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WREATH PRODUCT OF O^*-GROUPS THAT IS NOT IN O^*

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1. **Introduction.** It is well known that the wreath product of two ordered groups is an ordered group. In [2] Fuchs asks if the same is true for O^*-groups. Here we construct an example to show that the wreath product of an infinite cyclic group with a free metabelian group is not an O^*-group.

Following Fuchs [3] we shall call a group G an O^*-group if every partial order on G can be extended to a full order on G. We shall use the following characterization of O^*-groups given by Ohnishi in [4].

F is an O^*-group if and only if the following two conditions hold:

(i) If $1 \neq a \in G$ then $1 \notin S(a)$ where $S(a)$ is the semigroup generated by all conjugates of a in G.

(ii) If $b \in S(a)$ and $c \in S(a)$ then $S(b) \cap S(c) \neq \varnothing$.

Property (ii) is inherited by homomorphic images. Hence in order to show that a group G is not an O^*-group it is enough to show that some homomorphic image of G does not satisfy (ii).

2. THEOREM 1. *Every free metabelian group is an* O^*-group.

Proof. By a result of Baumslag [1] every free polynilpotent and in particular free metabelian group is an O-group. By Theorem 1 of [5] every metabelian O-group is an O^*-group.

3. THEOREM 2. *Let $A=\langle a\rangle$ be an infinite cyclic group and G be a free metabelian group on two generators. Then A* wr *G is not an O^*-group.*

Proof. If $D=\langle x, y; x^2=y^2=1\rangle$ is the infinite dihedral group it is clear that A wr D is a homomorphic image of A wr G. Therefore it is enough to prove the following:

LEMMA. *$W=A$* wr *D does not satisfy property* (ii).

It is well known that if R is the integral group ring of D then W is a semi-direct product of the additive group of R and D. In this notation it is sufficient to show that for all r, s of the form

$$\sum \zeta_g g(g \in D; \zeta_g \geq 0)$$

one has

$$(1+x)r \neq (1+y)s \text{ in } R.$$

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Suppose that r and s are any two elements say $r=1+g_2+\cdots+g_n$, $s=h_1+h_2+\cdots+h_m$, where g_i's and h_i's need not be distinct. Then $(1+x)(1+g_2+\cdots+g_n)=(1+y)(h_1+h_2+\cdots+h_m)$. Therefore the sets

$$I = \{1, g_2, \ldots, g_n, x, xg_2, \ldots, xg_n\},$$
$$II = \{h_1, \ldots, h_m, yh_1, \ldots, yh_m\}$$

are the same. $1 \in I$ hence for some i_1, $h_{i_1}=1$ or y. In either case $y \in II$. Now let $y(xy)^k \in II$ where k is a positive integer. As $x \neq y(xy)^k$, there is some j_k such that

$$g_{j_k} = y(xy)^k \quad \text{or} \quad (xy)^{k+1}.$$

In either case $(xy)^{k+1} \in I$. Therefore there exists some i_k such that

$$h_{i_k} = (xy)^{k+1} \quad \text{or} \quad y(xy)^{k+1}.$$

In either case $y(xy)^{k+1} \in II$. Thus $y(xy)^k \in II$ for all positive integers k. As II is a finite set, this gives that $(xy)^m=1$ for some $m>0$. A contradiction. Hence the result.

References

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