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Essential Commutants of Semicrossed Products

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Abstract. Let $\alpha\colon G \curvearrowright M$ be a spatial action of a countable abelian group on a "spatial" von Neumann algebra M and let S be its unital subsemigroup with $G = S^{-1}S$. We explicitly compute the essential commutant and the essential fixed-points, modulo the Schatten p-class or the compact operators, of the w*-semicrossed product of M by S when M' contains no non-zero compact operators. We also prove a weaker result when M is a von Neumann algebra on a finite dimensional Hilbert space and $(G, S) = (\mathbb{Z}, \mathbb{Z}_+)$, which extends a famous result due to Davidson (1977) for the classical analytic Toeplitz operators.

1 Introduction

Let A be a (not necessarily self-adjoint) operator algebra on a Hilbert space $\mathcal{H}$. We denote by $\mathfrak{S}_p = \mathfrak{S}_p(\mathcal{H})$ the Schatten p-class operators on $\mathcal{H}$ with $1 \le p < \infty$ and by $\mathfrak{S}_\infty = \mathfrak{S}_\infty(\mathcal{H})$ the compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H})$ on $\mathcal{H}$. We also denote by $\mathcal{I}(A)$ the set of all isometries in A. In this paper we investigate the following two sets:

$$\mathrm{Esscom}_p(A) = \{X \in \mathcal{B}(\mathcal{H}) \mid aX - Xa \in \mathfrak{S}_p(\mathcal{H}) \text{ for } a \in A\},$$
$$\mathrm{Essfix}_p(A) = \{X \in \mathcal{B}(\mathcal{H}) \mid v^*Xv - X \in \mathfrak{S}_p(\mathcal{H}) \text{ for } v \in \mathcal{I}(A)\},$$

called the *essential commutant* and the *essential fixed-points* of A modulo the $*$-ideal $\mathfrak{S}_p$, respectively. Clearly, $\mathrm{Esscom}_p(A)$ is contained in $\mathrm{Essfix}_p(A)$, and these two sets coincide when A is a C*-algebra that contains the identity operator, since any unital C*-algebra is the linear span of its unitary elements. Johnson and Parrott [8] and Popa [13] proved that $\mathrm{Esscom}_\infty(A) = A' + \mathcal{K}$ holds when A is a von Neumann algebra, and Hoover [7] proved that $\mathrm{Esscom}_p(A) = A' + \mathfrak{S}_p$ when A is a C*-algebra but $p \neq \infty$. On the other hand, for a non-self-adjoint algebra these two sets do not coincide in general, and thus the computation of them is non-trivial. Such non-trivial results, among others, are $\mathrm{Esscom}_\infty(T(H^\infty)) = T(H^\infty + C) + \mathcal{K}(H^2)$, due to Davidson [2] and $\mathrm{Essfix}_\infty(T(H^\infty)) = T(L^\infty) + \mathcal{K}(H^2)$, due to Xia [16], where $T(H^\infty)$, $T(H^\infty + C)$, and $T(L^\infty)$ are the sets of all Toeplitz operators on the Hardy space $H^2 = H^2(\mathbb{T})$ whose symbols are in the bounded Hardy space $H^\infty = H^\infty(\mathbb{T})$, the Douglas algebra $H^\infty + C$ (with $C = C(\mathbb{T})$), and $L^\infty = L^\infty(\mathbb{T})$, respectively (see [3, Chapters 6 and 7]).

In this paper, we study the essential commutant and the essential fixed-points for w*-semicrossed products. The notation below follows [9]. Let M be a von Neumann

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algebra on a Hilbert space $\mathcal{H}$, let G be a countable abelian group acting on M by α, and let S be a subsemigroup of G which contains the unit of G and generates G. We also assume that the action $\alpha\colon G \curvearrowright M$ is *spatial*; that is, α is implemented by a unitary representation $\{u_g \mid g \in G\}$ of G on $\mathcal{H}$. We denote by the same symbol α the action of G on M' implemented by $\{u_g \mid g \in G\}$. Then we can construct the left and right w*-crossed products $G\bar{\ltimes}_\alpha M$ and $M'\bar{\rtimes}_\alpha G$ on $\mathbb{L}^2 := \mathcal{H} \otimes \ell^2(G)$ with $(G\bar{\ltimes}_\alpha M)' = M'\bar{\rtimes}_\alpha G$. The left and right reduced w*-semicrossed products $S\bar{\ltimes}_\alpha M$ and $M\bar{\rtimes}_\alpha S$ are constructed as σ-weakly closed subalgebras of $G\bar{\ltimes}_\alpha M$ and $M\bar{\rtimes}_\alpha G$, respectively. Let P denote the orthogonal projection from $\mathbb{L}^2$ onto $\mathbb{H}^2 := \mathcal{H} \otimes \ell^2(S)$ and define the Toeplitz map $T\colon \mathcal{B}(\mathbb{L}^2) \to \mathcal{B}(\mathbb{H}^2)$ by $T_X := PX|_{\mathbb{H}^2}$ for $X \in \mathcal{B}(\mathbb{L}^2)$. The left and the right w*-semicrossed products $S\bar{\times}_\alpha M$ and $M\bar{\times}_\alpha S$ are constructed as σ-weakly closed subalgebras of $\mathcal{B}(\mathbb{H}^2)$, and in fact, coincide with $T(S\bar{\ltimes}_\alpha M)$ and $T(M\bar{\rtimes}_\alpha S)$, respectively (see Proposition 2.6), where, for a subset $\mathfrak{F}$ of $\mathcal{B}(\mathbb{L}^2)$, the $T(\mathfrak{F})$ stands for $\{T_F \mid F \in \mathfrak{F}\}$. In the typical case of $(M, \mathcal{H}, G, S) = (\mathbb{C}, \mathbb{C}, \mathbb{Z}, \mathbb{Z}_+)$, the "left" and the "right" algebras $(G\bar{\ltimes}_\alpha M, S\bar{\ltimes}_\alpha M, S\bar{\times}_\alpha M)$ and $(M\bar{\rtimes}_\alpha G, M\bar{\rtimes}_\alpha S, M\bar{\times}_\alpha S)$ coincide and become $(L^\infty, H^\infty, T(H^\infty))$. The concept of w*-semicrossed products and reduced ones were introduced by Kakariadis [9] for σ-weakly closed operator algebras and their normal endomorphisms and for von Neumann algebras and their automorphisms, *i.e.*, the case of $(G, S) = (\mathbb{Z}, \mathbb{Z}_+)$, respectively. The latter coincides with the adjoint of analytic crossed products studied by McAsey, Muhly, and Saito [11]. Toeplitz operators associated with analytic crossed products were studied by Saito [14], and the algebras of analytic Toeplitz operators in this sense essentially coincide with w*-semicrossed products, as above.

In Section 2 we define these objects and prove some elementary properties. When M' contains no non-zero compact operators, we can explicitly compute $\mathrm{Esscom}_p(S\bar{\times}_\alpha M)$ and $\mathrm{Essfix}_p(S\bar{\times}_\alpha M)$ as follows.

Theorem 1.1 *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, let $\alpha\colon G \curvearrowright M$ be a spatial action of a discrete countable abelian group, and let S be a subsemigroup of G that contains the unit of G and generates G. If M' contains no non-zero compact operators, then*

$$\mathrm{Essfix}_p(S\bar{\times}_\alpha M) = T(M'\bar{\rtimes}_\alpha G) + \mathfrak{S}_p(\mathbb{H}^2),$$
$$\mathrm{Esscom}_p(S\bar{\times}_\alpha M) = M'\bar{\times}_\alpha S + \mathfrak{S}_p(\mathbb{H}^2)$$

hold for every $1 \le p \le \infty$.

Corollary 1.2 *The assertion of Theorem 1.1 holds when $\mathcal{H}$ is the standard Hilbert space $L^2(M)$ and M is either diffuse, $\mathcal{B}(\ell^2)$ of infinite dimension, or a (possibly infinite) direct sum of them.*

Another immediate corollary of Theorem 1.1 is the double commutant theorem for $S\bar{\times}_\alpha M$ and $S\bar{\times}_\alpha M'$ in the Calkin algebra under the assumption that $M \cap \mathcal{K} = M' \cap \mathcal{K} = \{0\}$. The proofs of Theorem 1.1 and Corollary 1.2 will be given in Section 3. In Section 4, we consider the case where $\mathcal{H}$ is finite dimensional (and hence so is M) and $(G, S) = (\mathbb{Z}, \mathbb{Z}_+)$, and can prove the following theorem.

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Theorem 1.3 *Let M be a von Neumann algebra on a finite dimensional Hilbert space $\mathcal{H}$ and let α be a $*$-automorphism on M implemented by a unitary operator on $\mathcal{H}$. Then we have*

$$\mathrm{Esscom}_p(\mathbb{Z}_+\bar{\ltimes}_\alpha M) \subset T(M'\bar{\rtimes}_\alpha\mathbb{Z}) + \mathfrak{S}_p(\mathbb{H}^2) \quad \text{for } 1 \le p \le \infty.$$

This theorem is just [2, Theorem 2] for the classical analytic Toeplitz operators specialized to $M = \mathbb{C}$ and $p = \infty$, though our proof is slightly improved with the help of several ideas in [12,16]. We do not consider the essential fixed-points because there is a technical obstruction to translating the argument in [16] into our setting (see Remark 4.7). In the final section, we examine whether or not the assertion of Theorem 1.3 still holds when M is an arbitrary von Neumann algebra of standard form.

2 Preliminaries

Remark 2.1 For a given $v \in \mathcal{I}(A)$ we consider the unital completely positive map $\Psi_v\colon X \mapsto v^*Xv$ on $\mathcal{B}(\mathcal{H})$. Then $\mathrm{Essfix}_p(A)$ is nothing but the set of operators "fixed modulo $\mathfrak{S}_p$" by $\{\Psi_v \mid v \in \mathcal{I}(A)\}$. It is immediate that $\mathrm{Esscom}_p(A) \subset \mathrm{Essfix}_p(A)$. Moreover, these two sets coincide if A is a C*-algebra, since any C*-algebra is generated by its unitary elements. By [7, 8, 13] any von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ satisfies $\mathrm{Esscom}_p(M) = \mathrm{Essfix}_p(M) = M' + \mathfrak{S}_p$ for every $1 \le p \le \infty$.

Next, let us recall w*-semicrossed products and the Toeplitz maps associated with them. Our notation and formulation follow [9]. Note that those do not completely agree with the usual ones for crossed products. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let G be a countable discrete abelian group. Assume that we have a spatial action $\alpha\colon G \curvearrowright M$ implemented by a unitary representation $u\colon G \ni g \mapsto u_g \in \mathcal{B}(\mathcal{H})$; *i.e.*, we have $\alpha_g(x) = u_g x u_g^*$ for every $x \in M$ and $g \in G$. Since G acts on M' too by $\mathrm{Ad}\, u_g$, we also denote by the same symbol α_g the automorphism $\mathrm{Ad}\, u_g$ on M'. Let $G \ni g \mapsto \lambda_g \in \mathcal{B}(\ell^2(G))$ be the left regular representation of G and let $e_g \in \mathcal{B}(\ell^2(G))$ be the orthogonal projection onto $\mathbb{C}\delta_g$. Set $\mathbb{L}^2 := \mathcal{H} \otimes \ell^2(G)$ and define representations $\pi\colon M \to \mathcal{B}(\mathbb{L}^2)$ and $\lambda(\,\cdot\,), \rho(\,\cdot\,)\colon G \to \mathcal{B}(\mathbb{L}^2)$ by

$$\pi(x) = \sum_{g\in G} \alpha_g(x) \otimes e_g, \quad \lambda(g) = 1 \otimes \lambda_g, \quad \rho(g) = u_g^* \otimes \lambda_g$$

for $x \in M$ and $g \in G$. The *left* and *right w*-crossed products* of M by G are defined to be the von Neumann algebras $G\bar{\ltimes}_\alpha M = \pi(M) \vee \{\lambda(g) \mid g \in G\}''$ and $M\bar{\rtimes}_\alpha G := M \otimes \mathbb{C}1 \vee \{\rho(g) \mid g \in G\}''$, respectively. It is well known that $(G\bar{\ltimes}_\alpha M)' = M'\bar{\rtimes}_\alpha G$ (see *e.g.*, [15, Theorem 1.21]). We also note that when M is of standard form, each action $\alpha\colon G \curvearrowright M$ is spatial thanks to [4, Theorem 3.2].

Definition 2.1 (Kakariadis [9]) Let $M \subset \mathcal{B}(\mathcal{H})$ and let $\alpha\colon G \curvearrowright M$ be as above. For a given subsemigroup S of G that contains the unit ι of G and generates G, the *left* and *right reduced w*-semicrossed product* $S\bar{\ltimes}_\alpha M$ and $M\bar{\rtimes}_\alpha S$ of M by S are defined to be the σ-weakly closed subalgebras of $G\bar{\ltimes}_\alpha M$ and $M\bar{\rtimes}_\alpha G$ generated by $\pi(M)$ and $\{\lambda(s) \mid s \in S\}$ and by $M \otimes \mathbb{C}1$ and $\{\rho(s) \mid s \in S\}$, respectively.

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Let P be the projection onto $\mathbb{H}^2 := \mathcal{H}\otimes\ell^2(S) \subset \mathbb{L}^2$. The *Toeplitz map* $T\colon \mathcal{B}(\mathbb{L}^2) \to \mathcal{B}(\mathbb{H}^2)$ is defined by $T_X := PX|_{\mathbb{H}^2}$ for $X \in \mathcal{B}(\mathbb{L}^2)$. The *left* and *right* w^**-semicrossed product* $S\bar{\ltimes}_\alpha M$ and $M\bar{\rtimes}_\alpha S$ of M by S are defined to be σ-weakly closed subalgebras of $\mathcal{B}(\mathbb{H}^2)$ generated by $T(\pi(M))$ and $\{T_{\lambda(s)} \mid s \in S\}$ and by $M \otimes \mathbb{C}1_{\ell^2(S)}$ and $\{T_{\rho(s)} \mid s \in S\}$, respectively. Here, for a subset $\mathfrak{F}$ of $\mathcal{B}(\mathbb{L}^2)$ the $T(\mathfrak{F})$ stands for $\{T_F \mid F \in \mathfrak{F}\}$.

Remark 2.2 Since T is normal and multiplicative on $S\bar{\ltimes}_\alpha M$ and $M\bar{\rtimes}_\alpha S$, one has

$$T(S\bar{\ltimes}_\alpha M) \subset S\bar{\ltimes}_\alpha M \subset \overline{T(S\bar{\ltimes}_\alpha M)}^{\sigma\text{-w}} \quad \text{and} \quad T(M\bar{\rtimes}_\alpha S) \subset M\bar{\rtimes}_\alpha S \subset \overline{T(M\bar{\rtimes}_\alpha S)}^{\sigma\text{-w}}.$$

Moreover, for any $x, z \in (S\bar{\ltimes}_\alpha M) \cup (M\bar{\rtimes}_\alpha S)$ and $Y \in \mathcal{B}(\mathbb{L}^2)$, we have $T_x^* T_Y T_z = T_{x^*Yz}$. We also note that P is in $\pi(M)'$, and hence one has

$$\mathrm{Essfix}_p\big(T(\pi(M))\big) = \mathrm{Esscom}_p\big(T(\pi(M))\big) = T(\pi(M)') + \mathfrak{S}_p.$$

We will use these facts throughout.

Remark 2.3 Define a unitary operator $W := \sum_{g\in G} u_g \otimes e_g$ on $\mathbb{L}^2$. It is easily seen that $W^*\pi(x)W = x \otimes 1$ and $W^*\lambda(g)W = \rho(g)$ for $x \in M$ and $g \in G$. Moreover, since W commutes with P, $\widehat{W} := T_W$ is also unitary on $\mathbb{H}^2$, and hence one has $\widehat{W}^* T_{\pi(x)} \widehat{W} = x \otimes 1_{\ell^2(S)}$ and $\widehat{W}^* T_{\lambda(g)} \widehat{W} = T_{\rho(g)}$ for $x \in X$ and $g \in G$. Therefore, the "left" and "right" algebras are unitarily equivalent.

Proposition 2.4 *Let M, G, S, α be as in Definition 2.1. Then we have*

$$S\bar{\ltimes}_\alpha M = \{x \in G\bar{\ltimes}_\alpha M \mid x(1 \otimes e_\iota) = Px(1 \otimes e_\iota)\},$$
$$M\bar{\rtimes}_\alpha S = \{x \in M\bar{\rtimes}_\alpha G \mid x(1 \otimes e_\iota) = Px(1 \otimes e_\iota)\}.$$

Proof Put $\Omega := \{x \in G\bar{\ltimes}_\alpha M \mid x(1\otimes e_\iota) = Px(1\otimes e_\iota)\}$. Clearly, $S\bar{\ltimes}_\alpha M$ is contained in Ω. Let $\widehat{G}$ be the dual group of G equipped with the normalized Haar measure μ, and let $\widehat{\alpha}\colon \widehat{G} \to \mathrm{Aut}(G\bar{\ltimes}_\alpha M)$ be the dual action. By [10, Corollary 4.3.2], it suffices to show that for any $x \in \Omega$ its spectrum $\mathrm{sp}_{\widehat{\alpha}}(x)$ is contained in S (see [10, Definition 2.1]). Indeed, if g is in $G \setminus S$, then it is not hard to see that the function $\widehat{G} \ni \gamma \mapsto \gamma(g)^{-1} \in \mathbb{C}$ belongs to the annihilator of x. Hence, for any $h \in \mathrm{sp}_{\widehat{\alpha}}(x)$ we have $\int_{\widehat{G}} \gamma(g)^{-1}\gamma(h)d\mu(\gamma) = 0$, which implies $g \neq h$. Since $g \in G \setminus S$ is arbitrary, we have $h \in S$, and hence $\mathrm{sp}_{\widehat{\alpha}}(x)$ is contained in S. By the preceding remark, one has $M\bar{\rtimes}_\alpha S = \{x \in M\bar{\rtimes}_\alpha G \mid x(1 \otimes e_\iota) = Px(1 \otimes e_\iota)\}$. ∎

Recall that a semigroup S is *right amenable* if S has a *right invariant mean*; that is, there exists a state ψ on $\ell^\infty(S)$ that satisfies that $\psi(f) = \psi(r_s f)$ for $f \in \ell^\infty(S)$ and $s \in S$, where $r_s f(t) = f(ts)$, $t \in S$. It is known (see [5, Theorem 17.5]) that every abelian semigroup is (right) amenable.

Proposition 2.5 *If S is a right amenable semigroup and $\sigma\colon S \to \mathcal{B}(\mathcal{H})$ is a unitary representation, then $\{\sigma(s) \mid s \in S\}' \cap \overline{\mathrm{co}}^{\mathrm{w}}\{\sigma(s)^* x\sigma(s) \mid s \in S\} \neq \varnothing$ for every $x \in \mathcal{B}(\mathcal{H})$.*

Proof Let ψ be a right invariant mean on S. Fix $x \in \mathcal{B}(\mathcal{H})$. For $\xi \in \mathfrak{S}_1(\mathcal{H})$, define $f_\xi \in \ell^\infty(S)$ by $f_\xi(s) = \mathrm{Tr}(\sigma(s)^* x\sigma(s)\xi), s \in S$. Then there exists $y \in \mathcal{B}(\mathcal{H}) \cong$

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$\mathfrak{S}_1(\mathcal{H})^*$ such that $\mathrm{Tr}(y\xi) = \psi(f_\xi)$ for $\xi \in \mathfrak{S}_1(\mathcal{H})$. Since $f_{\sigma(s)\xi\sigma(s)^*} = r_s f_\xi$ for $s \in S$ and ψ is right invariant, we have $\mathrm{Tr}(\sigma(s)^* y\sigma(s)\xi) = \psi(f_{\sigma(s)\xi\sigma(s)^*}) = \psi(f_\xi) = \mathrm{Tr}(y\xi)$, which implies that $y \in \{\sigma(s) \mid s \in S\}'$. Suppose that $y \notin \overline{\mathrm{co}}^{\mathrm{w}}\{\sigma(s)^* x\sigma(s) \mid s \in S\}$. By the Hahn–Banach separation theorem, there exist $\xi \in \mathfrak{S}_1(\mathcal{H})$ and a constant $c \in \mathbb{R}$ such that $\mathrm{Re}\,\psi(f_\xi) = \mathrm{Re}\,\mathrm{Tr}(y\xi) < c \leq \mathrm{Re}\,\mathrm{Tr}(\sigma(s)^* x\sigma(s)) = \mathrm{Re}\, f_\xi(s)$ for $s \in S$. However, by the Kreĭn–Mil'man theorem, ψ belongs to the weak* closed convex hull of $S \subset \ell^\infty(S)^*$, a contradiction. ∎

The following proposition gives us a Brown–Halmos type criterion ([1, Theorems 6 and 7]).

Proposition 2.6 *Let $M \subset \mathcal{B}(\mathcal{H})$, $S \subset G$, α be as in Definition 2.1. Then the following are true.*

(i) *For a given $X \in \mathcal{B}(\mathbb{H}^2)$, X belongs to $T(M' \bar{\rtimes}_\alpha G)$ (resp. $T(G \bar{\ltimes}_\alpha M')$) if and only if X commutes with $T(\pi(M))$ (resp. $M \otimes \mathbb{C}1_{\ell^2(S)}$) and satisfies that $T^*_{\lambda(s)} X T_{\lambda(s)} = X$ (resp. $T^*_{\rho(s)} X T_{\rho(s)} = X$) for every $s \in S$.*

(ii) *$T(M' \bar{\rtimes}_\alpha S) = M' \bar{\times}_\alpha S = (S \bar{\ltimes}_\alpha M)'$ and $T(S \bar{\ltimes}_\alpha M') = S \bar{\times}_\alpha M' = (M \bar{\times}_\alpha S)'$.*

Proof Let $X \in T(\pi(M))'$ be arbitrarily chosen in such a way that $T^*_{\lambda(s)} X T_{\lambda(s)} = X$ for every $s \in S$. Since $T(\pi(M))' = T(\pi(M)')$, there exists $x \in \pi(M)'$ such that $X = T_x$. By Proposition 2.5, we find y in $\overline{\mathrm{co}}^{\mathrm{w}}\{\lambda(s)^* P x P \lambda(s) \mid s \in S\} \cap \{\lambda(s) \mid s \in S\}'$. Note that y is in $M' \bar{\rtimes}_\alpha G = (G \bar{\ltimes}_\alpha M)' = \pi(M)' \cap \{\lambda(g) \mid g \in G\}'$, since PxP is in $\pi(M)'$, the $\lambda(s), s \in S$, normalize $\pi(M)'$, and S generates G. Since the Toeplitz map T is σ-weakly continuous, one has $T_y \in \overline{\mathrm{co}}^{\mathrm{w}}\{T^*_{\lambda(s)} X T_{\lambda(s)} \mid s \in S\} = \{X\}$, which implies that $X = T_y \in T(M' \bar{\rtimes}_\alpha G)$. Conversely, let x be in $M' \bar{\rtimes}_\alpha G \subset \pi(M)'$. Then $T_x \in T(\pi(M)') = T(\pi(M))'$. For any $s \in S$, we have $T^*_{\lambda(s)} T_x T_{\lambda(s)} = T_{\lambda(s)^* x \lambda(s)} = T_x$, which implies (i).

To see (ii) it suffices to prove that $(S \bar{\ltimes}_\alpha M)' = T(M' \bar{\rtimes}_\alpha S)$ by Remark 2.2. It is immediate to see that $T(M' \bar{\rtimes}_\alpha S) \subset (S \bar{\ltimes}_\alpha M)'$. Conversely, let $Y \in (S \bar{\ltimes}_\alpha M)'$ be arbitrary. By the preceding paragraph, there exists $a \in M' \bar{\rtimes}_\alpha G$ such that $Y = T_a$. By the assumption that $G = S^{-1}S$, for each $g \in G \setminus S$ there exist $s, t \in S$ such that $g = s^{-1}t$. Since a commutes with $\lambda(s), s \in S$, we have

$$\begin{aligned}\left\langle a\xi \otimes \delta_\iota, \eta \otimes \delta_g \right\rangle &= \left\langle (1-P)aP\xi \otimes \delta_\iota, \eta \otimes \delta_{s^{-1}t} \right\rangle \\ &= \left\langle P\lambda(s)(1-P)aP\xi \otimes \delta_\iota, \eta \otimes \delta_t \right\rangle \\ &= \left\langle [T_a, T_{\lambda(s)}]\xi \otimes \delta_\iota, \eta \otimes \delta_t \right\rangle = 0\end{aligned}$$

for every $\xi, \eta \in \mathcal{H}$, and hence $(1 \otimes e_g)a(1 \otimes e_\iota) = 0$. Therefore, a belongs to $M' \bar{\rtimes}_\alpha S$ by Proposition 2.4. ∎

3 Proof of Theorem 1.1

Throughout this section, let $M \subset \mathcal{B}(\mathcal{H})$ denote a von Neumann algebra, $\alpha\colon G \curvearrowright M$ a spatial action of discrete countable abelian group, and S a subsemigroup of G that contains the unit ι of G and generates G.

Lemma 3.1 *If M' (resp. M) contains no non-zero compact operators, then so does $\pi(M)'$ (resp. $(M' \otimes \mathbb{C}1)'$).*

Proof Assume that $M' \cap \mathcal{K} = \{0\}$. By Remark 2.3 it suffices to prove that $(M \otimes \mathbb{C}1)' \cap \mathcal{K}(\mathbb{L}^2) = \{0\}$. Let $K \in (M \otimes \mathbb{C}1)' \cap \mathcal{K}(\mathbb{L}^2)$ be arbitrary chosen. Since $(M \otimes \mathbb{C}1)' = M' \bar{\otimes} \mathcal{B}(\ell^2(G))$, the $(1 \otimes \lambda_g^* e_g)K(1 \otimes e_h \lambda_h)|_{\mathcal{H} \otimes \mathbb{C}\delta_\iota}$ belongs to $(M' \otimes \mathbb{C}e_\iota) \cap \mathcal{K}(\mathcal{H} \otimes \mathbb{C}\delta_\iota) \cong M' \cap \mathcal{K}(\mathcal{H}) = \{0\}$ for every $g, h \in G$, which implies $K = 0$. ■

Lemma 3.2 *The restrictions of the Toeplitz map to $\{\lambda(g) \mid g \in G\}'$ and $\{\rho(h) \mid h \in G\}'$ are isometries. Consequently, every isometry in $S \bar{\times}_\alpha M$ and $M \bar{\times}_\alpha S$ is of the form T_v with some isometry v in $S \bar{\ltimes}_\alpha M$ and $M \bar{\rtimes}_\alpha S$, respectively.*

Proof First, we prove that the restriction of T to $\{\lambda(g) \mid g \in G\}'$ is injective. Let $x \in \{\lambda(g) \mid g \in G\}'$ be chosen in such a way that $T_x = 0$. Since S generates G and G is abelian, for each $g, h \in G$ there exist $s, s', t, t' \in S$ such that $g = t^{-1}s$ and $h = t'^{-1}s'$. Since x commutes with $\lambda(g), g \in G$, one has

$$\begin{aligned}\left\langle x(\xi \otimes \delta_{t^{-1}s}), \eta \otimes \delta_{t'^{-1}s'} \right\rangle &= \left\langle x(\xi \otimes \lambda_t^* \delta_s), \eta \otimes \lambda_{t'}^* \delta_{s'} \right\rangle = \left\langle x(\xi \otimes \lambda_{t'} \delta_s), \eta \otimes \lambda_t \delta_{s'} \right\rangle \\ &= \left\langle T_x(\xi \otimes \delta_{t's}), \eta \otimes \delta_{ts'} \right\rangle = 0\end{aligned}$$

for every $\xi, \eta \in \mathcal{H}$, and hence $x = 0$. By Proposition 2.5 there exists

$$y \in \overline{\mathrm{co}}^{\mathrm{w}}\{\lambda(s)^* P x P \lambda(s) \mid s \in S\} \cap \{\lambda(g) \mid g \in G\}'.$$

Note that $\|y\| \leq \|T_x\|$. Since $P\lambda(s)P = \lambda(s)P$ for $s \in S$, one has

$$T_y \in \overline{\mathrm{co}}^{\mathrm{w}}\{T_{\lambda(s)}^* T_x T_{\lambda(s)} \mid s \in S\} = \overline{\mathrm{co}}^{\mathrm{w}}\{T_{\lambda(s)^* x \lambda(s)} \mid s \in S\} = \{T_x\}.$$

Since T is injective on $\{\lambda(g) \mid g \in G\}'$, it follows that $x = y$. Thus, $\|x\| \leq \|T_x\| \leq \|x\|$, and hence the restriction of T to $\{\lambda(g) \mid g \in G\}'$ is isometric.

To see the second assertion let $V \in \mathcal{I}(S \bar{\times}_\alpha M)$ be arbitrary. By Proposition 2.6 we find $v \in S \bar{\ltimes}_\alpha M$ in such a way that $V = T_v$. Then one has $1 = V^*V = T_v^* T_v = T_{v^*v}$. Since T is injective on $S \bar{\ltimes}_\alpha M \subset \{\rho(g) \mid g \in G\}'$, v itself must be isometry. Similarly, one can prove the same assertion for $\{\rho(g) \mid g \in G\}'$ and $M \bar{\times}_\alpha S$. ■

By the preceding lemma, it is immediate to see that $T(M' \bar{\rtimes}_\alpha G) + \mathfrak{S}_p$ is contained in $\mathrm{Essfix}_p(S \bar{\times}_\alpha M)$. Hence the next theorem completes the proof of Theorem 1.1.

Theorem 3.3 *Assume that $M' \cap \mathcal{K} = \{0\}$. For a given $X \in \mathcal{B}(\mathbb{H}^2)$ the following are true.*

(i) *The X belongs to $T(M' \bar{\rtimes}_\alpha G) + \mathfrak{S}_p$ if and only if X commutes with $T(\pi(M))$ modulo $\mathfrak{S}_p$ and satisfies that $T_{\lambda(s)}^* X T_{\lambda(s)} - X \in \mathfrak{S}_p$ for every $s \in S$.*

(ii) *The X belongs to $M' \bar{\times}_\alpha S + \mathfrak{S}_p$ if and only if X commutes with $T(\pi(M))$ and $\{T_{\lambda(s)} \mid s \in S\}$ modulo $\mathfrak{S}_p$.*

Proof First, we prove the "if" part of (i). Let $X \in \mathrm{Esscom}_p(T(\pi(M)))$ be arbitrarily chosen in such a way that $T_{\lambda(s)}^* X T_{\lambda(s)} - X \in \mathfrak{S}_p$ for every $s \in S$. By Remark 2.2 there exist $a \in \pi(M)'$ and $K \in \mathfrak{S}_p$ such that $X = T_a + K$. Note that $(T_a - T_{\lambda(s)}^* T_a T_{\lambda(s)})P =$

$P(a - \lambda(s)^* a\lambda(s))P \in \pi(M)'$, since $a, P \in \pi(M)'$ and $\lambda(s), s \in S$, normalize $\pi(M)'$. For each $s \in S$ one has

$$(T_a - T^*_{\lambda(s)} T_a T_{\lambda(s)})P = (X - T^*_{\lambda(s)} X T_{\lambda(s)})P - (K - T^*_{\lambda(s)} K T_{\lambda(s)})P$$

belongs to $\mathcal{K} \cap \pi(M)'$ by assumption. Thus, it follows from Lemma 3.1 that $T^*_{\lambda(s)} T_a T_{\lambda(s)} = T_a$ for every $s \in S$, and hence we have $T_a \in T(M' \bar{\rtimes}_\alpha G)$ by Proposition 2.6. Therefore, the $X = T_a + K$ is in $T(M' \bar{\rtimes}_\alpha G) + \mathfrak{S}_p$. The "only if" part of (i) also follows from Proposition 2.6.

Next, we prove (ii). Since $M' \bar{\times}_\alpha S = (S \bar{\times}_\alpha M)'$, the "only if" part is trivial. By the preceding paragraph, it suffices to prove that every $T_y \in T(M' \bar{\rtimes}_\alpha G)$ commuting with $\{T_{\lambda(s)} \mid s \in S\}$ modulo $\mathfrak{S}_p$ belongs to $M' \bar{\times}_\alpha S = T(M' \bar{\rtimes}_\alpha S)$. By Proposition 2.4, it suffices to show that $(1 \otimes e_g) y (1 \otimes e_\iota) = 0$ for every $g \in G \setminus S$. Note that $(1 \otimes e_g) y (1 \otimes e_\iota) \in \pi(M)'$, since $(1 \otimes e_h) \in \pi(M)'$ for every $h \in G$. As in the proof of Proposition 2.6 we can find $s, t \in S$ in such a way that $g = s^{-1} t$, and have $\langle y\xi \otimes \delta_\iota, \eta \otimes \delta_g \rangle = \langle [T_y, T_{\lambda(s)}]\xi \otimes \delta_\iota, \eta \otimes \delta_t \rangle$ for every $\xi, \eta \in \mathcal{H}$, which implies that $(1 \otimes e_g) y (1 \otimes e_\iota)$ is compact, and hence $(1 \otimes e_g) y (1 \otimes e_\iota) \in \pi(M)' \cap \mathcal{K} = \{0\}$. Therefore, we get $T_y \in M' \bar{\times}_\alpha S$. ■

Proof of Corollary 1.2 By [4, Theorem 3.2] every action $\alpha\colon G \curvearrowright M$ is spatial. Since M is anti-spatially isomorphic to M', it suffices to prove that $\mathcal{K} \cap M = \{0\}$. We first prove the case when $M = \mathcal{B}(\ell^2)$. Since

$$\big(\mathcal{B}(\ell^2), L^2(\mathcal{B}(\ell^2))\big) \cong \big(\mathcal{B}(\ell^2) \otimes \mathbb{C}1, \ell^2 \otimes \overline{\ell^2}\big)$$

and ℓ^2 is infinite dimensional, we have $\mathcal{K} \cap (\mathcal{B}(\ell^2) \otimes \mathbb{C}1) = \{0\}$. When M is diffuse, it is clear that $\mathcal{K} \cap M = \{0\}$. The general case follows from [4, Lemma 2.6], which guarantees that each central projection $q \in M$ enjoys $(Mq, qL^2(M)) \cong (Mq, L^2(Mq))$. ■

4 Proof of Theorem 1.3

Let $(M, \mathcal{H}, \alpha)$ be as in Theorem 1.3. We first point out that the same assertions of the lemmas below hold true for $\rho(1)$, since $\lambda(1)$ and $\rho(1)$ are unitarily equivalent; see Remark 2.3. Remark that $\lambda(n)$ converge to 0 weakly, and hence $\lambda(n)^* K \lambda(n)$ converge to 0 strongly for every compact operator K. Also, note that $(1 - P)\lambda(n)$ converges to 0 strongly. These facts are frequently used throughout.

The following two lemmas seem well known, but we give their proofs for the reader's convenience.

Lemma 4.1 *There exists a $*$-isomorphism from $L^\infty = L^\infty(\mathbb{T})$ onto $\{\lambda(1)\}''$ sending z to $\lambda(1)$.*

Proof Let U be the bilateral shift on $L^2(\mathbb{T})$ with respect to the standard basis $\{z^n \mid n \in \mathbb{Z}\}$ and define the unitary transformation $V\colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ by $V\delta_n := z^n$, $n \in \mathbb{Z}$. Then one has $\lambda(1) = 1 \otimes \lambda_1 = 1 \otimes V^* U V$. Since the von Neumann algebra generated by U is known to be L^∞, the correspondence $L^\infty \ni f \mapsto 1 \otimes V^* f V \in \{\lambda(1)\}''$ obviously gives the desired $*$-isomorphism. ■

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We denote by Φ the quotient map from $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra. Recall that the *essential norm* $\|X\|_{\mathrm{e}}$ of $X \in \mathcal{B}(\mathcal{H})$ is defined to be

$$\|X\|_{\mathrm{e}} = \inf\{\|X - K\| \mid K \in \mathcal{K}(\mathcal{H})\}.$$

Lemma 4.2 *$[\lambda(1), P]$ is a compact operator and the mapping $C(\mathbb{T}) \ni f \mapsto \Phi(T_{\widehat{f}}) \in C^*(\Phi(T_{\lambda(1)}))$ gives a $*$-isomorphism, where $L^\infty \ni f \mapsto \widehat{f} \in \{\lambda(1)\}''$ is the $*$-isomorphism in the preceding lemma.*

Proof Since the range of $P\lambda(1)(1 - P)$ is $\mathcal{H} \otimes \mathbb{C}\delta_0$ and $\mathcal{H}$ is finite dimensional, $P\lambda(1) - \lambda(1)P = P\lambda(1)(1 - P)$ is compact. Hence, $\Phi \circ T|_{C^*(\lambda(1))}$ forms a $*$-homomorphism. By the preceding lemma we have $C^*(\lambda(1)) \cong C(\mathbb{T})$, and hence it suffices to show that $\|T_x\|_{\mathrm{e}} = \|x\|$ for $x \in C^*(\lambda(1))$. Indeed, for any $x \in \mathbb{Z}\bar{\ltimes}_\alpha M$ and $K \in \mathcal{K}(\mathbb{H}^2)$, it follows from [14, Poposition 3.4] and the compactness of K that $\rho(n)^*(PxP - K)\rho(n)$ converges to x strongly. By the lower-semicontinuity of operator norm, we have $\|T_x - K\| \geq \|x\|$. Since K is arbitrary, we get $\|T_x\|_{\mathrm{e}} = \|x\|$. ■

Lemma 4.3 *If $a \in \pi(M)'$ satisfies that $[T_a, T_{\lambda(1)}] \in \mathcal{K}$, then every σ-weak cluster point of $\{\lambda(n)^* PaP\lambda(n)\}_{n\geq 0}$ belongs to $M' \bar{\rtimes}_\alpha \mathbb{Z}$.*

Proof Let b be a σ-weak cluster point of $\{\lambda(n)^* PaP\lambda(n)\}_{n\geq 0}$. Then there exists a subnet Λ of $\mathbb{N}$ such that $b = \sigma\text{-w-}\lim_{n\in\Lambda} \lambda(n)^* PaP\lambda(n)$. Since $PaP \in \pi(M)'$ and $\lambda(1)$ normalizes $\pi(M)'$, one has $b \in \pi(M)'$. Hence, it suffices to show that the b commutes with $\lambda(1)$. Since $[PaP, P\lambda(1)P]$ is compact by assumption, one has

$$\begin{aligned}
&\lambda(1)(\lambda(n)^* PaP\lambda(n)) \\
&\quad = \lambda(n)^* P\lambda(1) PaP\lambda(n) \\
&\quad = \lambda(n)^* PaP\lambda(1)P\lambda(n) + \lambda(n)^*[P\lambda(1)P, PaP]\lambda(n) \\
&\quad = \lambda(n)^* PaP\lambda(n)\lambda(1) + \lambda(n)^*[P\lambda(1)P, PaP]\lambda(n) + \lambda(n)^* PaP\lambda(1)(1 - P)\lambda(n) \\
&\quad \xrightarrow{n\in\Lambda} b\lambda(1) \quad \text{strongly,}
\end{aligned}$$

which implies that $[b, \lambda(1)] = 0$. ■

We denote by $\|X\|_p \in [0, +\infty]$ the Schatten p-norm of $X \in \mathcal{B}(\mathcal{H})$ with $1 \leq p < \infty$ and define $\|X\|_\infty := \|X\|$, the operator norm of X. Recall the fact that the norms $\|\cdot\|_p$ are lower-semicontinuous with respect to the weak operator topology (see *e.g.*, [6, Proposition 2.11]).

Lemma 4.4 *Let $b \in \{\lambda(1)\}'$, let $K \in \mathcal{K}$, and let $1 \leq p \leq \infty$ be given and set $X := T_b + K$. Assume that x is an element in the $*$-algebra generated by $\lambda(1)$ and that there exists a constant $\delta > 0$ such that $\|T_x X - T_{bx}\|_p > \delta$. Then $\|[T_{x\lambda(n)}, X]\|_p > \delta$ and $x\lambda(n) \in \mathbb{Z}_+ \bar{\ltimes}_\alpha M$ hold for all sufficiently large $n \in \mathbb{N}$.*

Proof Since x is a polynomial of $\lambda(1)$ and $\lambda(1)^*$, there exists $n_0 \in \mathbb{N}$ such that $x\lambda(n_0)$ is in $\mathbb{Z}_+ \bar{\ltimes}_\alpha M$. For $n \geq n_0$ one has that

$$\begin{aligned} T^*_{\lambda(n)}[T_x T_{\lambda(n)}, X] &= T^*_{\lambda(n)} T_x T_{\lambda(n)} X - T^*_{\lambda(n)}(T_b + K) T_x T_{\lambda(n)} \\ &= T_x X - T^*_{\lambda(n)b} T_{x\lambda(n)} - T^*_{\lambda(n)} K T_x T_{\lambda(n)} \\ &= T_x X - T_{bx} - T^*_{\lambda(n)} K T_x T_{\lambda(n)} \end{aligned}$$

converges to $T_x X - T_{bx}$ strongly as $n \to \infty$, since K is compact. Thus, by the lower-semicontinuity of $\|\cdot\|_p$, there exists $n_1 > n_0$ such that

$$\|[T_{x\lambda(n)}, X]\|_p \geq \|T^*_{\lambda(n)}[T_x T_{\lambda(n)}, X]\|_p > \delta$$

as long as $n \geq n_1$. ■

First, we deal with the case where $p = \infty$. Although Claim 4.5 can be shown in the same way as the proof of [2, Theorem 2], we give a somewhat simplified proof based on the techniques used in [12, 16]. The method of our proof may be essentially known, but we could not find a suitable reference that explicitly explains such an argument.

Proof of Theorem 1.3 when $p = \infty$ Let X be in $\mathrm{Esscom}_\infty(\mathbb{Z}_+ \bar{\times}_\alpha M)$. Since X is in $\mathrm{Esscom}_\infty(T(\pi(M))) = T(\pi(M)') + \mathcal{K}$, there exists $a \in \pi(M)'$ such that $X - T_a$ is compact. Let Λ be a subsequence of $\mathbb{N}$ such that the limit

$$b = \sigma\text{-w-}\lim_{n \in \Lambda} \lambda(n)^* PaP \lambda(n)$$

exists. Lemma 4.3 states that this b must be in $M' \bar{\rtimes}_\alpha \mathbb{Z}$. If $T_a - T_b$ is compact, then so is $X - T_b$. Since T_a is in $\mathrm{Esscom}_\infty(\mathbb{Z}_+ \bar{\times}_\alpha M)$, that is, $[T_a, Y]$ is compact for every $Y \in \mathbb{Z}_+ \bar{\times}_\alpha M$, the proof will be complete after establishing the following claim. ■

Claim 4.5 If $T_a - T_b$ is not compact, then there exists Z in $\mathbb{Z}_+ \bar{\ltimes}_\alpha M$ such that $[T_Z, T_a]$ is not compact.

Proof Set $\delta := \|T_a - T_b\|_{\mathrm{e}} > 0$ and $\mathcal{A} := C^*(\Phi(T_a - T_b), \Phi(T_{\lambda(1)}))$. For $\gamma \in \mathbb{T}$ let $\mathcal{J}_\gamma$ be the closed ideal of $\mathcal{A}$ generated by $\{\Phi(T_{\widehat{f}}) \mid f(\gamma) = 0\}$ and let $\varphi_\gamma \colon \mathcal{A} \to \mathcal{A}/\mathcal{J}_\gamma$ be the quotient map. Since $\Phi(T_a - T_b)$ commutes with $\Phi(T_{\lambda(1)})$ and $C^*(\Phi(T_{\lambda(1)}))$ is isomorphic to $C(\mathbb{T})$, by [3, Theorem 7.47] $\sum^{\oplus}_{\gamma \in \mathbb{T}} \varphi_\gamma \colon \mathcal{A} \to \sum^{\oplus}_{\gamma \in \mathbb{T}} \mathcal{A}/\mathcal{J}_\gamma$ is injective. Thus, there exists $\gamma \in \mathbb{T}$ such that $\|\Phi(T_a - T_b) + \mathcal{J}_\gamma\| > \delta/2$. Now for any $f \in C(\mathbb{T})$ with $f(\gamma) = 1$, we have

$$\|T_{\widehat{f}} T_a - T_{\widehat{f}b}\|_{\mathrm{e}} = \|T_{\widehat{f}}(T_a - T_b)\|_{\mathrm{e}} = \|\Phi(T_a - T_b) + \Phi(T_{\widehat{f-1}}(T_a - T_b))\|, > \delta/2$$

since $\Phi(T_{\widehat{f-1}}(T_a - T_b)) \in \mathcal{J}_\gamma$.

Let us construct a sequence $\{p_k\}_k \subset \mathbb{Z}_+ \bar{\ltimes}_\alpha M$ such that $\lim_{k \to \infty} \|[T_{p_k}, T_a]\| \geq \delta/2$ and the sum $\sum_{k=1}^\infty p_k$ converges strongly. Set $B^\gamma_n := \{\gamma' \in \mathbb{T} \mid |\gamma' - \gamma| < n^{-1}\}$ and $c(f) := \|T_{\widehat{f}} T_a - T_{\widehat{f}b}\|$ for $f \in L^\infty(\mathbb{T})$. Note that $c(\,\cdot\,)$ is σ-weakly lower-semicontinuous. For $n + 2 < k$ one can take $f_{n,k} \in C(\mathbb{T})$ in such a way that $0 \leq f_{n,k} \leq 1$, $f = 0$ on $B^\gamma_k \cup (\mathbb{T} \setminus B^\gamma_n)$ and $f = 1$ on $B^\gamma_{n+1} \setminus B^\gamma_{k-1}$. Note that $\{f_{n,k}\}_k$ converges almost everywhere to a function $f_n \in C(\mathbb{T})$ with $f_n(\gamma) = 1$. Hence, by the lower-semicontinuity of $c(\,\cdot\,)$ together with Lebesgue's dominated convergence

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theorem, there exists $n' > n$ such that $c(f_{n,n'}) > \delta/2$. We can inductively choose $n_1 < n'_1 < n_2 < n'_2 < \cdots$ in such a way that $g_k := f_{n_k,n'_k}$ satisfies that $0 \le g_k \le 1$, $\{g_k\}_k$ have pairwise disjoint supports, and $c(g_k) > \delta/2$. Let h_k be a trigonometric polynomial such that $\|g_k - h_k\|_\infty < 2^{-k}$ and $c(h_k) > \delta/2$. By Lemma 4.4 one can choose m_k such that $\|[T_{\widehat{h_k}\lambda(m_k)}, T_a]\| > \delta/2$ and $\widehat{h_k}\lambda(m_k) \in \mathbb{Z}_+\bar{\ltimes}_\alpha M$. Set $p_k := \widehat{h_k}\lambda(m_k)$; then $\|[T_{p_k}, T_a]\| \le 2\|h_k\|_\infty\|a\| \le 4\|a\|$. Thus by passing to a subsequence if necessary, we may and do assume that $\lim_{k\to\infty}\|[T_{p_k}, T_a]\|$ exists. Let χ_k be the characteristic function of $\mathrm{supp}(g_k)$. Then $\widehat{\chi}_k$, $k \in \mathbb{N}$, are mutually orthogonal projections in $\{\lambda(1)\}''$ satisfying that $\|p_k(1-\widehat{\chi}_k)\| = \|h_k(1-\chi_k)\|_\infty < 2^{-k}$ and $\|p_k\widehat{\chi}_k\| = \|\widehat{\chi}_k p_k \widehat{\chi}_k\| \le 2$ for every $k \in \mathbb{N}$. For any $\xi \in \mathbb{L}^2$ and $k < l$, we have that

$$\Big\|\sum_{i=k}^{l} p_i\xi\Big\| \le \Big\|\sum_{i=k}^{l} \widehat{\chi}_i p_i \widehat{\chi}_i\xi\Big\| + \Big\|\sum_{i=k}^{l} p_i(1-\widehat{\chi}_i)\xi\Big\| \le 2\Big\|\sum_{i=k}^{l}\widehat{\chi}_i\xi\Big\| + 2^{-(k-1)}\|\xi\|$$

converges to 0 as $k, l \to \infty$, and hence the $\{p_k\}_k$ is the desired sequence. Note that p_k and p_k^* converge to 0 strongly. Since $T_a \in \mathrm{Esscom}_\infty(\mathbb{Z}_+\bar{\ltimes}_\alpha M)$, we can apply [12, Lemma 2.1] to compact operators $[T_{p_k}, T_a]$ and obtain a subsequence $\{p_{k(i)}\}_{i=1}^\infty$ such that

$$\sum_{i=1}^{\infty}[T_{p_{k(i)}}, T_a] \text{ converges strongly and } \Big\|\sum_{i=1}^{\infty}[T_{p_{k(i)}}, T_a]\Big\|_{\mathrm{e}} \ge \delta/2.$$

Letting $Z := \sum_{i=1}^\infty p_{k(i)}$, we have $Z \in \mathbb{Z}_+\bar{\ltimes}_\alpha M$ and $[T_Z, T_a] \notin \mathcal{K}$, which implies the claim. ∎

We then treat the case where $p \neq \infty$. The next lemma originates in [12, Lemma 2.1].

Lemma 4.6 *Let $\mathcal{H}_1$ be a Hilbert space and fix $1 \le p < \infty$. Assume that a sequence $\{K_n\}_n \subset \mathfrak{S}_p(\mathcal{H}_1)$ satisfies the following conditions:*

(i) $\|K_n\|_p > 2$ *for every* $n \in \mathbb{N}$;
(ii) $\sup_n \|K_n\| < C_1$ *for some* $C_1 > 0$;
(iii) K_n *and* K_n^* *converge to* 0 *strongly.*

Then there exists a subsequence $\{K_{n_k}\}_{k=1}^\infty$ such that $\sum_{k=1}^\infty K_{n_k}$ converges strongly and $\sum_{k=1}^\infty K_{n_k} \notin \mathfrak{S}_p(\mathcal{H}_1)$.

Proof Let $\mathcal{H}_0$ be the separable Hilbert space generated by $\bigcup_{n=1}^\infty(\mathrm{Ker}\,K_n)^\perp$. Choose an orthonormal basis $\{e_n\}_{n=1}^\infty$ of $\mathcal{H}_0$, and let R_n be the orthogonal projection onto the linear span of $e_1, \dots, e_n$. We claim that there exist mutually orthogonal finite rank projections $\{Q_k\}_{k=1}^\infty$ and a subsequence $\{K_{n_k}\}_{k=1}^\infty$ such that

(a) $\|Q_k K_{n_k} Q_k\|_p > 1$,
(b) $\|K_{n_k} Q_k^\perp\|_p < 3^{-k}$ and $\|Q_k^\perp K_{n_k}\|_p < 3^{-k}$,
(c) $\|K_{n_k} R_k\| < 2^{-k}$,

with $Q_k^\perp := I - Q_k$. Assume that we have chosen $Q_1, \dots, Q_k$ and $n_1, \dots, n_k$. Put $Q := \sum_{j=1}^k Q_j$. Since $K_n, K_n^* \to 0$ strongly and Q is finite rank, there exists $n_{k+1} > n_k$ such that $\|K_{n_{k+1}} R_{k+1}\| < 2^{-k-1}$, $\|K_{n_{k+1}} Q\|_p < 3^{-k-1}$, and $\|Q K_{n_{k+1}}\|_p < 3^{-k-1}$. Thus this $K_{n_{k+1}}$ satisfies the desired (c). Write $K := K_{n_{k+1}}$ for short. We have

$$2 < \|K\|_p \le \|Q^\perp K Q^\perp\|_p + \|QKQ^\perp\|_p + \|KQ\|_p < \|Q^\perp K Q^\perp\|_p + 1,$$

implying that $\|Q^\perp KQ^\perp\|_p > 1$. Let $F_j \le Q^\perp$, $j \in J$, be an increasing net of finite rank projections that converges to $Q^\perp$ strongly. Since K is in $\mathfrak{S}_p$, one has $\|F_iKF_i - Q^\perp KQ^\perp\|_p$, $\|KF_i^\perp\|_p$, and $\|F_i^\perp K\|_p$ converge to 0. By the lower-semicontinuity of norm, we can find $j \in J$ in such a way that $Q_{k+1} := F_j < Q^\perp$ satisfies (a) and (b) for $K = K_{n_{k+1}}$. Hence we can construct the desired Q_k and K_{n_k} by induction.

Write $K_k := K_{n_k}$ for short. By (ii) and (b) we have

$$\Big\|\sum_{k=1}^n K_k\Big\| \le \Big\|\sum_{k=1}^n Q_kK_kQ_k\Big\| + \sum_{k=1}^n \Big\|Q_k^\perp K_kQ_k\Big\| + \sum_{k=1}^n\Big\|K_kQ_k^\perp\Big\| < C_1 + 2.$$

Hence $\sum_{k=1}^n K_k$ is norm bounded. If ξ in $R_n\mathcal{H}_0$ and $m \ge l \ge n$, then by (c) we have

$$\Big\|\Big(\sum_{k=1}^m K_k - \sum_{k=1}^l K_k\Big)\xi\Big\| \le \sum_{k=l+1}^m \|K_kR_n\|\|\xi\| \le \sum_{k=l+1}^m 2^{-k}\|\xi\| \le 2^{-l}\|\xi\|.$$

Since $\bigcup_{n=1}^\infty R_n\mathcal{H}_1$ is dense in $\mathcal{H}_0$ and each K_k is equal to 0 on $\mathcal{H}_0^\perp$, $\sum_{k=1}^\infty K_k$ converges strongly. Set $X = \sum_{k=1}^\infty K_k$. Since each Q_n is finite rank, $Q_n\sum_{k=1}^m K_kQ_n$ converges to Q_nXQ_n in norm. For each $n \in \mathbb{N}$ we have

$$\begin{aligned}\|Q_nXQ_n\|_p &= \lim_{m\to\infty}\Big\|Q_n\sum_{k=1}^m K_kQ_n\Big\|_p \ge \|Q_nK_nQ_n\|_p - \sum_{k\ne n}\|Q_nK_kQ_n\|_p\\ &\ge 1 - \sum_{k\ne n}\|K_k(I-Q_k)\|_p \ge 1 - \sum_{k\ne n}3^{-k} \ge \frac{1}{2},\end{aligned}$$

which implies that $X \notin \mathfrak{S}_p$, since Q_n, $n \in \mathbb{N}$, are mutually orthogonal. ∎

We can now complete the proof of Theorem 1.3.

Proof of Theorem 1.3 when $p \ne \infty$ Let $X \in \mathrm{Esscom}_p(\mathbb{Z}_+\bar{\times}_\alpha M)$ be arbitrarily chosen. Since $\mathrm{Esscom}_p(\mathbb{Z}_+\bar{\times}_\alpha M) \subset \mathrm{Esscom}_\infty(\mathbb{Z}_+\bar{\times}_\alpha M)$, there exists $b \in M'\bar{\rtimes}_\alpha\mathbb{Z}$ such that $X - T_b$ is compact. Suppose that $X - T_b \in \mathcal{K}\setminus\mathfrak{S}_p$. Define $c_p\colon L^\infty \to [0,+\infty]$ by $c_p(g) = \|T(\widehat{g})X - T(b\widehat{g})\|_p$. Note that c_p is lower-semicontinuous with respect to the weak operator topology and $c_p(1) = +\infty$. Suppose that for each $\gamma \in \mathbb{T}$ there exists $n \in \mathbb{N}$ such that $c_p(f) \le 2$ as long as $f \in C(\mathbb{T})$ satisfies that $0 \le f \le 1$ and $\mathrm{supp}(f) \subset B_n^\gamma$. Then, by the compactness of $\mathbb{T}$, there exist such $(\gamma_1, n_1), \dots, (\gamma_k, n_k) \in \mathbb{T}\times\mathbb{N}$ with $\mathbb{T} = \bigcup_{i=1}^k B_{n_i}^{\gamma_i}$. Taking a partition of the unity $\{\psi_i\}_{i=1}^k$ for the covering $\{B_{n_i}^{\gamma_i}\}_{i=1}^k$, we have $c_p(1) \le \sum_{i=1}^k c_p(\psi_i) \le 2k$, a contradiction. Hence, we can find $\gamma \in \mathbb{T}$ such that for every $n \in \mathbb{N}$ there exists $f_n \in C(\mathbb{T})$ such that $c_p(f_n) > 2$, $0 \le f_n \le 1$, and $\mathrm{supp}(f_n) \subset B_n^\gamma$. By a same approximation argument as in the case of $p = \infty$ together with Lemma 4.4, we obtain $p_k \in \mathbb{Z}_+\bar{\rtimes}_\alpha M$ such that p_k and p_k^* converge to 0 strongly, $\sum_{k=1}^\infty p_k$ converges strongly, and that $\|p_k\| \le 2$ and $\|[T_{p_k}, X]\|_p > 2$ for every $k \in \mathbb{N}$. Applying Lemma 4.6 to $K_k = [T_{p_k}, X]$ we obtain a subsequence $\{p_{k(i)}\}_i$ such that $Z := \sum_{i=1}^\infty p_{k(i)} \in \mathbb{Z}_+\bar{\rtimes}_\alpha M$ and $[T_Z, X] \notin \mathfrak{S}_p$. This is a contradiction; hence $X - T_b$ is in $\mathfrak{S}_p$. ∎

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Remark 4.7 To compute the essential fixed-points, there is a technical obstruction. In the proof of Lemma 4.3, it is crucial that that $\lambda(1)$ normalizes $\pi(M)'$. In [16] Xia computed $\mathrm{Essfix}_\infty(T(H^\infty))$ by using the finite Blaschke product w_n instead of the inner function z^n; see [16, Proposition 3]. However, unitary elements in $C^*(\lambda(1))$ do not normalize $\pi(M)'$ in general.

5 Condition $(\star)$

Let M be a von Neumann algebra on $L^2(M)$ and α be a $*$-automorphism of M. We say (M,α) satisfies condition $(\star)$ if

$$(\star)\qquad \mathrm{Esscom}_p(\mathbb{Z}_+\bar{\ltimes}_\alpha M) \subset T(M'\bar{\rtimes}_\alpha\mathbb{Z}) + \mathfrak{S}_p, \quad 1\le p\le\infty$$

holds true. Note that (M,α) satisfies $(\star)$ if and only if so does (M',α) if and only if

$$\mathrm{Esscom}_p(M\bar{\rtimes}_\alpha\mathbb{Z}_+) \subset T(\mathbb{Z}\bar{\ltimes}_\alpha M') + \mathfrak{S}_p, \quad 1\le p\le\infty$$

holds. We have already seen that (M,α) satisfies $(\star)$ when M is either diffuse, a type I_∞ factor, a direct sum of them (Corollary 1.2), or is finite dimensional (Theorem 1.3). In fact, we can prove the next theorem, asserting that condition $(\star)$ is satisfied in a more general case. For a given (M,α) it is known and not hard to see that M is decomposed uniquely as $M = M_{\mathrm{c}}\oplus M_\infty \oplus \sum_{n\ge 1}^{\oplus} M_n$, where M_{c} is diffuse, the M_∞ a direct sum of infinite type I factors, and $M_n = M_n(\mathbb{C})\otimes\ell^\infty(\mathfrak{X}_n)$, $n\ge 1$, with discrete sets $\mathfrak{X}_n$. The uniqueness of this decomposition guarantees that α is also decomposed as $\alpha = \alpha_{\mathrm{c}}\oplus\alpha_\infty\oplus\sum_{n\ge1}^{\oplus}\alpha_n$. It is not difficult to see that there exists a unique automorphism β_n of $\ell^\infty(\mathfrak{X}_n)$ and a unitary element $v_n\in M_n(\mathbb{C})\otimes\ell^\infty(\mathfrak{X}_n)$ such that $\alpha_n = \mathrm{Ad}\, v_n\circ(\mathrm{id}\otimes\beta_n)$. Remark that β_n induces a unique bijection θ_n on $\mathfrak{X}_n$. With these notation we have the following theorem.

Theorem 5.1 *Assume that every orbit of θ_n forms a finite set. Then (M,α) satisfies condition $(\star)$.*

To prove this theorem we need the following lemma.

Lemma 5.2 *Let $\{(M_i,\alpha_i)\}_{i\in I}$ be a family of von Neumann algebras and their $*$-automorphisms. Set $M := \sum_{i\in I}^{\oplus} M_i$ and $\alpha := \sum_{i\in I}^{\oplus}\alpha_i$. If every (M_i,α_i) satisfies condition $(\star)$, then so does (M,α).*

Proof Let $X\in\mathrm{Esscom}_p(\mathbb{Z}_+\bar{\ltimes}_\alpha M)$ be arbitrarily chosen. Let e_i be the central support of $\pi(M_i)$ in $\pi(M)$ and put $A := \{e_i \mid i\in I\}''$. Note that $\lambda(1)$ and P commute with A. Then there exists $a\in\pi(M)'\subset A'$ such that $X - T_a\in\mathfrak{S}_p$. Since T_a is also in $\mathrm{Esscom}_p(\mathbb{Z}_+\bar{\ltimes}_\alpha M)$ and $\mathbb{Z}_+\bar{\ltimes}_\alpha M = \sum_{i\in I}^{\oplus}\mathbb{Z}_+\bar{\ltimes}_{\alpha_i}M_i$, we have $T_{ae_i}\in\mathrm{Esscom}_p(\mathbb{Z}_+\bar{\ltimes}_{\alpha_i}M_i)$. By assumption, there exist $b_i\in M_i'\bar{\rtimes}_{\alpha_i}\mathbb{Z}$ and $K_i\in\mathfrak{S}_p(e_iL^2(M))$ such that $T_{ae_i} = T_{b_i}+K_i$. Set $b := \sum_{i\in I} b_i$ and $K := \sum_{i\in I}K_i$. Since

$$b\in\sum_{i\in I}^{\oplus} M_i'\bar{\rtimes}_{\alpha_i}\mathbb{Z} = M'\bar{\rtimes}_\alpha\mathbb{Z},$$

it suffices to prove that $K\in\mathfrak{S}_p$. Since $T_{ae_i} - T^*_{\lambda(n)e_i}T_{ae_i}T_{\lambda(n)e_i}\to K_i$ strongly as $n\to\infty$, the lower-semicontinuity of $\|\cdot\|_p$ enables us to find $n_i\in\mathbb{N}$ in such a way

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that

$$\|[T_{\lambda(n_i)e_i}, T_{ae_i}]\|_p \geq \|T_{ae_i} - T^*_{\lambda(n_i)e_i} T_{ae_i} T_{\lambda(n_i)e_i}\|_p > 2^{-1}\|K_i\|_p.$$

Then $x := \sum_{i\in I} \lambda(n_i)e_i$ belongs to $\mathbb{Z}_+\bar{\ltimes}_\alpha M$. Since $T_a \in \mathrm{Esscom}_p(\mathbb{Z}_+\bar{\times}_\alpha M)$, one has $[T_x, T_a] = \sum_{i\in I}[T_{\lambda(n_i)e_i}, T_{ae_i}] \in \mathfrak{S}_p$. Hence, it follows from the inequality above that $K \in \mathfrak{S}_p$. ■

Proof of Theorem 5.1 By Corollary 1.2 and Lemma 5.2 we may and do assume that $M = M_n$ and $\alpha = \mathrm{id} \otimes \beta_n$. Decompose $\mathfrak{X}_n$ into the disjoint θ_n-orbits $\mathfrak{X}_{n,j}$, $j \in J_n$. Set $M_{n,j} := M_n(\mathbb{C}) \otimes \ell^\infty(\mathfrak{X}_{n,j})$, which sits inside $M_n = M_n(\mathbb{C}) \otimes \ell^\infty(\mathfrak{X}_n)$ naturally. Then $M_n = \sum^{\oplus}_{j\in J_n} M_{n,j}$, and clearly $\alpha(M_{n,j}) = M_{n,j}$ holds for every $j \in J_n$. Consequently, one has

$$(M_n, \alpha_n) = \sum\nolimits^{\oplus}_{j\in J_n}(M_{n,j}, \alpha|_{M_{n,j}}).$$

By assumption each $\mathfrak{X}_{n,j}$ is a finite set, and hence $M_{n,j}$ is finite dimensional. Therefore, the desired assertion follows from Theorem 1.3 thanks to Lemma 5.2. ■

Here a question naturally arises.

Question 5.3 Let σ be the $*$-automorphism on $\ell^\infty(\mathbb{Z})$ induced from the translation $n \in \mathbb{Z} \mapsto n + 1 \in \mathbb{Z}$. Does $(\ell^\infty(\mathbb{Z}), \sigma)$ satisfy condition $(\star)$?

In fact, the positive answer to the question enables us to get rid of the assumption from Theorem 5.1 as follows. We use the notation in the proof of Theorem 5.1. Thanks to Lemma 5.2 and Theorem 5.1, it suffices to prove that for every infinite θ_n-orbit $\mathfrak{X}_{n,j}$, $(M_{n,j}, \alpha|_{M_{n,j}})$ satisfies condition $(\star)$. Then $(M_{n,j}, \alpha|_{M_{n,j}})$ can be identified with $(M_n(\mathbb{C})\otimes\ell^\infty(\mathbb{Z}), \mathrm{Ad}\, v_{n,j} \circ \mathrm{id} \otimes\sigma)$ for a unitary element $v_{n,j} \in M_n(\mathbb{C})\otimes\ell^\infty(\mathfrak{X}_{n,j})$, and hence we may assume that $(M, \alpha) = (M_n(\mathbb{C}) \otimes \ell^\infty(\mathbb{Z}), \mathrm{id} \otimes\sigma)$. Write $\ell^\infty := \ell^\infty(\mathbb{Z})$ and $\ell^2 := \ell^2(\mathbb{Z})$ for simplicity. The standard form of $M_n(\mathbb{C}) \otimes \ell^\infty$ becomes $M_n(\mathbb{C}) \otimes \mathbb{C}1 \otimes \ell^\infty$ on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \ell^2$. Hence, $\mathbb{Z}\bar{\ltimes}_{\mathrm{id}\otimes\sigma}(M_n(\mathbb{C}) \otimes \ell^\infty)$ and $(M_n(\mathbb{C})\otimes\ell^\infty)'\bar{\rtimes}_{\mathrm{id}\otimes\sigma}\mathbb{Z}$ become $M_n(\mathbb{C})\otimes\mathbb{C}1\otimes(\mathbb{Z}\bar{\ltimes}_\sigma\ell^\infty)$ and $\mathbb{C}1\otimes M_n(\mathbb{C})\otimes(\ell^\infty\bar{\rtimes}_\sigma\mathbb{Z})$, respectively. It is easily seen that

$$\mathrm{Esscom}_p\big(M_n(\mathbb{C}) \otimes \mathbb{C}1 \otimes (\mathbb{Z}_+\bar{\times}_\sigma\ell^\infty)\big) \subset \mathbb{C}1 \otimes M_n(\mathbb{C}) \otimes \mathrm{Esscom}_p(\mathbb{Z}_+\bar{\times}_\sigma\ell^\infty) + \mathfrak{S}_p.$$

Therefore, if Question 5.3 had an affirmative answer, then (M, α) would satisfy condition $(\star)$, since

$$\begin{aligned}\mathrm{Esscom}_p(\mathbb{Z}_+\bar{\times}_\alpha M) &= \mathrm{Esscom}_p\big(M_n(\mathbb{C}) \otimes \mathbb{C}1 \otimes (\mathbb{Z}_+\bar{\times}_\sigma\ell^\infty)\big)\\ &\subset \mathbb{C}1 \otimes M_n(\mathbb{C}) \otimes \mathrm{Esscom}_p(\mathbb{Z}_+\bar{\times}_\sigma\ell^\infty) + \mathfrak{S}_p\\ &\subset \mathbb{C}1 \otimes M_n(\mathbb{C}) \otimes T(\ell^\infty\bar{\rtimes}_\sigma\mathbb{Z}) + \mathfrak{S}_p\\ &= T(M'\bar{\rtimes}_\alpha\mathbb{Z}) + \mathfrak{S}_p.\end{aligned}$$

Finally, we should remark that the canonical implementing unitary operator of σ is nothing but the bilateral shift on $\ell^2(\mathbb{Z})$ with respect to the standard basis. Hence Question 5.3 seems operator theoretic rather than operator algebraic.

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