.1, 9.2 and 9.3. In the last subsection § 9.4, we make a brief discussion of the generalized Jacobian of a curve.

9.1 Zariski sheafification preserves transfers

In [MVW06, Theorem 22.15], Zariski sheafification of a homotopy invariant presheaf with transfers is shown to have transfers. The proof actually shows the following result.

Theorem 9.1.1 [MVW06, Theorem 22.15]. *If $F \in \mathbf{PST}$ is MV-effaceable, then F_{Zar} has a unique structure of presheaf with transfers such that $F \to F_{\mathrm{Zar}}$ is a morphism in $\mathbf{PST}$.*

Combined with Theorem 8.1.6, we obtain the following theorem.

Theorem 9.1.2. *If $F \in \mathbf{PST}$ has weak reciprocity, F_{Zar} has a unique structure of presheaf with transfers such that $F \to F_{\mathrm{Zar}}$ is a morphism in $\mathbf{PST}$.*

Remark 9.1.3. Étale and Nisnevich analogues of the above theorem hold for any $F \in \mathbf{PST}$ (without assuming reciprocity). See [MVW06, 6.17 and 14.1].

9.2 Zariski sheafification preserves reciprocity

In [MVW06, Theorem 22.2], it is proved that Zariski sheafification of a homotopy invariant presheaf with transfers is homotopy invariant. The same argument does not work for reciprocity sheaves. The proofs of the following results are based on a different idea.

Lemma 9.2.1. *If $F \in \mathbf{PST}$ has weak reciprocity, so does F_{Zar} (note that the statement makes sense by Theorem 9.1.2).*

Proof. Let $S \in \mathbf{Sm}$ and $X/S \in rel\mathcal{C}(S)$ with a good compactification $X \hookrightarrow \overline{X}$, and take $a \in F_{\mathrm{Zar}}(X)$. We need to prove that a has a weak modulus $Y \subset \overline{X}$ such that $X = \overline{X}\backslash|Y|$. We may assume that X and S are connected. By Corollary 7.1.3, $F_{\mathrm{Zar}}(S) \to F_{\mathrm{Zar}}(V)$ is injective for any dense open subset $V \subset S$. Hence, we may prove the assertion after the base change to the generic point η of S. Now the theorem follows from Proposition 5.2.1 since the condition (2) of the proposition is insensible to Zariski sheafification. □

Theorem 9.2.2. *Assume that k is perfect. If $F \in \mathbf{PST}$ has reciprocity, then F_{Zar} has reciprocity.*

Proof. Let $\rho : F \to F_{\mathrm{Zar}}$ be the canonical morphism. Let $X \subset \overline{X}$ be an open immersion such that $\overline{X}$ is an integral proper variety over k and $X \in \mathbf{Sm}$ is quasi-affine. Let $a \in F_{\mathrm{Zar}}(X)$. We need to show that a has a modulus $Y \subset \overline{X}$ such that $|Y| = \overline{X} - X$.

There exist a Zariski open covering $X = \bigcup_{i=1}^{r} U_i$ and $a_i \in F(U_i)$ such that $\rho(a_i) = a|_{U_i}$ for each i. By assumption, a_i has a modulus $Y_i \subset \overline{X}$ such that $|Y_i| = \overline{X} - U_i$. Then Y_i is a modulus for $\rho(a_i) \in F_{\mathrm{Zar}}(U_i)$. Put $Y := Y_1 \times_{\overline{X}} \cdots \times_{\overline{X}} Y_r$. Since $X = \bigcup_{i=1}^{r} U_i$, we have $|Y| = \overline{X} - X$.

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We shall prove that Y is a modulus for a. By Corollary 7.1.3 and Lemma 5.1.7, $F_{\mathrm{Zar}}(S) \to F_{\mathrm{Zar}}(V)$ is injective for any dense open subset $V \subset S$. Hence, it suffices to verify the condition in Remark 4.1.2.

Put $K = k(S)$ and let $\varphi : \overline{C} \to \overline{X} \times K$ and $\psi : C \to X$ be as in Remark 4.1.2. We need to show that $\langle g, \psi^*(a)\rangle_{C/K} = 0 \in F(K)$ for all $g \in G(\overline{C}, \varphi^* Y)$, where $\varphi^* Y$ is the pullback of $Y \times K$ by φ. For each i, we put $C_i := \varphi^{-1}(U_i \times K) \subset \overline{C}$ and $I = \{i \mid C_i \neq \emptyset\}$. Note that $i \in I$ if and only if $\varphi(\overline{C}) \not\subset (\overline{X} - U_i) \times K$. Since $X = \bigcup_{i=1}^r U_i$, we have $C = \bigcup_{i \in I} C_i$. Let $i \in I$. We write ψ_i for the composition $C_i \to U_i \times K \xrightarrow{pr} U_i$. We have $\psi_i^*(\rho(a_i)) = \psi^*(a)|_{C_i}$. As $\rho(a_i)$ has modulus Y_i, we have $\langle G(\overline{C}, \varphi^* Y_i), \psi^*(a)|_{C_i}\rangle_{C_i/K} = 0 \in F_{\mathrm{Zar}}(K)$. Hence, the claim is a consequence of the following lemma. □

Lemma 9.2.3. *Let K be as above and $\overline{C}$ be a normal integral proper curve over K, C an open dense subscheme of $\overline{C}$ and $C = \bigcup_{i=1}^r C_i$ a Zariski covering. Let $F \in \mathbf{PST}$ and $a \in F(C)$. Suppose that for each i we are given an effective divisor $Y_i \subset \overline{C}$ satisfying $|Y_i| = \overline{C} - C_i$ and $\langle G(\overline{C}, Y_i), a|_{C_i}\rangle_{C_i/K} = 0 \in F(K)$. Then we have $\langle G(\overline{C}, Y), a\rangle_{C/K} = 0$ in $F(K)$, where $Y := Y_1 \times_{\overline{C}} \cdots \times_{\overline{C}} Y_r \subset \overline{C}$.*

Proof. This is an easy consequence of Proposition 5.2.1. □

9.3 Comparison of Zariski/Nisnevich sheafification

In [MVW06, Theorem 22.2], it is proved that $F_{\mathrm{Zar}} = F_{\mathrm{Nis}}$ if F is a homotopy invariant presheaf with transfers. The proof actually shows the following.

Theorem 9.3.1 [MVW06, Theorem 22.2]. *Let F be a Zariski sheaf with transfers. If F is MV-effaceable, then one has $F = F_{\mathrm{Nis}}$.*

Theorems 8.1.6 and 9.2.2 and Lemma 9.2.1 imply the following.

Theorem 9.3.2. *Let $F \in \mathbf{PST}$.*

(a) *If F has weak reciprocity, we have $F_{\mathrm{Zar}} = F_{\mathrm{Nis}}$.*
(b) *Assume that k is perfect. If F has reciprocity, so does F_{Nis}.*

9.4 Generalized Jacobian

Let $\overline{C}$ be a smooth projective geometrically connected curve over k, D an effective divisor on $\overline{C}$ and $C := \overline{C} - D$. The map $\mathbb{Z}_{\mathrm{tr}}(C) \to \mathbb{Z}$ induced by the structure map $\overline{C} \to \operatorname{Spec} k$ factors through $\deg : h(\overline{C}, D) \to \mathbb{Z}$ (see Theorem 2.1.5). We write $h(\overline{C}, D)^0$ for its kernel.

Proposition 9.4.1. *Suppose that k is perfect and that C has a k-rational point. Then Rosenlicht's generalized Jacobian $J := \mathrm{Jac}(\overline{C}, D)$ is isomorphic to the Zariski sheafification of $h(\overline{C}, D)^0$.*

Proof. Let $s : \mathbb{Z}_{\mathrm{tr}}(J) \to J$ be the canonical map constructed in [SS03, Proof of Lemma 3.2]. (For $S \in \mathbf{Sm}$ and $Z \in \mathbb{Z}_{\mathrm{tr}}(J)(S) = \mathbf{Cor}(S, J)$ integral, $s(Z)$ is given by the composition $S \xrightarrow{\text{(i)}} \mathrm{Sym}^d(Z) \xrightarrow{\text{(ii)}} \mathrm{Sym}^d(J) \xrightarrow{\text{(iii)}} J$, where d is the degree of Z over S, (i) is given by [SV96, p. 81], (ii) is induced by $Z \to J$ and (iii) is given by the addition map of J.) Let $\tilde{a} : \mathbb{Z}_{\mathrm{tr}}(C) \to \mathbb{Z}_{\mathrm{tr}}(J)$ be the map induced by the universal map $a : C \to J$ with respect to a k-rational point. Since a has modulus D in the sense of Rosenlicht–Serre, Remark 4.4.5 shows that $s\tilde{a}$ has modulus

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D in our sense. Thus, we get an induced map $u : h(\overline{C}, D)^0 \to J$ (see Remark 2.1.6). For any field $K \in \widetilde{\mathbf{Sm}}$, $u(K) : h(\overline{C}, D)^0(K) \to J(K)$ is an isomorphism as both groups are isomorphic to $\mathrm{Div}(C \times K)/G(\overline{C} \times K, D \times K)$. Note that $J = J_{\mathrm{Zar}}$ and $h(\overline{C}, D)^0_{\mathrm{Zar}}$ have reciprocity by Theorems 4 and 7(2). Now the proposition follows from Theorem 6(3) applied to the kernel and cokernel of u. □

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We thank Rin Sugiyama for pointing out a mistake in our initial proof of (2.12).

Appendix. Kähler differentials and de Rham–Witt differentials have reciprocity

Kay Rülling[4]

In this appendix we prove that the absolute Kähler differentials of any degree are reciprocity presheaves. If the ground field k is perfect, ditto for the Kähler differentials relative to k. If k has positive characteristic, we show that the de Rham–Witt complex on a finite level of Bloch–Deligne–Illusie is a complex of reciprocity presheaves.

Appendix A. Kähler differentials

We use the notation from § 1. For a k-scheme X, we denote by $\Omega^j_{X/k}$, $j \geqslant 0$, the sheaf of Kähler differentials of degree j relative to k and by $\Omega^j_X = \Omega^j_{X/\mathbb{Z}}$ the sheaf of absolute Kähler differentials of degree j.

We recall some constructions from [CR11].

A.1 Pullback

Let $f : X \to Y$ be a morphism of k-schemes; then there is a natural pullback map

$$f^* : H^i(Y, \Omega^j_{Y/k}) \to H^i(X, \Omega^j_{X/k}) \quad \text{for all } i, j \geqslant 0,$$

which is functorial in the obvious sense.

A.2 Pushforward

Let $f : X \to Y$ be a morphism in $\mathbf{Sm}$ of pure relative dimension r, $Z \subset X$ a closed subset such that $f_{|Z} : Z \to Y$ is proper and $Z' \subset Y$ a closed subset with $Z \subset f^{-1}(Z')$. Then there is a pushforward morphism (see [CR11, 2.3])

$$f_* : H^{i+r}_Z(X, \Omega^{j+r}_{X/k}) \to H^i_{Z'}(Y, \Omega^j_{X/k}) \quad \text{for all } i, j \geqslant 0,$$

which is functorial in the obvious sense.

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A.3 Cycle class

Assume that k is a perfect field. Let $X \in \mathbf{Sm}$ be integral and $V \subset X$ a closed integral subscheme of codimension c. Then (see e.g. [CR11, Proposition 3.1.1]) there is an element

$$cl(V) \in H^c_V(X, \Omega^c_{X/k}),$$

which is unique with the following property: for any (or some) open subset $U \subset X$ such that the closed immersion $i : V \cap U \hookrightarrow U$ is a regular embedding of codimension c, the restriction of $cl(V)$ to U equals the image of 1 under the pushforward $i_* : H^0(V \cap U, \mathcal{O}_V) \to H^c_{V\cap U}(U, \Omega^c_{U/k})$.

A.4 Correspondence action

Assume that k is a perfect field. Let $X, Y \in \mathbf{Sm}$ be equidimensional and $Z \subset X \times Y$ a closed integral subscheme which is proper over X. Set $r := \dim Z - \dim X$. We define the action of Z on Hodge cohomology

$$Z^* : H^{i+r}(Y, \Omega^{j+r}_{Y/k}) \to H^i(X, \Omega^j_{X/k}) \quad \text{for all } i, j \geqslant 0$$

as the composition

$$H^{i+r}(Y, \Omega^{j+r}_{Y/k}) \xrightarrow{p_Y^*} H^{i+r}(X \times Y, \Omega^{j+r}_{X\times Y/k}) \xrightarrow{\cup cl(Z)} H^{i+\dim Y}_Z(X \times Y, \Omega^{j+\dim Y}_{X\times Y/k}) \xrightarrow{p_{X*}} H^i(X, \Omega^j_{X/k}),$$

where we denote by $p_X, p_Y : X \times Y \to X, Y$ the projection maps. If $f : X \to Y$ is a morphism and $Z := \Gamma_f \subset X \times Y$ is its graph, then $Z^* = f^*$; if f is also proper and $Z^t \subset Y \times X$ denotes the transpose of Z, then $(Z^t)^* = f_*$: see [CR11, 3.2.1].

The following theorem is a particular case of [CR11, Theorem 3.1.8] together with [CR11, 1.3.18, Lemma 1.3.19].

Theorem A.4.1. *Assume that k is a perfect field. Then the correspondence action from § A.4 above induces the structure of a presheaf with transfers on the presheaf*

$$\mathbf{Sm} \ni X \mapsto H^i(X, \Omega^j_{X/k}) \in (k\textbf{-vector spaces})$$

for all $i, j \geqslant 0$.

Remark A.5. It follows from [CR11, Theorem 1.2.3] and [CR11, 1.3.18, Lemma 1.3.19] that any weak cohomology theory in the sense of [CR11, 1.1.9] which satisfies the conditions of [CR11, Theorem 1.2.3] defines a graded presheaf with transfers. In [CR11, Theorem 3.1.8], it is proven that Hodge cohomology defines such a weak cohomology theory.

A.6 The absolute case

Let k be a field and $k_0 \subset k$ its prime field. Let $\mathcal{I}$ be the set of smooth k_0-subalgebras of k. The ordering by inclusion makes $\mathcal{I}$ a filtered partially ordered set. For each $X \in \mathbf{Sm}$, we find a k_0-algebra $A \in \mathcal{I}$ and a smooth separated A-scheme X_A with $X_A \otimes_A k = X$. Fix such an A for each X and set $X_B := X_A \otimes_A B \in \mathbf{Sm}_{k_0}$ for each $B \in \mathcal{I}$ containing A. For $X \in \mathbf{Sm}$, we have

$$H^i(X, \Omega^j_X) = \varinjlim_{B\in\mathcal{I}} H^i(X_B, \Omega^j_{X_B/k_0}).$$

For $Z \subset X \times Y$ as in § A.4, we find a k_0-algebra $A \in \mathcal{I}$ such that there exists a closed integral subscheme $Z_A \subset X_A \times_A Y_A$ which is proper over X_A, flat over A and satisfies $Z_A \otimes_A k = Z$. For

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$B \in \mathcal{I}$ with $B \supset A$, we set $Z_B := Z_A \otimes_A B$. Let $f : \operatorname{Spec} B' \to \operatorname{Spec} B$ be the map induced by an inclusion $B \subset B'$ in $\mathcal{I}$; then

$$Z_{B'}^* \circ (\mathrm{id}_Y \times f)^* = (\mathrm{id}_X \times f)^* \circ Z_B^* : H^{i+r}(Y_B, \Omega^{j+r}_{Y_B/k_0}) \to H^i(X_{B'}, \Omega^j_{X_{B'}/k_0})$$

for all $i, j \geqslant 0$. (Indeed, by [CR11, Theorem 3.1.8], the two compositions are given by the correspondences $(\Gamma_{\mathrm{id}_Y \times f} \circ Z_{B'})$ and $(Z_B \circ \Gamma_{\mathrm{id}_X \times f})$, which are both equal to $Z_{B'}$ viewed as correspondences from $X_{B'}$ to Y_B via the closed immersion $X_{B'} \times_{B'} Y_{B'} = X_{B'} \times_B Y_B \subset X_{B'} \times_{k_0} Y_B$. Here we view correspondences as elements in the Chow groups with supports; see [CR11, 1].) Therefore, we can define the action of Z on absolute Hodge cohomology

$$Z^* : H^{i+r}(Y, \Omega_Y^{j+r}) \to H^i(X, \Omega_X^j)$$

by the formula

$$\varinjlim_{B \in \mathcal{I}} Z_B^* : \varinjlim_{B \in \mathcal{I}} H^{i+r}(Y_B, \Omega^{j+r}_{Y_B/k_0}) \to \varinjlim_{B \in \mathcal{I}} H^i(X_B, \Omega^j_{X_B/k_0}).$$

We obtain the following result.

Corollary A.6.1. *Let k be an arbitrary field. Then the correspondence action from § A.6 above induces the structure of a presheaf with transfers on the presheaf*

$$\mathbf{Sm} \ni X \mapsto H^i(X, \Omega_X^j) \in (k\textbf{-vector spaces})$$

for all $i, j \geqslant 0$.

Theorem A.6.2. *The presheaf with transfers*

$$\mathbf{Sm} \ni X \mapsto H^0(X, \Omega_X^i), \quad i \geqslant 0,$$

has reciprocity in the sense of Definition 2.1.3. If k is perfect, the same is true with Ω_X^i replaced by $\Omega^i_{X/k}$.

Proof. First, assume that k is perfect; we show that $\Omega^i_{-/k}$ has reciprocity. Let $X \in \mathbf{Sm}$ be quasi-affine and take $a \in H^0(X, \Omega^i_{X/k})$. Choose an open immersion $X \hookrightarrow \bar{X}$ of X into an integral and proper k-scheme $\bar{X}$ such that $\bar{X} \backslash X$ is the support of a Cartier divisor Y_0. Then for some large enough integer n the form a is the restriction of a section in $H^0(\bar{X}, \Omega^i_{\bar{X}/k}(nY_0))$, where we write $\Omega^i_{\bar{X}/k}(nY_0) := \Omega^i_{\bar{X}/k} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{\bar{X}}(nY_0)$. By Corollary 2.1.7, it suffices to prove that

$$Y := (n+1)Y_0 \quad \text{is a modulus for } a. \tag{A.1}$$

To this end, take $S \in \mathbf{Sm}$, consider a diagram $(\varphi : \bar{C} \to \bar{X} \times S)$ as in (2.1) (we will use the notation from (2.1) freely) and a function $f \in G(\bar{C}, \gamma_\varphi^* Y)$. We have to show that

$$(\varphi_* \operatorname{div}_{\bar{C}}(f))^*(a) = 0 \quad \text{in } H^0(S, \Omega^i_{S/k}). \tag{A.2}$$

Clearly, we can assume that S is connected. Further, $\Omega^i_{S/k}$ is locally free and hence restriction to open subsets is injective. Therefore (cf. Remark 4.1.2), we can replace S by a non-empty open subset. Using the perfectness of k, we can thus assume that $\bar{C}$ is smooth and connected over k, the map $p_\varphi : \bar{C} \to S$ is proper and flat of pure relative dimension one and the support of $\operatorname{div}_{\bar{C}}(f)$ is a disjoint union of smooth prime divisors $|\operatorname{div}_{\bar{C}}(f)| = \bigsqcup_i Z_i$. Set $C := \varphi^{-1}(X \times S)$

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and denote by $b \in H^0(C, \Omega^i_{C/k})$ the pullback of a to C. By assumption, b extends to a section $H^0(\bar{C}, \Omega^i_{\bar{C}/k}(n\gamma_\varphi^* Y_0))$. We have to show that

$$\mathrm{div}_{\bar{C}}(f)^*(b) = 0 \quad \text{in } H^0(S, \Omega^i_{S/k}), \tag{A.3}$$

where we consider $\mathrm{div}_{\bar{C}}(f)$ as an element in $\mathrm{Cor}(S, C)$ via the transpose of the graph map $C \to S \times C$ of p_φ. For Z a prime divisor in the support of $\mathrm{div}_{\bar{C}}(f)$, denote by $i : Z \hookrightarrow S \times C$ the induced closed immersion and by $p_S, p_C : S \times C \to S, C$ the projections. We get

$$Z^*(b) = p_{S*}(p_C^*(b) \cup cl(Z)) = p_{S*}(p_C^*(b) \cup i_*(1)) = \mathrm{Tr}_{Z/S}(b_{|Z}), \tag{A.4}$$

where we denote by $\mathrm{Tr}_{Z/S}$ the pushforward along the finite morphism $Z \to S$ and the first equality holds by definition, the second equality by the characterization of the cycle class in § A.3 and the last equality follows from the projection formula [CR11, Proposition 1.1.16]. Writing $\mathrm{div}_{\bar{C}}(f) = \sum_j n_j Z_j$, we therefore have to show that

$$\sum_j n_j \mathrm{Tr}_{Z_j/S}(b_{|Z_j}) = 0 \quad \text{in } H^0(S, \Omega^i_{S/k}). \tag{A.5}$$

By the functoriality of the pushforward, the map $\mathrm{Tr}_{Z_j/S}$ equals the composition

$$H^0(Z_j, \Omega^i_{Z_j/k}) \xrightarrow{i_{j*}} H^1_D(C, \Omega^{i+1}_{C/k}) \to H^1(\bar{C}, \Omega^{i+1}_{\bar{C}/k}) \xrightarrow{p_{\varphi*}} H^0(S, \Omega^i_{S/k});$$

here $D := |\mathrm{div}_{\bar{C}}(f)|$ and $i_j : Z_j \hookrightarrow C$ is the closed immersion. We claim that

$$\sum_j n_j \, i_{j*}(b_{|Z_j}) = -\delta\left(\frac{df}{f} \wedge b\right) \quad \text{in } H^1_D(C, \Omega^{i+1}_{C/k}),$$

where $\delta : H^0(C \backslash D, \Omega^{i+1}_{C/k}) \to H^1_D(C, \Omega^{i+1}_{C/k})$ is the connecting homomorphism. Indeed, it suffices to check this equality after restricting to an open subset of C which contains all the generic points of D; in particular, we can assume that the Z_j are the zero loci of sections in $H^0(C, \mathcal{O}_C)$ and then the claim follows from [CR11, Proposition 2.2.19]. Since $b \in H^0(\bar{C}, \Omega^i_{\bar{C}/k}(n\gamma_\varphi^* Y_0))$ and $f \in G(\bar{C}, (n+1)\gamma_\varphi^* Y_0)$, the section $(df/f) \wedge b$ extends to a section in $H^0(\bar{C} \backslash D, \Omega^{i+1}_{\bar{C}/k})$. Therefore, the image of $\delta((df/f) \wedge b)$ in $H^1(\bar{C}, \Omega^{i+1}_{\bar{C}/k})$ is zero, which implies the vanishing of (A.5).

Now let k be an arbitrary field with prime field k_0. We want to show that $\Omega^i_{-/\mathbb{Z}}$ has reciprocity. Let $X \in \mathbf{Sm}$ be quasi-affine and take $a \in H^0(X, \Omega^i_X)$. With the notation from § A.6, we find a smooth k_0-algebra $A \in \mathcal{I}$ such that a comes via pullback from an element $a_A \in H^0(X_A, \Omega^i_{X_A/k_0})$. If $B \in \mathcal{I}$ contains A, we denote by a_B the pullback of a_A to X_B. Choose an open immersion $X_A \hookrightarrow \bar{X}_A$ with $\bar{X}_A$ an integral and proper A-scheme such that $\bar{X}_A \backslash X_A$ is the support of a Cartier divisor $Y_{0,A}$. There exists an integer $n \geqslant 0$ such that a_A is the restriction of a section in $H^0(\bar{X}_A, \Omega^i_{\bar{X}_A/k_0}(nY_{0,A}))$. Set $\bar{X} := \bar{X}_A \otimes_A k$ and $Y := (n+1)Y_{0,A} \otimes_A k$; we claim that Y is a modulus for a. Take $S \in \mathbf{Sm}$, $(\varphi : \bar{C} \to \bar{X} \times S)$ and $f \in G(\bar{C}, \gamma_\varphi^* Y)$ as above. Then there exist a $B \in \mathcal{I}$ containing A and $S_B \in \mathbf{Sm}_B$ $(\varphi_B : \bar{C}_B \to \bar{X}_B \times_B S_B)$ as in (2.1) (only that the cartesian product is over B) and $f_B \in G(\bar{C}_B, \gamma_{\varphi_B}^* Y_B)$, which give S, φ, f when pulled back over k. It suffices to show that

$$(\varphi_{B*} \, \mathrm{div}_{\bar{C}_B}(f_B))^*(a_B) = 0 \quad \text{in } H^0(S_B, \Omega^i_{S_B/k_0}).$$

Notice that this is not exactly the same situation as in the first case since $\bar{X}_B$ is not proper over k_0. Nevertheless, the same argument as above reduces us to prove the vanishing of (A.3) with $S, \bar{C}, f, k$ replaced by $S_B, \bar{C}_B, f_B, k_0$, which follows from the first case. □

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Appendix B. De Rham–Witt differentials

In this section, k is a perfect field of characteristic $p > 0$ and we denote by **qpSm** the category of smooth and quasi-projective k-schemes.

B.1 De Rham–Witt complex

For a k-scheme X, we denote by $W_n\Omega^{\cdot}_X$ the de Rham–Witt complex of Bloch–Deligne–Illusie of length n; see [Ill79]. We denote by $W_n\Omega^j_X$ the degree-j part. Recall that $W_n\Omega^0_X = W_n\mathcal{O}_X$ is the sheaf of Witt vectors of length n on X and $W_1\Omega^j_X = \Omega^j_X$. Also recall that the de Rham–Witt complex comes with morphisms of sheaves of abelian groups $R : W_{n+1}\Omega^j_X \to W_n\Omega^j_X$ (the restriction), $F : W_{n+1}\Omega^j_X \to W_n\Omega^j_X$ (the Frobenius), $V : W_n\Omega^j_X \to W_{n+1}\Omega^j_X$ (the Verschiebung) and $d : W_n\Omega^j_X \to W_n\Omega^{j+1}_X$ (the differential) satisfying various relations (see [Ill79, I, Proposition 2.18]).

B.2 Correspondence action

If we restrict to the category **qpSm**, then we have analogues of the pullback map § A.1, the pushforward § A.2 and the cycle map § A.3 for Hodge–Witt cohomology (i.e. in §§ A.1–A.3 replace $\Omega^{\cdot}_{-/k}$ by $W_n\Omega^{\cdot}_-$ and in §§ A.2 and A.3 restrict to **qpSm**). The construction of the cycle map and the pushforward uses essentially the results from [Eke84]. For the cycle map and the pushforward for proper maps, this is carried out in [Gro85, II]; for the pushforward with projective supports and the compatibilities, see [CR12, §§ 2 and 3].

Let $X, Y \in$ **qpSm** be connected and $Z \subset X \times Y$ a closed integral subscheme which is projective over X. Set $r := \dim Z - \dim X$. We define the action of Z on Hodge–Witt cohomology

$$Z^* : H^{i+r}(Y, W_n\Omega^{j+r}_Y) \to H^i(X, W_n\Omega^j_X) \quad \text{for all } i, j \geqslant 0$$

as in § A.4 to be the composition

$$\begin{aligned} H^{i+r}(Y, W_n\Omega^{j+r}_Y) &\xrightarrow{p_Y^*} H^{i+r}(X \times Y, W_n\Omega^{j+r}_{X\times Y}) \\ &\xrightarrow{\cup cl(Z)} H^{i+d_Y}_Z(X \times Y, W_n\Omega^{j+d_Y}_{X\times Y}) \xrightarrow{p_{X*}} H^i(X, W_n\Omega^j_X). \end{aligned}$$

If $f : X \to Y$ is a morphism and $Z := \Gamma_f \subset X \times Y$ is its graph, then $Z^* = f^*$; if f is also projective and $Z^t \subset Y \times X$ denotes the transpose of Z, then $(Z^t)^* = f_*$: see [CR12, Proposition 3.4.7].

Theorem B.2.1. *The correspondence action from § B.2 above induces the structure of a presheaf with transfers on the presheaf*

$$\mathbf{qpSm} \ni X \mapsto H^i(X, W_n\Omega^j_X) \in (W_n(k)\text{-}\mathbf{modules})$$

for all $i, j \geqslant 0$. Furthermore, the correspondence action is compatible with the maps R, F, V, d from § B.1 in the obvious sense.

Proof. This is actually a special case of [CR12, Theorem 3.4.6] only that there the statement is proved in the limit, i.e. for the presheaf $X \mapsto H^i(X, W\Omega^i_X)$. The proof on the finite level goes through except for one place in the proof of [CR12, Theorem 3.4.3] (which is used in the proof of [CR12, Theorem 3.4.6]), where a *pro-argument* is used. There the situation is the following: we are given closed immersions between smooth schemes $D' \hookrightarrow D \xrightarrow{i} Y$ with $c := \operatorname{codim}(D', D)$ and $1 = \operatorname{codim}(D, Y)$. Then it is shown that the pushforward $i_* : H^c_{D'}(D, W\Omega^c_D) \to H^{c+1}_{D'}(Y, W\Omega^{c+1}_Y)$ is injective on its Frobenius invariant submodule $H^c_{D'}(D, W\Omega^c_D)^F$. Replace this with the following argument: by [CSS83, 1.4, Lemma 2], we have a short exact sequence on $D_{\acute{e}t}$

$$0 \to W_n\Omega^c_{D,\log} \to W_n\Omega^c_D \xrightarrow{1-F} W_n\Omega^c_D/dV^{n-1}\Omega^{c-1}_D \to 0.$$

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We get a short exact sequence

$$H^{c-1}_{D'}(D, W_n\Omega^c_D/dV^{n-1}\Omega^{c-1}_D) \to H^c_{D'}(D, W_n\Omega^c_{D,\log}) \to H^c_{D'}(D, W_n\Omega^c_D)^F \to 0, \qquad \text{(B.1)}$$

where the group on the right is defined as the kernel of $1-F$ on $H^c_{D'}$. By [Ill79, I, Corollary 3.9] (and with the notation from there), we have a short exact sequence of sheaves of abelian groups

$$0 \to \frac{\Omega^c_D}{B_n\Omega^c_D} \xrightarrow{V^{n-1}} \frac{W_n\Omega^c_D}{dV^{n-1}\Omega^{c-1}_D} \xrightarrow{R} W_{n-1}\Omega^c_D \to 0.$$

The two outer sheaves are Cohen–Macaulay by [Ill79, I, Corollary 3.9] and hence so is the sheaf in the middle. In particular, the cohomology group on the left of (B.1) vanishes. Thus,

$$H^c_{D'}(D, W_n\Omega^c_D)^F = H^c_{D'}(D, W_n\Omega^c_{D,\log}) \cong \mathbb{Z}/p^n\mathbb{Z},$$

where the second isomorphism is the composition of [Gro85, II, Theorem 3.5.8] and [Gro85, (3.5.19)], and similarly

$$H^{c+1}_{D'}(Y, W_n\Omega^{c+1}_Y)^F = H^{c+1}_{D'}(Y, W_n\Omega^{c+1}_{Y,\log}) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

Via these identifications, i_* sends $1 \in \mathbb{Z}/p^n\mathbb{Z}$ to itself and hence

$$i_* : H^c_{D'}(D, W_n\Omega^j_D)^F \hookrightarrow H^{c+1}_{D'}(Y, W_n\Omega^{c+1}_D)$$

is injective. Now the rest of the proof of [CR12, Theorem 3.4.3] and of [CR12, Theorem 3.4.6] goes through. Notice that the compatibility of the correspondence action with R, F, V, d requires Ekedahl's notion of a Witt-dualizing system; see [Eke84, CR12]. □

Theorem B.2.2. *The presheaf*

$$\mathbf{Sm} \ni X \mapsto H^0(X, W_n\Omega^i_X) \in (W_n(k)\textbf{-modules})$$

has the structure of a presheaf with transfers and has reciprocity, for all $i \geqslant 0$ and $n \geqslant 1$.

Proof. Since $W_n\Omega^i_{(-)}$ is a Zariski sheaf and has transfers on smooth and quasi-affine schemes by Theorem B.2.1, we can glue the transfers to obtain $W_n\Omega^i_{(-)} \in \mathbf{PST}$. We prove that $W_n\Omega^i_{(-)}$ has reciprocity. Let $X \in \mathbf{Sm}$ be quasi-affine and $a \in H^0(X, W_n\Omega^i_X)$ a section. Choose a compactification $X \hookrightarrow \bar{X}$ as in the proof of Theorem A.6.2 with $\bar{X}\backslash X$ the support of the Cartier divisor Y_0; we can assume that $\bar{X}$ is projective. Denote by $W_n\mathcal{O}_{\bar{X}}(Y_0)$ the invertible $W_n\mathcal{O}_{\bar{X}}$-module whose isomorphism class in $H^1(\bar{X}, W_n\mathcal{O}^\times_{\bar{X}})$ is the image of the class of $\mathcal{O}_{\bar{X}}(Y_0)$ in $H^1(\bar{X}, \mathcal{O}^\times_{\bar{X}})$ under the map induced by the Teichmüller lift $[-] : H^1(\bar{X}, \mathcal{O}^\times_{\bar{X}}) \to H^1(\bar{X}, W_n\mathcal{O}^\times_{\bar{X}})$. More precisely, if t is a local coordinate of Y_0 on some open $U \subset \bar{X}$, then $W_n\mathcal{O}_{\bar{X}}(Y_0)_{|U} = W_n(\mathcal{O}_U)\cdot 1/[t]$. In particular,

$$W_n\Omega^i_X = \varinjlim_r W_n\Omega^i_{\bar{X}}(rY_0),$$

where we set $W_n\Omega^i_{\bar{X}}(rY_0) := W_n\Omega^i_{\bar{X}} \otimes_{W_n\mathcal{O}_{\bar{X}}} W_n\mathcal{O}_{\bar{X}}(rY_0)$. Hence, we find an integer $r \geqslant 1$ such that a is the restriction of a section in $H^0(\bar{X}, W_n\Omega^i_{\bar{X}}(rY_0))$. By Corollary 2.1.7, it suffices to prove that

$$Y := sY_0 \text{ is a modulus for } a \quad \text{for } s \geqslant p^{n-1}r + 1. \qquad \text{(B.2)}$$

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To this end, take $S \in \mathbf{Sm}$ and $(\varphi : \bar{C} \to \bar{X} \times S)$ as in (2.1) and a function $f \in G(\bar{C}, \gamma_\varphi^* Y)$. By [Ill79, I, Corollary 3.9], restriction to dense open subsets is injective on $W_n\Omega^i_S$. Thus, as in the proof of Theorem A.6.2, we can assume that S, φ, f have the same properties as in the proof of Theorem A.6.2 between (A.2) and (A.3); further, we can achieve that all schemes in question are quasi-projective. Denote by b the pullback of a to C. By assumption, b extends to a section of $H^0(\bar{C}, W_n\Omega^i_{\bar{C}}(r\gamma_\varphi^* Y_0))$. As in the proof of Theorem A.6.2, (A.5), we are reduced to showing that

$$\sum_j n_j \mathrm{Tr}_{Z_j/S}(b_{|Z_j}) = 0 \quad \text{in } H^0(S, W_n\Omega^i_S), \tag{B.3}$$

where we write $\mathrm{div}_{\bar{C}}(f) = \sum_j n_j Z_j$ and $\mathrm{Tr}_{Z_j/S} : H^0(Z_j, W_n\Omega^i_{Z_j}) \to H^0(S, W_n\Omega^i_S)$ is the pushforward along the finite map $Z_j \to S$. By the functoriality of the pushforward, the map $\mathrm{Tr}_{Z_j/S}$ equals the composition (with $D = |\mathrm{div}_{\bar{C}}(f)|$ and $i_j : Z_j \hookrightarrow C$ the closed immersion)

$$H^0(Z_j, W_n\Omega^i_{Z_j}) \xrightarrow{i_{j*}} H^1_D(C, W_n\Omega^{i+1}_C) \to H^1(\bar{C}, W_n\Omega^{i+1}_{\bar{C}}) \xrightarrow{p_{\varphi *}} H^0(S, W_n\Omega^i_S).$$

As in the proof of Theorem A.6.2, the following equality follows from [CR12, Proposition 2.4.1] (see also [Gro85, II, 3.4]):

$$\sum_j n_j \, i_{j*}(b_{|Z_j}) = -\delta\left(\frac{d[f]}{[f]} b\right) \quad \text{in } H^1_D(C, W_n\Omega^{i+1}_C),$$

where $\delta : H^0(C \backslash D, W_n\Omega^{i+1}_C) \to H^1_D(C, W_n\Omega^{i+1}_C)$ is the connecting homomorphism and $[f] \in W_n(k(\bar{C}))$ denotes the Teichmüller lift of f. Thus, it suffices to show that $d[f]/[f]b$ extends to a section of $H^0(\bar{C} \backslash D, W_n\Omega^{i+1}_{\bar{C}})$ to conclude the vanishing of (B.3). To this end, it suffices to show that $d([f])b$ is regular around any point of Y_0. Let A be the local ring of $\bar{C}$ at a point of Y_0 and $t \in A$ an equation for Y_0. Then we can write $b = b_0/[t]^r$, for some $b_0 \in W_n\Omega^i_A$, and $f = 1 + t^s g$, for some $g \in A$. By [Rül07, Lemma 3.4],

$$[f] = [1] + \sum_{j=0}^{n-1} V^j([t]^s g_j) \quad \text{in } W_n(A) \text{ for some } g_j \in W_{n-j}(A).$$

For $j \in \{0, \dots, n-1\}$ and $h = g_j$, we have (using the standard identities for R, F, V, d)

$$\begin{aligned} d(V^j([t]^s h))b &= d(V^j([t]^s h)b) - V^j([t]^s h)\, db \\ &= dV^j([t]^s h F^j(b)) - V^j([t]^s h)\left(\frac{db_0}{[t]^r} - r b_0 \frac{1}{[t]^r}\frac{d[t]}{[t]}\right) \\ &= dV^j([t]^{s-p^j r} h F^j(b_0)) - V^j([t]^{s-p^j r} h F^j(db_0)) \\ &\quad + V^j([t]^{s-(p^j r+1)} h r F^j(b_0)\, d[t]). \end{aligned}$$

This expression is regular by the choice of s and hence so is $d([f])b$. This finishes the proof. □

Remark B.3. One can remove the perfectness assumption on k in this section using the same method as in § A.6.

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