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# Condensed Domains

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*Abstract.* An integral domain $D$ with identity is condensed (resp., strongly condensed) if for each pair of ideals $I$, $J$ of $D$, $IJ = \{ij\ ;\ i \in I, j \in J\}$ (resp., $IJ = iJ$ for some $i \in I$ or $IJ = Ij$ for some $j \in J$). We show that for a Noetherian domain $D$, $D$ is condensed if and only if $\mathrm{Pic}(D) = 0$ and $D$ is locally condensed, while a local domain is strongly condensed if and only if it has the two-generator property. An integrally closed domain $D$ is strongly condensed if and only if $D$ is a Bézout generalized Dedekind domain with at most one maximal ideal of height greater than one. We give a number of equivalencies for a local domain with finite integral closure to be strongly condensed. Finally, we show that for a field extension $k \subseteq K$, the domain $D = k + XK[[X]]$ is condensed if and only if $[K:k] \leq 2$ or $[K:k] = 3$ and each degree-two polynomial in $k[X]$ splits over $k$, while $D$ is strongly condensed if and only if $[K:k] \leq 2$.

## 1 Introduction

D. F. Anderson and D. E. Dobbs [4] introduced the concept of a condensed integral domain. An integral domain $D$ is *condensed* if for each pair of ideals $I$ and $J$ of $D$, $IJ = \{ij\ ;\ i \in I, j \in J\}$. They showed that a condensed domain $D$ has $\mathrm{Pic}(D) = 0$ and that a Noetherian condensed domain $D$ has $\dim D \leq 1$. Also, they showed that $k[[X^2, X^3]]$, with $k$ field, is a condensed domain. Later, Anderson, J. T. Arnold and Dobbs [3] showed that an integrally closed domain is condensed if and only if it is Bézout. Next, C. Gottlieb [9] introduced a class of condensed domains, the strongly condensed domains. An integral domain $D$ is *strongly condensed (SC)* if for each pair of ideals $I$ and $J$ of $D$, either $IJ = Ij$ for some $j \in J$ or $IJ = iJ$ for some $i \in I$. Gottlieb showed that if $(D, M)$ is a local domain whose integral closure $(D', M')$ is a DVR and a finite $D$-module with $D/M = D'/M'$ and $D$ contains a value-two element of $D'$, then $D$ is strongly condensed. Hence $k[[X^2, X^n]]$, $k$ a field, is condensed for any natural number $n$.

In Section 2 we study condensed domains. We begin by examining the role that $\mathrm{Pic}(D) = 0$ plays. In this regard, see Theorems 2.1 and 2.4 and Corollaries 2.3 and 2.5. Then, we characterize those field extensions $k \subseteq K$ for which $k + XK[[X]]$ is a condensed domain (see Corollary 2.10).

In Section 3 we study strongly condensed domains. We give characterizations for the general (resp., integrally closed, Noetherian, local) SC domains (see Proposition 3.3, Theorem 3.4, Corollary 3.6, Theorems 3.7 and 3.8 and Corollary 3.9). Also, as an extension of the main result in [9], we give a number of equivalencies for a local domain with finite integral closure to be SC (see Theorem 3.11).

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## 2 Condensed Domains

Let $D$ be an integral domain and $I$, $J$ ideals of $D$. We say that $I$, $J$ is a *condensed pair* (in $D$) if $IJ = \{ij\ ;\ i \in I,\ j \in J\}$. Following [4], we say that $D$ is *condensed* if $I$, $J$ is a condensed pair for every $I$ and $J$. In [4], they proved that an integral domain $D$ is condensed if and only if every pair of doubly-generated ideals of $D$ is condensed. They also proved that any overring of a condensed integral domain is condensed.

An important result proved in the previously mentioned paper was that if $D$ is a condensed integral domain, then $\mathrm{Pic}(D) = 0$. A related result is the following. Recall that a domain is *h-local* if each nonzero ideal is contained in only finitely many maximal ideals and each nonzero prime ideal is contained in a unique maximal ideal. Hence a one-dimensional Noetherian domain is $h$-local. Also, a domain $D$ is *atomic*, if every nonzero nonunit of $D$ is a product of irreducible elements (atoms). It is well known that a Noetherian domain is atomic. Let $D$ be a domain. Call a nonzero ideal $I$ of $D$ (resp., a nonzero nonunit $x \in D$) *unidirectional* [2] if $I$ (resp., $xD$) is contained in a unique maximal ideal.

***Theorem 2.1*** *Let $D$ be an atomic domain. Then the following statements are equivalent.*

(1) *Every pair of comaximal ideals of $D$ is condensed,*
(2) *Every pair of distinct maximal ideals of $D$ is condensed,*
(3) *Each atom of $D$ is unidirectional,*
(4) *$D$ is $h$-local with $\mathrm{Pic}(D) = 0$,*
(5) *every nonzero nonunit of $D$ is a product of unidirectional elements,*
(6) *for each nonzero nonunit $x \in D$ and each maximal ideal $M$ containing $x$, $xD_M \cap D$ is a principal unidirectional ideal.*

**Proof** Let $D$ be an arbitrary (not necessarily atomic) domain. The equivalence of (4), (5) and (6) is given in [2, Corollary 3.6], (1) $\Rightarrow$ (2) is clear and (4) $\Rightarrow$ (1) is a consequence of the lemma below.

(2) $\Rightarrow$ (3). Let $x$ be an atom of $D$. Suppose $x$ is in two distinct maximal ideals $M$ and $N$. Then $x \in M \cap N = MN$, so $x = mn$ where $m \in M$ and $n \in N$. A contradiction. Finally, (3) $\Leftrightarrow$ (5) is clear, provided $D$ is atomic. ∎

***Lemma 2.2*** *Let $D$ be an $h$-local domain with $\mathrm{Pic}(D) = 0$ and let $I$, $J$ be nonzero ideals of $D$. If $ID_M$, $JD_M$ is a condensed pair in $D_M$ for each maximal ideal $M$ of $D$ containing $I + J$, then $I$, $J$ is a condensed pair in $D$.*

**Proof** Clearly, $ID_M$, $JD_M$ is a condensed pair for each maximal ideal $M$ of $D$ not containing $I + J$. Let $0 \neq x \in IJ$. Let $M_1, \dots, M_n$ be the maximal ideals containing $x$. Since $ID_{M_i}$, $JD_{M_i}$ is a condensed pair, $x = a_i b_i$ for some $a_i \in ID_{M_i}$ and $b_i \in JD_{M_i}$. By the equivalence (5) $\Leftrightarrow$ (6) of Theorem 2.1, $xD_{M_i} = (A_i)_{M_i}(B_i)_{M_i}$, where $A_i = a_i D_{M_i} \cap D$, $B_i = b_i D_{M_i} \cap D$ are principal unidirectional ideals or equal to $D$. Set $A = A_1 \cdots A_n$, $B = B_1 \cdots B_n$. Then the relations $xD = AB$, $A \subseteq I$, $B \subseteq J$ hold locally and hence globally. Since $A$ and $B$ are principal ideals, $x = ab$ for some $a \in I$ and $b \in J$. ∎

***Corollary 2.3*** *For a one-dimensional Noetherian domain $D$,* $\mathrm{Pic}(D) = 0$ *if and only if* $MN = \{mn \,;\, m \in M, n \in N\}$ *for all distinct maximal ideals $M$ and $N$ of $D$.*

The next theorem essentially reduces the study of atomic condensed domains to the local case.

***Theorem 2.4*** *Let $D$ be an atomic domain. Then $D$ is condensed if and only if $D$ is h-local,* $\mathrm{Pic}(D) = 0$ *and $D_M$ is condensed for each maximal ideal $M$ of $D$.*

**Proof** Apply Theorem 2.1 and Lemma 2.2. ■

***Corollary 2.5*** *Let $D$ be a Noetherian domain. Then $D$ is condensed if and only if* $\mathrm{Pic}(D) = 0$ *and $D_M$ is condensed for each maximal ideal $M$ of $D$.*

For the next results we use the following notation. If $k \subseteq K$ is a field extension and $V, W$ are $k$-subspaces of $K$, we set $P(V, W) = \{vw \,;\, v \in V, w \in W\}$ and $VW$ the $k$-subspace of $K$ generated by $P(V, W)$.

***Proposition 2.6*** *For a field extension $k \subseteq K$, the following assertions are equivalent:*

(a) *the domain $D = k + XK[[X]]$ is condensed,*
(b) *$VW = P(V, W)$ for all $k$-subspaces $V, W$ of $K$, and*
(c) *For every $\alpha, \beta \in K$, $1 + \alpha\beta = (a + b\alpha)(c + d\beta)$ for some $a, b, c, d \in k$.*

**Proof** Clearly, if $V$ is a $k$-subspace of $K$ and $n \geq 1$, then $I_n(V) = VX^n + X^{n+1}K[[X]]$ is an ideal of $D$. Conversely, every nonzero proper ideal $J$ of $D$ has this form. Indeed, let $f \in J$ be of minimal order, say $\mathrm{ord}(f) = n \geq 1$. Then $X^{n+1}K[[X]] \subseteq fD \subseteq J$. Clearly, the set $V$ consisting of 0 and all leading coefficients of the power series of $J$ having order $n$ is a $k$-subspace of $K$ and $J \subseteq I_n(V)$. Since $X^{n+1}K[[X]] \subseteq J$, we have $VX^n \subseteq J$, so $I_n(V) \subseteq J$. Thus $J = I_n(V)$. Now let $V, W$ be nonzero $k$-subspaces of $K$ and $n, m \geq 1$. An easy computation shows that $I_n(V)I_m(W) = I_{n+m}(VW)$ and $\{fg \,;\, f \in I_n(V), g \in I_m(W)\} = P(V, W)X^{n+m} + X^{n+m+1}K[[X]]$. Hence the assertions (a) and (b) are equivalent. Clearly, (b) $\Rightarrow$ (c). To prove the converse, let $V$, $W$ be nonzero $k$-subspaces of $K$. It suffices to show that $v_1w_1 + v_2w_2 \in P(V, W)$, for all nonzero $v_1, v_2 \in V$ and $w_1, w_2 \in W$. By (c), $1 + v_2w_2/v_1w_1 = (a + bv_2/v_1)(c + dw_2/w_1)$ for some $a, b, c, d \in k$, so $v_1w_1 + v_2w_2 = (av_1 + bv_2)(cw_1 + dw_2) \in P(V, W)$. ■

***Definition 2.7*** We say that a field extension $k \subseteq K$ is *vs-closed* if it satisfies any (and hence all) of the equivalent conditions of the previous proposition.

Note that when $k \subseteq K$ is vs-closed, $[K:k] \leq 3$. Indeed, if $[K:k] \geq 4$, there exist $\alpha, \beta \in K$ such that $1, \alpha, \beta, \alpha\beta$ are $k$-independent. Now, if $k \subseteq K$ is vs-closed, then $1 + \alpha\beta = (a + b\alpha)(c + d\beta)$ for some $a, b, c, d \in k$, that is, $(ac - 1) + bc\alpha + ad\beta + (bd - 1)\alpha\beta = 0$, so $ac = bd = 1$ and $ad = bc = 0$, hence $0 = abcd = 1$, a contradiction. Clearly, $k \subseteq K$ is vs-closed, if $[K:k] \leq 2$.

We notice that the implication (a) $\Rightarrow$ (b) in Proposition 2.6 holds in a more general setting.

***Proposition 2.8*** *If $(A, M)$ is a quasilocal condensed domain and $(B, N)$ a quasilocal overring of $A$ dominating $A$ (i.e., $N \cap A = M$), then the residue field extension $A/M \subseteq B/N$ is* vs*-closed.*

**Proof** Set $k = A/M$ and $K = B/N$. Let $\alpha, \beta$ be nonzero elements of $K$. There exist $a, b, c \in A \setminus \{0\}$ such that $\alpha = a/c + N$ and $\beta = b/c + N$. Now $c^2 + ab \in (cA + aA)(cA + bA)$, so $c^2 + ab = (mc + na)(pc + qb)$ for some $m, n, p, q \in A$. Dividing by $c^2$ and taking mod $N$, we get $1 + \alpha\beta = (\bar{m} + \bar{n}\alpha)(\bar{p} + \bar{q}\beta)$. Hence $k \subseteq K$ is vs-closed. ∎

***Theorem 2.9*** *A field extension $k \subseteq K$ is* vs*-closed if and only if $[K : k] \leq 2$ or $[K : k] = 3$ and each degree-two polynomial in $k[X]$ splits over $k$.*

**Proof** By the paragraph after Definition 2.7, we may assume that $[K : k] = 3$. By Proposition 2.6, $k \subseteq K$ is vs-closed if and only if for every $\alpha, \beta \in K \setminus k$, $V = P(k + k\alpha, k + k\beta)$ is a $k$-subspace of $K$. Since $[K : k] = 3$, this means $V = K$, because $V$ strictly contains $k + k\alpha$. Let $\alpha, \beta$ be as above. Obviously $K = k(\alpha)$. Let $f = X^3 - pX^2 - nX - m$ be the minimal polynomial of $\alpha$ over $k$. Now $\{1, \alpha, \alpha^2\}$ is a $k$-basis of $K$, so $\beta = e_0 + e_1\alpha + e_2\alpha^2$, for some $e_0, e_1, e_2 \in k$. Since $k + k\beta = k + k(e_1\alpha + e_2\alpha^2)$ it suffices to consider only the following two cases (a) $\beta = q\alpha + \alpha^2$ with $q \in k$ and (b) $\beta = \alpha$. Consider the case (a). Since $\alpha^3 = m + n\alpha + p\alpha^2$, every element of $V$ has the form

$$(a + b\alpha)\big(c + d(q\alpha + \alpha^2)\big) = ac + mbd + (qad + bc + nbd)\alpha + \big(ad + (p + q)bd\big)\,\alpha^2$$

with $a, b, c, d \in k$. Hence $V = K$ if and only if the following map is surjective:

$$\theta\colon k^4 \to k^3, \quad \theta(a, b, c, d) = \big(ac + mbd, qad + bc + nbd, ad + (p + q)bd\big).$$

Clearly $\theta = \eta\mu$, where $\mu\colon k^4 \to k^4$, $\eta\colon k^4 \to k^3$ are given by

$$\mu(a, b, c, d) = (ac, bd, ad, bc), \quad \eta(u, v, w, x) = \big(u + mv, qw + x + nv, w + (p + q)v\big).$$

The image of $\mu$ is $\{(u, v, w, x) \; ; \; uv = wx\}$. Indeed, if $uv = wx$ and $w \neq 0$, then $(u, v, w, x) = \mu(1, v/w, u, w)$, while $(0, v, 0, x) = \mu(0, 1, x, v)$ and $(u, 0, 0, x) = \mu(u, x, 1, 0)$. The converse inclusion is obvious. Let $\beta, \gamma, \delta \in k$. Solving in $u$, $v$, $w$, $x$ the system of equations $\eta(u, v, w, x) = (\beta, \gamma, \delta)$, we get

$$u = \beta - ms, \quad v = s, \quad w = \delta - (p + q)s, \quad x = (pq + q^2 - n)s - \delta q + \gamma$$

with $s \in k$ arbitrary. So, in case (a), the surjectivity of $\theta$ means there exists $s \in k$ such that $uv = wx$. The equality $uv = wx$ gives the equation in $s$,

$$js^2 + [\beta + (p + q)(\gamma - 2\delta q) + n\delta]s + \delta(\delta q - \gamma) = 0 \tag{2.1}$$

whose leading coefficient $j = q^3 + 2pq^2 + (p^2 - n)q - pn - m = (p + q)^3 - m(p + q)^2 - n(p + q) - m = f(p + q)$ is nonzero because $f$ is the minimal polynomial

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of $\alpha$ over $k$, so it has no root in $k$. The case when $\beta = \alpha$ can be done similarly. Using the same notation, we get $\eta(u, v, w, x) = (u, w + x, v)$, $(u, v, w, x) = (\beta, \delta, s, \gamma - s)$ and finally the degree-two equation

$$s^2 - \gamma s + \beta\delta = 0. \tag{2.2}$$

These two equations show that $k \subseteq K$ is vs-closed if each degree-two polynomial in $k[X]$ splits over $k$. Conversely, if $k \subseteq K$ is vs-closed, then equation (2.2) has roots in $k$ for every $\beta, \gamma, \delta \in k$, so each degree-two polynomial in $k[X]$ splits over $k$. We notice that the "only if" part of our result is also a consequence of [3, Theorem 5]. ■

***Corollary 2.10*** *For a field extension $k \subseteq K$, the domain $k + XK[[X]]$ is condensed if and only if $[K : k] \leq 2$ or $[K : k] = 3$ and each degree-two polynomial in $k[X]$ splits over $k$.*

***Example 2.11*** Let $\mathbf{B}$ be the field of all complex numbers which can be constructed by straight-edge and compass from 0 and 1 (see [12, page 210]). By [12, page 213], each degree-two polynomial in $\mathbf{B}[X]$ splits over $\mathbf{B}$ and $[\mathbf{B}(\sqrt[3]{3}) : \mathbf{B}] = 3$, so $\mathbf{B} + X\mathbf{B}(\sqrt[3]{3})[[X]]$ is a local condensed domain, *cf.* Corollary 2.10.

## 3 Strongly Condensed Domains

Let $D$ be an integral domain and $I$, $J$ ideals of $D$. We say that $I$, $J$ is a *strongly condensed* (SC) *pair* if $IJ = iJ$ for some $i \in I$ or $IJ = Ij$ for some $j \in J$. Following [9], we say that $D$ is *strongly condensed* (SC) if $I$, $J$ is an SC pair for every $I$ and $J$. Obviously, an SC domain is condensed and if $D$ is SC, then so is each ring of quotients of $D$. A rank-one non-discrete valuation domain is condensed and not SC, because in an SC domain every idempotent ideal is principal. Let $D$ be a domain with quotient field $K$. If $I$ is an ideal of $D$, we denote by $I : I$ the overring of $D$ consisting of all elements $x \in K$ with $xI \subseteq I$. It is the largest overring of $D$ in which $I$ is an ideal.

***Lemma 3.1*** *Let $D$ be a domain and $I$, $J$ nonzero ideals of $D$.*

(a) *$IJ = iJ$ for some $i \in I$ if and only if the extension of $I$ in $J : J$ is a principal ideal generated by some element of $I$. In this case, $I : I \subseteq J : J$.*
(b) *If $I$ is a principal ideal of $I : I$, then $I^2 = iI$ for some $i \in I$.*

**Proof** (a) Assume that $IJ = iJ$ with $0 \neq i \in I$. We have $(I/i)J = J$, so $I/i \subseteq J : J$, hence $I \subseteq i(J : J)$, that is, $I(J : J) = i(J : J)$. Also, $I : I \subseteq IJ : IJ = iJ : iJ = J : J$. Conversely, if $I(J : J) = i(J : J)$ with $i \in I$, then $IJ = iJ$ because $J(J : J) = J$.

(b) If $I = i(I : I)$ with $i \in I : I$, then $i \in I$ and $I^2 = iI(I : I) = iI$. ■

It is well known that the complete integral closure $\tilde{D}$ of a domain $D$ is the union of all $I : I$ for $I$ nonzero ideal of $D$.

***Lemma 3.2*** *Let $D$ be a domain and $\tilde{D}$ its complete integral closure. The set $\{I : I\ ;\ I$ a nonzero ideal of $D\}$ is linearly ordered if and only if the set of intermediate rings between $D$ and $\tilde{D}$ is linearly ordered.*

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**Proof** The "if" part is clear. For the converse, it suffices to see that whenever $x, y \in \tilde{D}$, the subrings $D[x]$ and $D[y]$ are comparable. Let $x, y \in \tilde{D}$. There exists a nonzero $d \in D$ such that $I = dD[x] \subseteq D$ and $J = dD[y] \subseteq D$. Then $I$ and $J$ are ideals of $D$. Moreover, $I : I = D[x] : D[x] = D[x]$, because $D[x]$ is a subring of $\tilde{D}$. Similarly, $J : J = D[y]$. By hypothesis, $I : I$ and $J : J$ are comparable. ■

The SC condition can be characterized in the following way.

***Proposition 3.3*** *Let $D$ be a domain and $\tilde{D}$ its complete integral closure. Then $D$ is* SC *if and only if the following two conditions hold:*

(a) *every nonzero ideal $I$ of $D$ is a principal ideal of $I : I$, and*
(b) *the set of intermediate rings between $D$ and $\tilde{D}$ is linearly ordered.*

**Proof** The "only if" part follows from Lemmata 3.1 and 3.2. Conversely, assume that $D$ satisfies the two conditions (a), (b). Let $I$, $J$ be nonzero ideals of $D$. By (a) and (b), we may assume that $I = i(I : I)$ for some $i \in I$ and $I : I \subseteq J : J$. Then $IJ = iJ$ by part (a) of Lemma 3.1. ■

For a domain $D$, let $\mathrm{Maxp}_1(D)$ denote the set of principal height-one maximal ideals of $D$. Note that $D$ is a PID if and only if $\mathrm{Max}(D) = \mathrm{Maxp}_1(D)$ or $D$ is a field. The following result shows that for a SC domain $D$, $\mathrm{Max}(D) \setminus \mathrm{Maxp}_1(D)$ has at most one element. We say that $\mathrm{Spec}(D)$ is *Noetherian* if $D$ satisfies the ascending chain condition for the radical ideals. It is well known that this implies that each radical ideal is a finite intersection of prime ideals and hence each proper ideal has only finitely many minimal prime ideals.

***Theorem 3.4*** *A domain $D$ is* SC *if and only if either $D$ is a* PID *or* $\mathrm{Spec}(D)$ *is Noetherian,* $\mathrm{Max}(D) = \mathrm{Maxp}_1(D) \cup \{M\}$ *and $D_M$ is* SC.

**Proof** The "only if" part. Let $M$ and $N$ be distinct maximal ideals of $D$. By Proposition 3.3, we may assume that $M : M \subseteq N : N$. We claim that $M$ is principal, so $D$ has at most one nonprincipal maximal ideal. Indeed, since $D$ is SC, $M^2 = mM$ for some $m \in M$, so $M = \sqrt{mD}$ and hence $mD$ is $M$-primary. Then $M = M(M : M) = m(M : M)$ by the proof of Lemma 3.1(a). So $M(N : N) = m(N : N)$ and hence $MN = mN$, again by the proof of Lemma 3.1(a). Localizing at $M$, we get $M_M = mD_M$, so $M = mD$ because $mD$ is $M$-primary.

Now, assume that $N$ is nonprincipal and $M = mD$ is of height greater than one. Hence $\cap_1^\infty m^n D \neq 0$. So if $0 \neq y \in \cap_1^\infty m^n D$, then $y(1/m^n) \in D$. So $D[1/m]$ is an overring of $D$ contained in the complete integral closure as is $N : N$, hence they are comparable, *cf.* Proposition 3.3. Now $D[1/m]$ is not contained in $N : N$, for $1/m \in N : N$ gives $(1/m)N \subseteq N$ and hence $N \subseteq mN \subseteq M$, a contradiction. So $N : N \subseteq D[1/m]$. As done above, there exists an $n \in N$ such that $N = n(N : N)$ and $nD$ is $N$-primary. Then $ND[1/m] = nD[1/m]$. Since $m \notin N$, $N = nD[1/m] \cap D = nD$, a contradiction.

Finally, suppose that $M = mD$ and $N = nD$ are both principal of height greater than one. Since the intersections of powers of $mD$ and $nD$ respectively are nonzero, $D[1/m]$ and $D[1/n]$ are incomparable overrings contained in the complete integral closure, a contradiction, *cf.* Proposition 3.3. The fact that $\mathrm{Spec}(D)$ is Noetherian follows from [6, Theorem 3.1.11], because, as done above, for every ideal $I$ of $D$, $\sqrt{I} = \sqrt{iD}$ for some $i \in I$.

The "if" part. Suppose there exists an ideal $M$ as in the statement, otherwise $D$ is a PID, hence SC. Since $\mathrm{Spec}(D)$ is Noetherian, each ideal of $D$ has only finitely many minimal prime ideals. Let $I$ be a nonzero ideal of $D$ and $p_1D, \dots, p_nD$ the principal height-one maximal ideals containing $I$. Since each $p_iD$ has the intersection of its powers equal to zero, we can write $I = xI'$ where $x \in D$ and either $I' = D$ or $M$ is the only maximal ideal containing $I'$. (For suppose $I \subseteq p_i^{s_i}D$, but $I \not\subseteq p_i^{s_i+1}D$. Then $I \subseteq p_1^{s_1}D \cap \cdots \cap p_n^{s_n} = p_1^{s_1} \cdots p_n^{s_n}D$. Take $x = p_1^{s_1} \cdots p_n^{s_n}$, so $I \subseteq xD$ and hence $I = xI'$ where $I' \supseteq I$. Since $I' \not\subseteq p_iD$, either $I' = D$ or $M$ is the only maximal ideal containing $I'$.)

Now, let $I$, $J$ be a pair of proper nonzero ideals of $D$. We show that $I$, $J$ is an SC pair. We can assume that neither is principal. Moreover, as noted above, we can assume that $M$ is the only maximal ideal containing $I \cap J$. Since $D_M$ is SC, we can assume that $I_M J_M = iJ_M$ for some $i \in I$ such that $M$ is the only maximal ideal containing $iD$ (see argument given above). So the equality $IJ = iJ$ holds locally and hence globally. ■

***Remark 3.5*** A condensed domain need not be $h$-local (take any non $h$-local Bézout domain, for instance the ring of entire functions, *cf.* [8, page 147]), but a SC domain is $h$-local.

We have already remarked that an integrally closed domain is condensed if and only if it is Bézout [3, Main Theorem]. However, the case of an integrally closed domain $D$ being SC is more delicate. Certainly an integrally closed SC domain being condensed is Bézout. According to [6, Proposition 5.3.8], a valuation domain $V$ is strongly discrete (*i.e.*, $V$ has no nonzero idempotent prime ideals) if and only if for each nonzero ideal $I$ of $V$, $I$ is a principal ideal in $I : I$. Consequently, by Proposition 3.3, an integrally closed quasilocal domain is SC if and only if it is a strongly discrete valuation domain. Recall that a domain $D$ is called *generalized Dedekind* if $D$ is a strongly discrete Prüfer domain (*i.e.*, $D_M$ is a strongly discrete valuation domain for each maximal ideal $M$ of $D$) with Noetherian spectrum (see [6, Chapter 5]). By [6, Lemma 5.8.2], each maximal ideal of a generalized Dedekind domain is invertible. These remarks and Theorem 3.4 give the following result.

***Corollary 3.6*** *Let $D$ be an integrally closed domain. Then $D$ is an* SC *domain if and only if $D$ is a Bézout generalized Dedekind domain with at most one maximal ideal of height greater than one.*

A. Facchini has shown [5, Theorem 5.3] that for every Noetherian tree $X$ with a least element there exists a generalized Dedekind domain $D$ whose prime spectrum $\big(\mathrm{Spec}(D), \subseteq\big)$ is order isomorphic to $X$. Applying this theorem for appropriate trees

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(*e.g.*, $\big(\{\varnothing, \{1\}, \{1,2\}, \{3\}\}, \subseteq\big)$), it follows there exist integrally closed SC domains which are neither PIDs nor valuation domains.

We thank B. Olberding for suggesting the following theorem.

***Theorem 3.7*** *Let $D$ be an integral domain. Then $D$ is an integrally closed* SC *domain if and only if every proper ideal $I$ of $D$ has the form $I = \mathrm{Pr}$ where $P$ is a prime ideal of $D$ and $r \in D$.*

**Proof** ($\Rightarrow$) We recall two results from [7]. A Prüfer domain $E$ is a generalized Dedekind domain if and only if every divisorial $I$ of $E$ can be written as $I = JP_1 \cdots P_n$ where $J$ is an invertible ideal and $P_1, \dots, P_n$ are pairwise-comaximal prime ideals [7, Theorem 3.3]. Moreover, for a generalized Dedekind domain, every nonzero ideal is divisorial if and only if every nonzero prime ideal is contained in a unique maximal ideal [7, Proposition 3.6]. Now, let $D$ be an integrally closed SC domain. By the previous corollary, $D$ a generalized Dedekind domain with at most one maximal ideal which is not principal of height one. So the cited results apply.

($\Leftarrow$) According to [14, Theorem 2.3], a domain whose every ideal can be written as a product of invertible ideals and prime ideals is a strongly discrete h-local Prüfer domain. Hence so is our $D$. By [6, Corollary 5.4.10], $D$ is a generalized Dedekind domain. We next show that $\mathrm{Pic}(D) = 0$. Let $M$ be a maximal ideal of $D$. Since $D$ is a generalized Dedekind domain, $M$ is invertible. Now $M^2 = \mathrm{Pr}$ for some prime ideal $P$ of $D$ and $r \in D$. Since $M^2 \subseteq P$, $M = P$. But $M$ is invertible, so $M = rD$ is principal. Let $I$ be an invertible ideal of $D$. Then $I = \mathrm{Pr}$ for some prime ideal $P$ of $D$ and $r \in D$. Now $I$ invertible gives $P$ is invertible. But then $P$ is maximal and so $I = \mathrm{Pr}$ is principal. Suppose that $D$ has two maximal ideals $M$ and $N$ of height greater than one. Since $M$ and $N$ are principal, $M' = \cap_n M^n$ and $N' = \cap_n N^n$ are nonzero prime ideals of $D$ lying directly below $M$ and $N$, respectively. Since $D$ is $h$-local, $M$ (resp., $N$) is the only maximal ideal containing $M'$ (resp., $N'$). Now $M'N' = \mathrm{Pr}$ for some prime ideal $P$ of $D$ and $r \in D$. Since $P \supseteq M'N'$ and $P$ cannot be principal (for then $M'$ and $N'$ would be principal), $P = M'$ or $P = N'$; say $P = M'$. So $M'N' = M'r$. But then $N'D_N = M'N'D_N = M'rD_N = rD_N$ is principal, a contradiction. So $D$ is a generalized Dedekind domain with at most one maximal ideal that is not principal of height one. By the previous corollary, $D$ is SC. ∎

Recall that an integral domain $D$ is said to be *stable* (or *SV-stable* in the terminology of [6]) if each nonzero ideal $I$ of $D$ is invertible in $I:I$. Thus Proposition 3.3 gives that a SC domain is stable. Recall that a module is called *serial* if its submodules are linearly ordered with respect to inclusion. The next result is a SC variant of Corollary 2.5.

***Theorem 3.8*** *Let $D$ be a Noetherian domain. The following assertions are equivalent:*

(a) *$D$ is* SC,
(b) *$D$ is a* PID *or $D$ has exactly one nonprincipal maximal ideal $M$ and $D_M$ is* SC, *and*
(c) $\dim D \leq 1$, $\mathrm{Pic}(D) = 0$ *and $D'/D$ is a serial $D$-module, where $D'$ is the integral closure of $D$.*

**Proof** (a) $\Leftrightarrow$ (b). *Cf.* Theorem 3.4.

(a) $\Rightarrow$ (c). By [4], $\dim D \leq 1$. Also, $\mathrm{Pic}(D) = 0$, *cf.* Theorem 2.4. By [15, Lemma 2.1], each $D$-submodule of $D'$ containing $D$ is a ring. By Lemma 3.2, we deduce that $D'/D$ is a serial $D$-module.

(c) $\Rightarrow$ (a). Since $D'/D$ is a serial $D$-module, condition (b) of Proposition 3.3 holds for $D$. By [15, Theorem 3.2], $D$ is stable, so, to verify condition (a) of Proposition 3.3, it suffices to show that $\mathrm{Pic}(I:I) = 0$ for each nonzero ideal $I$ of $D$. But this follows from [1, Theorem 12] and the remark following it. ■

If $m$ is a square free integer $\neq 0, 1$, then $D = \mathbf{Z}[\sqrt{m}]$ is SC if and only if its Picard group is zero. Indeed, the factor $D$-module $D'/D$ has at most two elements, hence it is serial, so Theorem 3.8 applies.

Let $D$ be a Noetherian integral domain. $D$ is said to have the *two-generator property* if every ideal of $D$ is generated by two elements. If $D$ has the two-generator property, then $D$ is stable. The converse is true if $D'$ is a finite $D$-module, but not in general, *cf.* [16, Example 5.4]. Now, let $D$ be a one-dimensional Noetherian domain. By [15, Theorem 3.2], $D$ has the two-generator property if and only if $D'/D$ is a distributive $D$-module (*i.e.*, $(D'/D)_M$ is a serial $D_M$-module for each maximal ideal $M$ of $D$). Consequently, if $D$ is a SC Noetherian domain, then $D$ has the two-generator property. In the local case we obtain the following corollary.

***Corollary 3.9*** *A local domain $D$ is* SC *if and only if $D$ has the two-generator property.*

In particular, the characteristic-two case of Nagata's example [13, Example 3, page 205] of a local one-dimensional domain whose integral closure is not a finite module is SC. Combining Corollary 3.9 with Theorem 2.4, we see that a domain $D$ having the two-generator property and $\mathrm{Pic}(D) = 0$ is condensed.

***Example 3.10*** There exist domains $D$ having the two-generator property and $\mathrm{Pic}(D) = 0$ (hence condensed) which are not SC. Indeed, let $\omega = \frac{1+\sqrt{5}}{2}$. Then $\omega$ is the fundamental unit of the PID $\mathbf{Z}[\omega]$. Combining [17, Theorem 4.1, Remark 4.7, Proposition 4.8, Theorem 4.9], it follows that $\mathbf{Z}[2\omega]$ and $\mathbf{Z}[5\omega]$, hence also $\mathbf{Z}[10\omega]$, have zero Picard group. Now $\mathbf{Z}[10\omega]$ has the two-generator property, but it is not SC because its overrings $\mathbf{Z}[2\omega]$ and $\mathbf{Z}[5\omega]$ are integral over $\mathbf{Z}[10\omega]$ and not comparable.

If $D$ is a local domain whose integral closure is finite over $D$, we get the following result.

***Theorem 3.11*** *Let $(D, M, k)$ be a local domain whose integral closure $D'$ is a finite $D$-module. The following assertions are equivalent:*

(a) *$D$ is* SC,
(b) *$D$ is stable (equivalently, $I, I$ is an* SC *pair for every ideal $I$ of $D$),*
(c) *$D$ has the two-generator property,*
(d) *$D$ is Gorenstein and $M$, $M$ is an* SC *pair,*
(e) *$D' = D + D\theta$ for some $\theta \in D'$,*

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(f) *$D'/MD'$ is isomorphic to one of the following $k$-algebras: $k$—in this case $D = D'$ is a* DVR, *a quadratic field extension of $k$, $k \times k$, or $k[X]/(X^2)$.*

*If $(D', M', K)$ is local, then the preceding assertions are also equivalent to:*

(g) *$D$ is condensed and $[K:k] \neq 3$,*
(h) *$K = k$ and $D$ contains a value-two element of $D'$ or $[K:k] = 2$ and $D$ contains a value-one element of $D'$.*

**Proof** Clearly (a) $\Leftrightarrow$ (c). *Cf.* Corollary 3.9.

(b) $\Leftrightarrow$ (c). *Cf.* [16, Theorem 2.4] (the paranthetical form of (b) comes from our assumption that $D'$ is semi-local, so $I:I$ is semi-local for each nonzero ideal $I$).

(c) $\Leftrightarrow$ (e). *Cf.* [10, Theorem 2.3].

(c) $\Leftrightarrow$ (d). *Cf.* [15, Theorem 3.2].

(f) $\Rightarrow$ (e). In each one of these cases $[D'/MD':k] \leq 2$, so [10, Proposition 1.1] applies.

(b) $\Rightarrow$ (f). By [11, Theorem 6], each $D$-submodule of $D'$ containing $D$ is a ring, each $k$-subspace of $D'/MD'$ is a ring. After an application of [11, Lemma 5], it suffices to notice that if $D'/MD'$ is a local ring with square zero radical and residue field $k$, then it is an $k$-epimorphic image of $k[X]/(X^2)$.

(a) $\Rightarrow$ (g). Because (a) is equivalent to (e).

(g) $\Rightarrow$ (h). By Proposition 2.8 applied for $D \subseteq D'$, $[K:k] \leq 3$, so $[K:k] \leq 2$. Since $D$ is condensed, it contains a value-two element of $D'$, *cf.* [9, Proposition 1]. So, it suffices to show that if $[K:k] \geq 2$ and $D$ is condensed, then $D$ contains a value-one element of $D'$. Let $\theta \in D'$ such that $K = k(\bar{\theta})$. Let $x$ be a prime element of $D'$. Since $D'$ is a finite $D$-module, there exists $N$ such that $x^N D' \subseteq D$. In $D$ we consider the ideals $I = (x^N, x^{N+1})$ and $J = (x^N, \theta x^{N+1})$. Then $(1+\theta)x^{2N+1} = x^{N+1}x^N + x^N(\theta x^{N+1}) \in IJ$. Since $D$ is condensed, $(1+\theta)x^{2N+1} = (fx^N + gx^{N+1})(hx^N + i\theta x^{N+1})$, for some $f, g, h, i \in D$. Hence $(1+\theta)x = (f+gx)(h+i\theta x)$, so $f \in xD'$ or $h \in xD'$. Assume that $f = xf'$ with $f' \in D'$. Then $1+\theta = (f'+g)(h+i\theta x)$ and $f' \notin xD'$, otherwise $\bar{1}+\bar{\theta} = \bar{g}\bar{h} \in k$, a contradiction. Now, assume that $h = xh'$ with $h' \in D'$. Then $1+\theta = (f+gx)(h'+i\theta)$ and $h' \notin xD'$, otherwise $\bar{1}+\bar{\theta} = \bar{f}\bar{i}\bar{\theta} \in k\bar{\theta}$, again a contradiction. Consequently, $f$ or $h$ has value one in $D'$.

(h) $\Rightarrow$ (f). Assume that $K = k$ and $D$ contains a value-two element of $D'$. Then $MD'$ contains the square of the prime element $q$ of $D'$. Since $K = k$, the $k$-algebra morphism $k[X] \to D'/MD'$ sending $X$ to $q + MD'$ is surjective, hence $D'/MD'$ is an epimorphic image of $k[X]/(X^2)$. Thus $D'/MD'$ is either $k[X]/(X^2)$ or $k$. If $[K:k] = 2$ and $D$ contains a value-one element of $D'$, then $D'/MD' = K$. We note that the first case of implication (h) $\Rightarrow$ (a) was proved in [9, Proposition 6]. ■

By the proof of implication (g) $\Rightarrow$ (h) in Theorem 3.11, if $k \subset K$ is a proper field extension and $n \geq 2$, then $k + X^n K[[X]]$ is not condensed.

We end by giving an example of a local condensed domain that is not SC; thus answering a question raised by Gottlieb [9].

***Example 3.12*** Let $\mathbf{B} \subset \mathbf{B}(\sqrt[3]{3})$ be the field extension in Example 2.11. Then $\mathbf{B} + X\mathbf{B}(\sqrt[3]{3})[[X]]$ is a local condensed domain which is neither SC nor Gorenstein.

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