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ASSOCIATED PRIME DIVISORS IN THE SENSE OF KRULL

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1. Introduction. In a recent paper by Douglas Underwood [**8**] several definitions of "associated prime divisors" were discussed and shown to be unique. In this note we produce a fifth type, which is due to W. Krull, and is found in his classical paper [**2**] and further discussed by B. Banaschewski in [**1**]. Historically this characterization considerably predates the other four definitions.

Throughout this note, R denotes a commutative ring with unity, and all ideals and elements are assumed to be in such a ring. We shall let upper case letters, most frequently the beginning of the alphabet, denote ideals and lower case letters, elements of R. On the whole, our terminology will be that of [**9**]. We do, however, take exception with [**9**] in two instances, viz. $\subset$ will denote set containment which may or may not be proper, and use the symbol $<$ for proper containment. We use the concept "ideal" in the somewhat restrictive sense, in that for us, an ideal is not the entire ring (sometimes in literature this is called a genuine ideal).

2. Preliminary remarks. We first introduce some terminology. Suppose S is a m.c. (nonempty multiplicatively closed) set and A is an ideal. Then the set $\{x \in R |$ there exists $s \in S$, such that $xs \in A\}$ is called the "isolated component of A determined by S" or more simply the S-component of A and is denoted by A_S. If P is a prime and $S = C(P)$ (set complement of P in R), then we let $A_S = A(P)$, i.e., $A(P) = \{x \in R |$ there exists $t \notin P$, such that $xt \in A\}$. (Recall that $A(P) = AR_P \cap R = A^{ec}$.) We shall let $Z(A)$ denote the set of all zero divisors modulo A, i.e., $Z(A) = \{x \in R |$ there exists $y \notin A$, such that $xy \in A\}$. Following the usage of N. McCoy in [**5**], an ideal B is called related to A, if $B \subset Z(A)$, i.e., if each element of B is not prime to A (cf. [**9**, p. 223]). Since the set complement of $Z(A)$ is the collection of all elements in R which are prime to A, it is a multiplicatively closed set, thus by Zorn's Lemma there are ideals contained in $Z(A)$ which contain A and are maximal with respect to this property. These ideals are called maximal associated prime divisors of A, which we shall denote by $MxPD$ (cf. [**7**, p. 19]). The prime ideals minimal in the collection of prime overideals of A are called minimal prime divisors or isolated primes of A. These will be denoted by $MnPD$. If A has a finite primary representation, then the prime ideals which occur as the radicals of the primary ideals in an irredundant representation are called associated primes of A (cf. [**9**, p. 211]).

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We now list the four definitions of "associated prime divisors" as found on p. 72 of [**8**].

(B): P is an associated prime divisor of A (in the Bourbaki sense) if $P = A : (x)$ for some $x \in R$.

$(Z\text{–}S)$: P is an associated prime divisor of A (in the Zarski-Samuel sense) if $A : (x)$ is a P-primary ideal for some $x \in R$.

(Bw): P is an associated prime divisor of A (in the weak Bourbaki sense), if P is a $MnPD$ of $A : (x)$ for some $x \in R$.

(N): P is an associated prime divisor of A (in the Negata sense), if PR_S is a $MxPD$ of AR_S for some multiplicatively closed set S.

Notation. If P is a prime containing an ideal A, then P is called a (B)-prime of A, if P is an associated prime divisor of A in the Bourbaki sense. Similarly for the other conditions.

We now state the theorem of Krull which motivated the definition.

THEOREM [**2**, p. 741]. *If A is an ideal with a finite primary decomposition, then a prime ideal P is one of the associated primes of A if and only if P is a $MxPD$ of $A(P)$.*

W. Krull gave the name *zu A gehörig* to any prime overideal possessing the property: P is an $MxPD$ of $A(P)$, (cf. [**2**, p. 742]) and B. Banaschewski the name *A-extremal* (cf. [**1**, p. 24]). However, in order to be consistent with the terminology due to Underwood we make the following

Definition 1. (K): P is an associated prime divisor of A (in the Krull sense) if P is a $MxPD$ of $A(P)$, thus P is called a (K)-prime of A if and only if P is a $MxPD$ of $A(P)$.

In [**8**] it was shown that if P is a prime containing A, then P a (B)-prime of A implies P is a $(Z\text{–}S)$-prime of A implies P is a (Bw)-prime of A implies P is a (N)-prime of A. We shall show that the (K)-primes of A fit in between the (Bw)-primes and the (N)-primes of A and are in general distinct. But first we record the following which are either easily verified or found in [**1**]:

(1) If U is the set of units in R, then for any ideal A, $Z(A) \cap U = \emptyset$.

(2) Suppose P is a prime containing A. Then $x \in A(P)$ if and only if $x/1 \in AR_P$.

(3) For any prime P, $x \notin P$ implies $x \notin Z(A(P))$, i.e., $(R \backslash P) \cap Z(A(P)) = \emptyset$.

(4) For any ideal A and primes P, P^* containing A, $A \subset A(P)$; $(A(P))(P) = A(P)$; if $P \subset P^*$, then $A(P^*) \subset A(P)$; $(A(P))(P^*) = (A(P^*))(P)$; if $P \subset P^*$, then $(A(P))(P^*) = A(P)$.

We now show that there are several ways to describe the property of being a (K)-prime more precisely.

PROPOSITION 1. *Let A be an ideal and P a prime containing A, then the following statements are equivalent:*

(1) *P is a (K)-prime of A.*
(2) $P \subset Z(A(P))$.
(3) $P = Z(A(P))$.
(4) *PR_P is a $MxPD$ of AR_P.*
(5) *For any multiplicatively closed set T such that $C(P) < T$, then $A_T > A(P)$.*
(6) *For any multiplicatively closed set T such that $A_T \subset A(P)$, then $T \subset C(P)$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are clear from the definitions and Remark 3 above. (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (3) are found in [**1**].

(2) $\Rightarrow$ (4): Suppose P is a prime containing A such that $P \subset Z(A(P))$. It is clear that no element in $R_P \backslash PR_P$ belongs to $Z(AR_P)$ (each such element is a unit). So suppose $p/x \in PR_P$ with $p \in P$. Now $p \in Z(A(P))$, so there exists $u \notin A(P)$ such that $pu \in A(P)$. But $u \notin A(P)$ implies $u/1 \notin AR_P$, thus $(p/x)(u/1) = pu/x \in AR_P$, and so $p/x \in Z(AR_P)$, i.e., PR_P is a $MxPD$ of AR_P.

(4) $\Rightarrow$ (2): Suppose PR_P is a $MxPD$ of AR_P and $p \in P$. Then there exists $u/t \in R_P \backslash AR_P$ such that $(p/1)(u/t) = pu/t \in AR_P$. We now show that $u \notin A(P)$ and $pu \in A(P)$. It is easily seen that $u \notin A(P)$, for if so, there exists $v \notin P$ such that $uv \in A$ and then $u/t = uv/tv \in AR_P$. Now since $pu/t \in AR_P$, there exists $a/t' \in AR_P$ such that $a \in A$ and $a/t' = pu/t$, i.e., there exists $s \in P$ such that $ats = put's$, hence $put's \in A$. But since $t', s \notin P$, this implies $t's \notin P$ and so $pu \in A(P)$.

We now prove the result mentioned above, more precisely:

THEOREM 1. *If P is a prime containing A, then*:
(1) *If P is a (Bw)-prime of A, then it is also a (K)-prime of A.*
(2) *If P is a (K)-prime of A, then it is also a (N)-prime of A.*

Proof. (2): Proposition 1, part (4) with the multiplicatively closed set $S = C(P)$.

(1): Now suppose P is a (Bw)-prime of A, then there exists $x \notin A$ such that P is a $MnPD$ of $A\colon (x)$. But in this case $(A\colon (x))(P)$ is P-primary. Let $p \in P$; then there exists an integer n such that $p^n \in (A\colon (x))(P)$, i.e., there exists $t \notin P$ such that $p^n tx \in A$ or $p^n x \in A(P)$. But since $A\colon (x) \subset P$, this implies that $x \notin A(P)$. Now let k be the least positive integer such that $p^k x \in A(P)$. Clearly $k \neq 0$ and if $k = 1$, then $p \in Z(A(P))$. Suppose that $k > 1$; then $p(p^{k-1}x) \in A(P)$, where $p^{k-1}x \notin A(P)$, i.e., p is a zero divisor modulo $A(P)$. But since p was any element of P, we have that each element of P is a zero divisor modulo $A(P)$, whence P is a (K)-prime of A.

We now exhibit two examples to show that in general the (K)-primes of an ideal are distinct from the (Bw)-primes and (N)-primes. First we find an ideal with an (N)-prime which is not a (K)-prime.

Example 1. Let R be a quasi-local (need not be Noetherian) integral domain containing a field K. By Nagata's theorem [**6**, p. 297] there is a ring T such that:

(i) there exists a maximal ideal M of T such that $T_M \cong R$,
(ii) for each maximal ideal M of T, T_M is isomorphic to R or K,
(iii) the total quotient ring of T is T itself.

Now let M be a maximal prime of T such that $T_M \cong R$. We now show that M is an N-prime of 0 (the zero ideal) but not a (K)-prime of 0. Let S be the m.c. set of units in T. Then by (iii), $T_S \cong T$ and so every element in M is a zero divisor; thus MT_S ($\cong M$) is a $MxPD$ of OT_S ($\cong 0$), hence M is an (N)-prime of 0. It is now seen that M is not a (K)-prime of 0, for by (4) of Proposition 1, M is a (K)-prime of 0 if and only if MT_M is a $MxPD$ of OT_M. But $MT_M \cong R, OT_M \cong 0$, and since R is an integral domain MT_M is not a $MxPD$ of 0.

Example 2. Let us now examine Example 1.3 of [8]. Let K be a field of characteristic 0, and R the polynomial ring in infinitely many indeterminates x_i over the field K. Suppose A is the ideal generated by the products x_ix_j for $i \neq j$. It was shown that $M = (x_1, x_2, \ldots)$ is an (N)-prime of A, but not a (Bw)-prime. We shall show that M is a (K)-prime of A, thus showing that the (K) condition on prime overideals does not imply the (Bw) condition. To see this we show that $A(M) = A$ and $Z(A) = M$.

Let $t \notin A$, then $t = a_0 + \sum^{n(1)} a_{1i}x_1{}^i + \ldots + \sum^{n(m)} a_{mi}x_m{}^i + X$ where each monomial term in X has a factor of the form x_ix_j for $i \neq j$. Now since $t \notin A$, either $a_0 \neq 0$ or $a_{ij} \neq 0$ for some i, j. In the first case we have $t = a_0 + [x_i]$, where each monomial term in $[x_i]$ has a factor of x_i for some i. Now if $a_0 = 0$, let p, q be minimal positive integers such that $a_{ij} \neq 0$. In this case $t = a_{pq}x_p{}^q + \sum_{q+1}^{n(p)} a_{pi}x_p{}^i + \ldots + \sum^{n(m)} a_{mi}x_m{}^i + X$. Now for each $u \notin M$, $u = b + [x_i]$, so tu is either $a_0b + [x_i]$, which does not belong to A, or $a_{pq}bx_p{}^q + [x_i]$, which also does not belong to A as $[x_i]$ does not contain any monomial term of the form $cx_p{}^q$. This means $t \notin A(M)$, thus $A = A(M)$, as $A \subset A(M)$ is always true. Now, it is seen that $Z(A) = M$, for suppose $f \in M$, then $f = \sum_1^n q_ix_i$, where $q_i \in R$. But $x_{n+1} \notin A$, and $x_{n+1} f \in A$. This means that $M \subset Z(A(M))$, i.e., M is a (K)-prime of A.

3. Basic properties of (K)-primes.

As this note is concerned mostly with (K)-primes of an ideal, we make the following

Definition 2. If A is an ideal, then $k(A)$ is the collection of all (K)-primes of A.

It is well known that if P is a $MnPD$ of A, that $A(P)$ is a P-primary ideal and in this case $P = Z(A(P))$; thus, since every ideal has at least one $MnPD$, $k(A)$ is a nonempty set for any ideal A.

The following remarks are easily verified from the definitions:

1. If P is a prime containing an ideal A and $s \in Z(A(P))$, then $s \in Z(A)$.
2. If P is a prime containing an ideal A, then $x \in A(P)$ if and only if the ideal quotient $A: x$ ($= A: (x)$) is not contained in P.
3. If P is a prime containing A and $x \notin P$, then $A: x \subset A(P)$.

In order to obtain the converse to statement 3, we must restrict ourselves to (K)-primes, i.e., PROPOSITION 2. *If P is a prime containing an ideal A, then $P \in k(A)$ if and only if the following condition holds: $x \notin P$ if and only if $A:x \subset A(P)$.*

Proof. Suppose first the condition holds. Then for each $p \in P$, $A: p \not\subset A(P)$, so there exists $t \notin A(P)$ such that $tp \in A$, whence $p \in Z(A(P))$. So $P \in k(A)$. Conversely, suppose $P \in k(A)$. If $x \notin P$, then x is not related to $A(P)$. Therefore $A(P): x = A(P)$, and since $A \subset A(P)$ we have $A: x \subset A(P): x = A(P)$. Now, if $x \in P \backslash A$, then $x \in Z(A(P))$, so there exists $y \notin A(P)$ such that $xy \in A(P)$. Then there exists $r \notin P$ such that $rxy \in A$. But $yr \notin A(P)$, since $y \notin A(P)$ and $r \notin P$. Thus yr belongs to A: x, but not $A(P)$.

In Example 1, M was a maximal ideal which was not a (K)-prime of 0. However, since T is its own total quotient ring, M consists of zero divisors, and thus M is related to the zero ideal. So M is a $MxPD$ of 0 which is not a (K)-prime of 0. Thus, to distinguish between the two possibilities we make the following:

Definition 3. An ideal is said to have the *MBD property* (called an *MDB ideal*), if each of its *MxPD* are (K)-primes.

Definition 4. An ideal A is said to have the *FB property* (called an *FB ideal*), if it has only a finite number of (K)-primes.

We shall adopt the technique of extending concepts from ideals to rings in the following manner: if T is a property on ideals (e.g., the *MDB* property), then R has property T if and only if each ideal in R has property T (e.g., R is called a *MDB* ring if each ideal in R is a *MDB* ideal). Furthermore, following the lead of Krull, we call a ring in which every ideal has a finite primary decomposition a Lasker ring, more precisely:

Definition 5. An ideal A is called a *Lasker ideal*, if A has a finite primary decomposition, i.e., A is representable as an intersection of finitely many primary ideals.

We now list some of the results found in [**1**] which are relevant to our discussion.

PROPOSITION 3. *For any ideal A, $Z(A) = \bigcup P \ (P \in k(A))$.*

PROPOSITION 4. *For any ideal A, $A = \bigcap A(P) \ (P \in k(A))$.*

PROPOSITION 5. *For any ideal A and prime P containing A, $k(A(P)) = \{P^* \in k(A) | P^* \subset P\}$.*

We close this section with a remark explaining why we have talked mainly about the P-components with respect to primes which contain the ideal A.

Remark. If R is a commutative ring with unity, then $A(P) = R$ for a prime P, if and only if $A \not\subset P$.

Proof. If $A \not\subset P$, then $A \cap (R \backslash P) \neq \emptyset$, so $A^e = R_P$, whence $A^{ec} = R$. Conversely if $A(P) = R$, then $1 \in A(P)$, so there exists $x \notin P$ such that $x \cdot 1 \in A$.

4. (K)-Primes and intersections. Krull observed that if P is a $MnPD$ of A, then the P-component ideal $A(P)$ is P-primary, and in fact, the intersection of all the P-primary ideals which contain A. We now show that the converse is also valid, i.e.,

PROPOSITION 6. *If A is an ideal, then P is a $MnPD$ of A if and only if $A(P)$ is a P-primary ideal.*

Proof. If P is an isolated prime divisor of A, then [**2**, p. 737] shows that $A(P)$ is P-primary. Conversely, if $A(P)$ is P-primary, then P is an isolated prime divisor of A. For if not, then there exists a prime ideal M such that $A \subset M < P$, but then we have $A \subset A(P) \subset A(M) \subset M < P$, and this is impossible unless $M = P$, since $\sqrt{A(P)} = P$.

We now record a useful property that will be used in the sequel.

PROPOSITION 7. *Let $A, B_1, \ldots, B_n$ be ideals in R such that $A = \bigcap^n B_i$. Then for any prime ideal P, $A(P) = \bigcap^n B_i(P)$.*

Proof. An element-wise proof can be constructed. Alternatively the result follows by recalling that $AR_P \cap BR_P = (A \cap B)R_P$ and $A^c \cap B^c = (A \cap B)^c$.

With the aid of the previous observation we can produce two "carry over" results with regard to ideal intersections.

PROPOSITION 8. *If P is a $MnPD$ of both A and B, then P is a (K)-prime of $A \cap B$.*

Proof. If P is an isolated prime of both A and B, then P is an isolated prime of $A \cap B$, hence a (K)-prime of $A \cap B$.

PROPOSITION 9. *If P is a prime such that $P \in k(A)$ and does not contain the ideal B, then $P \in k(A \cap B)$.*

Proof. Since P does not contain B, $B(P) = R$. Thus $(A \cap B)(P) = A(P)$ and so $P \in k(A \cap B)$.

PROPOSITION 10. *For any prime P containing an ideal A, $A(P) = \bigcap A(P^*)$ $(P^* \in k(A)$ and $P^* \subset P)$.*

Proof. The proof follows from Propositions 4 and 5.

We now record another result of Krull.

PROPOSITION 11. *Suppose $A \subset B$ are ideals such that each $MxPD$ of B is contained in a prime P, then $A(P) \subset B$.*

Proof. The result follows from [**2**, p. 733].

Along this same line of thought we have

Proposition 12. *If P is a prime in $k(A)$, and B is an ideal contained between A and $A(P)$, then P is contained in a MxPD of B.*

Proof. Let $x \in P$. Then $x \in Z(A(P))$, so there exists $y \notin A(P)$ such that $xy \in A(P)$. If $xy \in B$, we are through; so assume not. Then there exists $r \notin P$ such that $xyr \in A$. If $yr \notin B$, we are through; so suppose $yr \in B \subset A(P)$. Then there exists $q \notin P$ such that $yrq \in A$; but when r and q do not belong to P, then neither does their product, but this in turn makes $y \in A(P)$, which is a contradiction; so $yr \notin B$ and thus x is an element of $Z(B)$.

It is easy to see that the converses of Propositions 11 and 12 are not true. For let K be a field and $R = K[x, y]$. Suppose $A = (x^2)$, $B = (x^2, xy)$, and $P = (x)$. Then $A(P) = A$ and so $A(P) \subset B$. But y is related to B (since $x \notin B$ and $xy \in B$) and not in P. This shows that the converse of Proposition 11 does not hold, since B must have a $MxPD$ which is not contained in P. Now let A, B, and P be as above. Then $B = (x) \cap (x^2, xy, y^2)$. If $P^* = (x, y)$, then $A \subset B \subset P \subset P^*$. But since $A = A(P)$, we do not have B contained in $A(P)$, yet P is contained in a $MxPD$ of B.

Proposition 13. *Let A be an ideal with finitely many MxPD's. Then $A = \bigcap A(P)$ (P MxPD of A) is an irredundant intersection.*

Proof. Suppose not, say $P_1, \ldots, P_n$ are $MxPD$ of A, $B = \bigcap_2^n A(P_i)$, and $B = A$. Let x be any element of P_1. Then $x \in Z(A)$, so there exists $y \notin A$ such that $xy \in A$. Since $A = B$, this means that $y \notin A(P_j)$ for some $j > 1$, whence $x \in Z(A(P_j))$, which is possible only if $x \in P_j$. So we conclude that $P_1 \subset \bigcup_2^n P_j$, which means $P_1 \subset P_k$ for some $k > 1$. This contradicts the fact that $P_1, \ldots, P_n$ are distinct $MxPD$ of A, thus $B \neq A$.

An immediate consequence of this proposition is the fact that when A has a finite number of $MxPD$, but more than one, then necessarily A is a reducible ideal. We can, however, drop the finiteness, as the following observations show:

Lemma. *If A is any ideal with more than one MxPD, then for any MxPD, P, $A \neq A(P)$.*

Proof. Suppose P and P^* are two $MxPD$ of an ideal A, then $P \not\subset P^*$, so there exists $x \in P \backslash P^*$. Then $x \in Z(A)$; but $x \notin Z(A(P^*))$, thus $A \neq A(P^*)$.

Proposition 14. *If A is an ideal with more than one MxPD, then A is an intersection (possibly infinite) of proper overideals.*

Proof. The proof is clear since $A = \bigcap_{MxPD} A(P)$ and $A(P) \neq A$ for each $MxPD$, P.

We close this section with a note concerning the $MxPD$ of an intersection of component ideals.

Lemma. *Suppose B and C are ideals with only a finite number of MxPD. If $A = B \cap C$, then $P \subset Z(A)$ implies $P \subset Z(B)$ or $P \subset Z(C)$.*

Proof. $Z(B) = \bigcup P^*(P^*, MxPD$ of $B)$ and $Z(C) = \bigcup P^*(P^*, MxPD$ of $C)$. Thus $Z(A) \subset Z(B) \cup Z(C)$; for $x \in Z(A)$ implies there exists $y \notin A$ such that $xy \in A$, and $y \notin A$ means $y \notin B$ or $y \notin C$, so that $x \in Z(B)$ or $x \in Z(C)$. Let $P \subset Z(A)$. Then $P \subset Z(B) \cup Z(C) = \bigcup P^*$ $(P^*, MxPD$ of B or $C)$. This is a finite union of primes so P is contained in at least one of them, i.e., $P \subset Z(B)$ or $P \subset Z(C)$.

PROPOSITION 15. *Let A be an ideal and $P_1, \ldots, P_n$ (K)-primes of A such that each is maximal (with respect to set containment) in the set $\{P_1, \ldots, P_n\}$. Then the MxPD of $B = \bigcap^n A(P_i)$ are precisely these P_i's.*

Proof. We need only consider the case for $n = 2$, as a simple application of mathematical induction can be applied for $n > 2$. Let $P \subset Z(B)$, then $P \subset Z(A(P_1))$ or $P \subset Z(A(P_2))$, which means that $P \subset P_1$ or $P \subset P_2$. Now since $P_i \in k(A)$ and $A \subset B \subset A(P_i)$, we have (Proposition 12) $P_i \subset Z(B)$. Thus the $MxPD$ of B are precisely P_1 and P_2.

5. ***MDB*** **ideals.** In this section we turn our attention to displaying properties which insure that an ideal be a MDB ideal. We first note that if A has a unique $MxPD$, say P, then $(R \backslash P) \cap Z(A) = \emptyset$, for if that were not so, there would be another $MxPD$ distinct from P. With this observation we have:

PROPOSITION 16. *If A is an ideal with a unique $MxPD$, then A is a MDB ideal.*

Proof. Let P be the unique $MxPD$ of A and assume the contrary, that is, there exists $p \in P$ such that $p \notin Z(A(P))$. Now $p \in Z(A)$, so that there exists $r \notin A$ such that $rp \in A \subset A(P)$. Since $p \notin Z(A(P))$, this means $r \in A(P)$, so there exists $w \notin P$ such that $wr \in A$. But this is not possible, since $r \notin A$, $wr \in A$, implies $w \in Z(A) \subset P$. Hence $p \in Z(A))$.

We have the following converse:

PROPOSITION 17. *Suppose A is an ideal and P is a (K)-prime of A such that $A = A(P)$, then P is a unique $MxPD$ of A.*

Proof. Since $A = A(P)$ it follows that $Z(A) = Z(A(P))$. But P is a (K)-prime, hence $Z(A(P)) = P$. Thus $Z(A) = P$, so that P is the only $MxPD$ of A.

PROPOSITION 18. *If A is an ideal such that $k(A)$ has only a finite number of maximal elements, then A is an MDB ideal.*

Proof. Let $P_1, \ldots, P_n$ be the maximal elements in $k(A)$ and suppose P is a $MxPD$ of A. Then $P \subset Z(A) = \bigcup^n P_i$, and so $P \subset P_i$ for some i. But since P is a $MxPD$ and $P \subset P_i$, we have $P = P_i$, whence P is a (K)-prime of A.

COROLLARY. *If A is a Lasker ideal, then A is a MDB ideal.*

PROPOSITION 19. *A is a MDB ideal if and only if for each pair of $MxPD$, P, P^*, $x \in P \cap P^*$ implies $x \in Z(A(P)) \cap Z(A(P^*))$.*

Proof. If A is a MDB ideal, the condition is clearly true, so suppose the condition holds and P is a $MxPD$ of A. Let x be any element of P. If there exists a $MxPD$ P^* of A such that $x \in P^*$, then by the condition, x belongs to $Z(A(P))$, so we may assume that no other $MxPD$ of A also contains x. Now $x \in Z(A)$ implies there exists u not belonging to A such that ux belongs to A. Since $x \notin P^*$ for any other $MxPD$, we have $u \in A(P^*)$. Hence $u \notin A(P)$, since $A = \bigcap A(P^*)$ (P^*, $MxPD$ of A). This then means that $x \in Z(A(P))$. But this is true for every $x \in P$, so that $P = Z(A(P))$.

PROPOSITION 20. *If A is an ideal with finitely many $MxPD$ such that for each $MxPD$ P, $A(P)$ also has finitely many $MxPD$, then A is a MDB ideal.*

Proof. $A = \bigcap_{MxPD} A(P^*)$, which is a finite intersection. Furthermore, for each $MxPD$ P^* of A, the $MxPD$ of $A(P^*)$ are contained in P^*. Now using the previous lemma, for each $MxPD$ P of A, $P \subset Z(A(P^*))$ for some P^*. But since P and P^* are $MxPD$ of A and $P \subset P^*$ we have $P = P^*$. This means that $P = Z(A(P))$.

6. Multiplicatively closed sets and component ideals. We shall now discuss some results pertaining to general m.c. (multiplicatively closed) sets and their associated components of an ideal. The following are immediate consequences of Propositions 1 and 2 of [**1**].

PROPOSITION 21. *If S is a m.c. set not intersecting an ideal A and P is any prime containing A which is disjoint from S, then $A_S(P) = A(P)$.*

PROPOSITION 22. *If S is a m.c. set not intersecting A, then $A_S = \bigcap A(P^*)$, where the P^* are the $MxPD$ of $A(S)$.*

Recall that if S_1 and S_2 are m.c. sets, then so is their intersection. Now suppose A is an ideal and $x \in A_{S_1 \cap S_2}$. Then by definition of the component ideal, there is $t \in S_1 \cap S_2$ such that $tx \in A$. But t belongs to S_1 and S_2, so x belongs to A_{S_1} and A_{S_2}, thus:

PROPOSITION 23. *If S_1 and S_2 are m.c. sets not intersecting and ideal A, then $A_{S_1 \cap S_2} \subset A_{S_1} \cap A_{S_2}$.*

If we consider the special case when the m.c. sets are complements of primes containing A, then we have the converse containment relation:

PROPOSITION 24. *If P_1 and P_2 are primes containing A, then letting $S = (P_1 \cup P_2)^c$, we have $A_S = A(P_1) \cap A(P_2)$. So if we call A_S, $A(P_1 \cup P_2)$, we have $A(P_1 \cup P_2) = A(P_1) \cap A(P_2)$.*

Proof. Since $S = P_1^c \cap P_2^c$, we have by the previous proposition, $A(P_1 \cup P_2) \subset A(P_1) \cap A(P_2)$. Now let x belong to $A(P_1) \cap A(P_2)$. Then for $i = 1, 2$ there exists t_i not belonging to P_i such that xt_i belongs to A. If $t_1 \notin P_2$ or

$t_2 \notin P_1$ we are done, so suppose not. Then it is seen that $t_1 + t_2$ does not belong to P_1 nor P_2, thus x belongs to $A(P_1 \cup P_2)$ since $x(t_1 + t_2) \in A$.

In fact, if S is a m.c. set, such that its set complement is a finite union of prime ideals ("finitely saturated"), we have

PROPOSITION 25. *Let $P_1, \ldots, P_n$ be a collection of prime ideals containing A and $S = \bigcap^n P_1^c$. Then $A(S) = \bigcap^n A(P_i)$.*

Proof. By Proposition 5, $A_S = \bigcap A(P)$, where the intersection is taken over the $MxPD$ of A_S. Now P is a $MxPD$ of A_S, $P \cap S = \emptyset$, so $P \subset \bigcup^n P_i$. But this means that $P \subset P_i$ for some i, which in turn means that $A(P_i) \subset A(P)$, so that we have $\bigcap^n A(P_i) \subset A(P_i) \subset A(P)$. But this is true for each $MxPD$, so that $\bigcap^n A(P_i) \subset \bigcap_{MxPD} A(P)$. Furthermore $A_S \subset A(P_i)$ for each i, so we have the statement $A_S \subset (\bigcap^n A(P_i)) \subset (\bigcap_{MxPD} A(P)) = A_S$.

We shall close our remarks with a result resembling the Cohen theorem (every ideal is finitely generated if and only if every prime ideal is finitely generated) in that in the special case of rings in which every ideal has only a finite number of $MxPD$, a property on the set complement of prime ideals guarantees it for all m.c. sets. To this end we make the following

Definition 6 (Krull). An ideal A, is called a q-ideal, if for each m.c. set S not intersecting A, there is an element x such that $A_S = A: x$.

Definition 7. An ideal A is called a strong k-ideal, if for each prime ideal P, there is an element s such that $A(P) = A: s$.

PROPOSITION 26. *Suppose A is a MDB ideal with finitely many MxPD. Then A is a q-ideal if and only if A is a strong k-ideal.*

Proof. Clearly A a q-ideal implies that A a strong k-ideal. So suppose S is a m.c. set not intersecting A. Then $A_S = \bigcap A(P_i)$ $(MxPD)$. But A is a strong k-ideal, so that for each i there exists x_i such that $A(P_i) = A: x_i$. Since R is a MDB ideal, $x_i \notin P_i$ (cf. Proposition 2). Now $\bigcap A(P_i) = \bigcap A: x_i = A: (x_1, \ldots, x_n)$, but $x_i \notin P_i$ implies $(x_1, \ldots, x_n)$ is not contained in $\bigcup P_i$. So let r belong to $(x_1, \ldots, x_n)$ but not to $\bigcup P_i$. It is now seen that $A_S = A: r$, since $A_S \subset A: r$ and $t \in A: r$ implies $t \in A(P_i)$ for each i, and so $t \in \bigcap A(P_i) = A_S$.

REFERENCES

1. B. Banaschewski, *On the components of ideals in commutative rings*, Arch. Math. V. *12* (1961), 22–29.
2. W. Krull, *Idealtheorie in ringen ohne Endlichkeitsbedingung*, Math. Ann. *101* (1929), 729–744.
3. ——— *Uber einen Hauptsatz der allgemeinen Idealtheorie*, S. B. Heidelberg Akad. Wiss. Abhandl., *2* (1929), 11–16.
4. ——— *Uber Laskersche Ringe*, Rend. Circ. Mat. Palermo, Ser 2, *7* (1958), 155–165.
5. N. H. McCoy, *Rings and ideals*, Carus Math. Monographs No. 8 (Mathematical Association of America, Menascha, Wis., 1948).

6. M. Nagata, *Some remarks on prime divisors,* Memoirs of the College of Science Univ. of Kyoto, Series A, vol. XXXIII, No. *2* (1960), 297–299.
7. ——— *Local rings* (Interscience Publishers, New York, 1962).
8. D. H. Underwood, *On some uniqueness questions in primary representations of ideals,* J. Math. Kyoto Univ., Vol. 9, No. *1* (1969), 69–94.
9. O. Zariski and P. Samuel, *Commutative algebra* Vol. I (D. Van Nostrand Company, Inc. Princeton, 1958).

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