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Hilbert Functions and Homogeneous Generic Forms II

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Abstract. In this paper we study how the Hilbert function changes when we factor out generic homogeneous forms of an arbitrary degree.

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1. Introduction

Let k be a field and $R = \oplus_{i=0}^{\infty} R_i$ a homogeneous k-algebra. Denote by $\mathrm{H}(R, d)$ the Hilbert function of R, i.e., $\mathrm{H}(R, d) = \dim_k R_d$ for $d \geqslant 0$. In this paper we study how the Hilbert function changes when we factor out generic homogeneous forms of an arbitrary degree.

In [11], Green gives an upper bound for $\mathrm{H}(R/hR, d)$ for all $d \geqslant 1$, where h is a generic linear form in R. He applies this to obtain a new short proof of Macaulay's Theorem [17] and Gotzmann's Persistence Theorem [10], which are fundamental theorems on Hilbert functions. The main result in [12] by Herzog and Popescu is the Theorem in the Introduction, which generalizes Green's Theorem to generic forms h of arbitrary degree s in the case when char $k = 0$. In Section 2, we present a new short proof of this theorem and generalize it to an arbitrary characteristic (see Theorem 2.4). In particular, it follows that all applications of Herzog–Popescu's Theorem given in [12, Sect. 4] also hold for an arbitrary characteristic.

Strongly stable ideals play a special role in the study of Hilbert functions (see Section 3 for the definition of a strongly stable ideal). The reason is that one can often apply Gröbner's basis techniques to reduce the general case to the study of Hilbert functions of strongly stable ideals (see e.g. [1–3, 7, 8, 12, 14, 18]). In the case of strongly stable ideals, we generalize in [9] Green's bound to polynomial rings Q with restricted powers of the variables (see Section 3 for a definition). This provides an upper bound for $\mathrm{H}(R/hR, d)$ when $R = Q/I$, I is a strongly stable ideal and h is a generic linear form. In Section 3, we generalize this result to generic homogeneous forms of an arbitrary degree (see Theorem 3.2).

Results on Hilbert functions have applications in combinatorics for describing f-vectors and h-vectors (cf. [5, 13, 19]), and in algebraic geometry (cf. [4]).

2. Homogeneous Generic Forms in Arbitrary Characteristic

In order to state the results precisely, we need some notation. Let a and d be positive integers. There exist unique integers $m_d, m_{d-1}, \dots, m_\delta$ such that $m_d > m_{d-1} > \cdots > m_\delta \geqslant \delta$ and

$$a = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta}. \tag{1}$$

We call (1) the dth *Macaulay expansion* of a. Define an operator $a_{\langle d\rangle}$ on nonnegative integers a as follows. If $a = 0$, then $a_{\langle d\rangle} = 0$. If $a > 0$ and the dth Macaulay expansion of a is given by (1), then set

$$a_{\langle d\rangle} = \binom{m_d - 1}{d} + \binom{m_{d-1} - 1}{d-1} + \cdots + \binom{m_\delta - 1}{\delta}. \tag{2}$$

The right-hand side of (2) is not necessarily the dth Macaulay expansion of $a_{\langle d\rangle}$.

Green [11] shows that if h is a generic linear form in R, then for all $d \geqslant 1$ we have the inequality $\mathrm{H}(R/hR, d) \leqslant \mathrm{H}(R, d)_{\langle d\rangle}$. In the case when char $k = 0$, Herzog and Popescu [12] generalize Green's Theorem to generic forms h of arbitrary degree s. In order to state their theorem, we need to introduce more operators on nonnegative integers. Let $d > i \geqslant 0$. Define $a_{\langle d,i\rangle} = 0$ if $a = 0$ and

$$a_{\langle d,i\rangle} = \binom{m_d - i - 1}{d-i} + \binom{m_{d-1} - i - 1}{d-i-1} + \cdots + \binom{m_t - i - 1}{t-i}$$

if a is positive with dth Macaulay expansion given by (1), where $t = j$ if $j > i$ and $t = i + 1$ if $j \leqslant i$. We also define

$$a_{\langle\langle d,i\rangle\rangle} = a_{\langle d,i\rangle} + \binom{m_i - i}{0},$$

where $\binom{m_i - i}{0} = 1$ if $m_i \geqslant i$ and 0 otherwise.

Herzog and Popescu prove for all $d \geqslant s$ the upper bound $\mathrm{H}(R/hR, d) \leqslant \sum_{i=0}^{s-1} \mathrm{H}(R, d)_{\langle\langle d,i\rangle\rangle}$.

If R is isomorphic to a quotient of a polynomial ring $S = k[x_1, \dots, x_n]$ modulo a lexicographic ideal, then on the one hand we have that the inequalities in Green's and Herzog–Popescu's theorems become equalities, and on the other hand x_n^s is generic for R by [12, Prop. 1.4]. Also, it follows from Macaulay's Theorem [5, 13, 17, 19] that for any homogeneous ideal there exists a lexicographic ideal with the same Hilbert function. Therefore we can restate Green's Theorem and the main result of Herzog and Popescu in [12] as in Theorem 2.2 below.

Remark 2.1. The numerical inequalities in the original Green's and Herzog–Popescu's theorems are for $d \geqslant s$ and, in general, they are not true for $0 \leqslant d \leqslant s-1$. However, the inequalities in Theorem 2.2 hold for $0 \leqslant d \leqslant s-1$ because (in the hypothesis of the theorem) we have

$$\mathrm{H}(S/(I,h),d) = \mathrm{H}(S/I,d) = \mathrm{H}(S/L,d) = \mathrm{H}(S/(L,x_n^s),d).$$

THEOREM 2.2. *Let $I, L \subseteq S$ be homogeneous ideals with the same Hilbert function, such that L is lexicographic. Let h be a generic homogeneous form of degree $s \geqslant 1$.*

(1) [11, Thm. 1] *If $s = 1$, then* $\mathrm{H}(S/(I,h),d) \leqslant \mathrm{H}(S/(L,x_n),d)$ *for all* $d \geqslant 0$.
(2) [12, Thm. in Introduction] *If s is arbitrary and* char $k = 0$, *then for all* $d \geqslant 0$ *we have* $\mathrm{H}(S/(I,h),d) \leqslant \mathrm{H}(S/(L,x_n^s),d)$.

We present a new short proof of Theorem 2.2(2) and generalize it to arbitrary characteristic. We first prove the following simple lemma:

LEMMA 2.3. *Let T be a graded commutative finitely generated algebra over a field. For any homogeneous ideal $J \subseteq T$ and homogeneous form $g \in T_s$, $s \geqslant 1$, we have*

$$\mathrm{H}((J\colon g)/(0:g),d)$$
$$= \mathrm{H}(T/(0:g),d) - \mathrm{H}(T/J,d+s) + \mathrm{H}(T/(J,g),d+s) \quad \textit{for} \quad d \geqslant 0.$$

Proof. The exact sequence

$$0 \to (J\colon g)/(0:g) \to T/(0:g) \xrightarrow{g} T/J \to T/(J,g) \to 0$$

and the additivity property of the Hilbert function imply the desired result. □

THEOREM 2.4. *Let k be a field of any characteristic and $I, L \subseteq S$ homogeneous ideals with the same Hilbert function, such that L is lexicographic. If h is a generic homogeneous form for I of degree s, then for all $d \geqslant 0$ we have* $\mathrm{H}(S/(I,h),d) \leqslant \mathrm{H}(S/(L,x_n^s),d)$.

Proof. Let f_1 be a generic linear form for I. Then for all $d \geqslant 0$ Green's Theorem provides the upper bound

$$\mathrm{H}(S/(I,f_1),d) \leqslant \mathrm{H}(S/(L,x_n),d). \tag{3}$$

By Lemma 2.3 it follows that (3) is equivalent to the inequality

$$\mathrm{H}((I\colon f_1),d) \leqslant \mathrm{H}((L\colon x_n),d) \tag{4}$$

for $d \geqslant 0$. Note that $(L\colon x_n)$ is again a lexicographic ideal. By Macaulay's Theorem [17] and (4) we can choose a lexicographic ideal K such that $K \subseteq (L\colon x_n)$ and

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$\mathrm{H}((I\colon f_1), d) = \mathrm{H}(K, d)$ for all d. Let f_2 be a generic linear form for $(I\colon f_1)$. Applying Green's Theorem to the ideal $(I\colon f_1)$ we get the inequality

$$\mathrm{H}(((I\colon f_1)\colon f_2), d) \leqslant \mathrm{H}((K\colon x_n), d) \tag{5}$$

for $d \geqslant 0$. Note that for any ideal $J \subseteq S$ and elements $g_1, g_2 \in S$ we have the equality $((J\colon g_1)\colon g_2) = (J\colon g_1 g_2)$. Also, since $K \subseteq (L\colon x_n)$, it follows that $(K\colon x_n) \subseteq ((L\colon x_n)\colon x_n)$. Therefore we obtain from (5) that

$$\begin{aligned} \mathrm{H}((I\colon f_1 f_2), d) &= \mathrm{H}(((I\colon f_1)\colon f_2), d) \leqslant \mathrm{H}((K\colon x_n), d) \\ &\leqslant \mathrm{H}(((L\colon x_n)\colon x_n), d) \leqslant \mathrm{H}((L\colon x_n^2), d) \end{aligned}$$

for $d \geqslant 0$. Proceeding in this way we can find linear forms $f_1, \ldots, f_s$, such that

$$\mathrm{H}((I\colon f_1 \ldots f_s), d) \leqslant \mathrm{H}((L\colon x_n^s), d)$$

for $d \geqslant 0$. (Here for $2 \leqslant i \leqslant s$ we choose f_i to be a generic linear form for the ideal $(I\colon f_1 \ldots f_{i-1})$.) Again by Lemma 2.3 this is equivalent to

$$\mathrm{H}(S/(I, f_1 \ldots f_s), d) \leqslant \mathrm{H}(S/(L, x_n^s), d)$$

for $d \geqslant 0$. Since h is generic for I, it follows that

$$\mathrm{H}(S/(I, h), d) \leqslant \mathrm{H}(S/(I, f_1 \ldots f_s), d) \leqslant \mathrm{H}(S/(L, x_n^s), d)$$

for $d \geqslant 0$, which completes the proof of Theorem 2.4. □

3. Polynomial Rings with Restricted Powers of the Variables

Let $2 \leqslant a_1 \leqslant \cdots \leqslant a_n \leqslant \infty$ and set $Q = k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$, where $x_i^\infty = 0$ for $1 \leqslant i \leqslant n$. We say that Q is a *polynomial ring with restricted powers of the variables*. As in the case of polynomial rings, it is well known that for any homogeneous ideal in Q, there exists a lexicographic ideal with the same Hilbert function [6, 15, 16].

For a monomial $m \in Q$, denote by $\phi(m)$ the largest index of a variable appearing in m. An ideal I in Q is called *stable* if I is generated by monomials and for any monomial $m \in I$ we have $x_i m / x_{\phi(m)} \in I$ for $1 \leqslant i \leqslant \phi(m)$. The ideal I is called *strongly stable* if it is monomial and $x_i m / x_j \in I$ whenever m is a monomial in I, $x_j | m$, and $i \leqslant j$. In the case when I is strongly stable, Theorem 2.2(1) was generalized in [9] to the rings Q. More precisely, we have

THEOREM 3.1 ([9, Thm. 2.1(1)]). *Let $I, L \subseteq Q$ be homogeneous ideals with the same Hilbert function, such that I is strongly stable and L is lexicographic. Then*

$$\mathrm{H}(Q/(I, x_n), d) \leqslant \mathrm{H}(Q/(L, x_n), d) \quad \textit{for all} \quad d \geqslant 0.$$

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We generalize Theorem 3.1 to generic homogeneous forms of arbitrary degree:

THEOREM 3.2. *Let $I, L \subseteq Q$ be homogeneous ideals with the same Hilbert function, such that I is strongly stable and L is lexicographic. If h is a homogeneous form of degree s which is generic for I, then*

$$\mathrm{H}(Q/(I,h),d) \leqslant \mathrm{H}(Q/(L,x_n^s),d) \quad \textit{for all} \quad d \geqslant 0.$$

Proof. It suffices to prove the theorem when we replace h by x_n^s. The proof is by induction on s. If $s = 1$, then Theorem 3.2 follows from Theorem 3.1. Now assume that we have already proved that $\mathrm{H}(Q/(I, x_n^{s-1}), d) \leqslant \mathrm{H}(Q/(L, x_n^{s-1}), d)$ for $d \geqslant 0$, which by Lemma 2.3 is equivalent to

$$\mathrm{H}((I : x_n^{s-1})/x_n^{a_n-s+1}, d) \leqslant \mathrm{H}((L : x_n^{s-1})/x_n^{a_n-s+1}, d) \tag{6}$$

for $d \geqslant 0$. Let $K \subseteq (L : x_n^{s-1})/x_n^{a_n-s+1}$ be a lexicographic ideal such that $\mathrm{H}(K,d) = \mathrm{H}((I : x_n^{s-1})/x_n^{a_n-s+1}, d)$ for $d \geqslant 0$; the existence of such an ideal follows from (6) and the Clements–Lindström Theorem [6]. The ideal $(I : x_n^{s-1})/x_n^{a_n-s+1} \subseteq Q/x_n^{a_n-s+1}$ is strongly stable, so applying Theorem 3.1 to it we obtain

$$\mathrm{H}(((I : x_n^{s-1}) : x_n)/x_n^{a_n-s}, d) \leqslant \mathrm{H}((K : x_n)/x_n^{a_n-s}, d). \tag{7}$$

Now we proceed as in the proof of Theorem 2.4. Namely, since $K \subseteq (L : x_n^{s-1})/x_n^{a_n-s+1}$, it follows that $(K : x_n)/x_n^{a_n-s} \subseteq ((L : x_n^{s-1}) : x_n)/x_n^{a_n-s}$. Also, since for any ideal $J \subseteq Q$ the equality $(J : x_n^s) = ((J : x_n^{s-1}) : x_n)$ holds, we obtain from (7) that for $d \geqslant 0$

$$\begin{aligned} &\mathrm{H}((I : x_n^s)/x_n^{a_n-s}, d) \\ &\quad = \mathrm{H}(((I : x_n^{s-1}) : x_n)/x_n^{a_n-s}, d) \leqslant \mathrm{H}((K : x_n)/x_n^{a_n-s}, d) \\ &\quad \leqslant \mathrm{H}(((L : x_n^{s-1}) : x_n)/x_n^{a_n-s}, d) = \mathrm{H}((L : x_n^s)/x_n^{a_n-s}, d). \end{aligned} \tag{8}$$

Lemma 2.3 and (8) imply that the desired inequality

$$\mathrm{H}(Q/(I, x_n^s), d) \leqslant \mathrm{H}(Q/(L, x_n^s), d)$$

holds for $d \geqslant 0$. □

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