CORRESPONDENCE.

THE RELATION BETWEEN THE RATES OF MORTALITY, OR DECREMENT, FROM PARTICULAR CAUSES, AND THE RATES FROM ALL CAUSES, WITH SOME REMARKS UPON THE PREVALENT MISUSE OF THE TERM "RATE."

To the Editors of the Journal of the Institute of Actuaries.

DEAR SIRS,—With reference to Mr. D. S. Savory's letter *J.I.A.*, vol. li, p. 65, regarding the mortality due to the war, I beg to submit the following remarks dealing with the subject somewhat generally.

The distinction between a probability and a rate is one that deserves to be more clearly emphasized. The term "rate of mortality", as generally employed, might be defined as meaning the proportion dying in a year out of a number exposed to the risk of death for the duration of a year or until death within the year, the words "per annum" being understood. This particular definition has most conveniently led to the function q_x in an ordinary mortality table becoming invested with a dual capacity; for q_x not only represents the annual rate of mortality as above defined, but also the probability of dying within a year, at age x.

When, however, we come to split q_x into its component parts, the distinction is a real one. For we can subdivide q_x, the total probability of dying in a year, into the partial, mutually exclusive, and additive probabilities of dying in a year from cancer, consumption, &c., whose sum equals exactly the total probability of dying in a year; but we cannot so split up the rate of mortality q_x into partial rates of mortality, for the rates of mortality from different diseases are proportions of dissimilar things and are therefore not additive, *e.g.*, the rate of mortality from consumption is the proportion dying from consumption of those exposed for a year to that particular risk, while the rate of mortality from cancer is a proportion of another thing.

Coming now to the particular problem (which is somewhat similar to that dealt with by Dr. Sprague in *J.I.A.*, vol. xxi, p. 406, *see* also Ackland, vol. xxxiii, p. 194): Given the values q_{ng}, the rate of mortality from normal and war causes combined, and q_n, the rate of mortality due to normal causes, it is desired to find an expression for the rate of mortality due to the war.

Let d_n and d_g be the number dying during a war year from normal and war causes respectively, out of l persons alive at the commencement of the year; then $\frac{d_n}{l}$ and $\frac{d_g}{l}$ will be the probabilities of dying from the respective causes mentioned. Let these probabilities be designated q'_n and q'_g, as distinguished from q_n and q_g, the corresponding rates of mortality.

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Then
$$q_n = \frac{d_n}{l - \frac{1}{2} d_g}$$

Dividing both numerator and denominator by l, we have
$$q_n = \frac{q'_n}{1 - \frac{1}{2} q'_g}$$

Whence
$$q'_n = q_n\left(1 - \frac{1}{2} q'_g\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{A}$$

Similarly
$$q_g = \frac{d_g}{l - \frac{1}{2} d_n}$$

Whence
$$q'_g = q_g\left(1 - \frac{1}{2} q'_n\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{B}$$

It is easily seen that
$$q_{ng} (\text{or } q'_{ng}) = q'_n + q'_g \quad . \quad . \quad . \quad . \quad . \quad \text{C}$$

Substituting first A and then B, we derive from C
$$q'_g = \frac{q_{ng} - q_n}{1 - \frac{1}{2} q_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{D}$$

$$q'_n = \frac{q_{ng} - q_g}{1 - \frac{1}{2} q_g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{E}$$

From B,
$$q_g = \frac{2q'_g}{2 - q'_n} :$$

whence, substituting for q'_g and q'_n the values given in D and E,
$$q_g = \frac{2(q_{ng} - q_n)}{1 - \frac{1}{2} q_n} \times \frac{1 - \frac{1}{2} q_g}{2 - q_{ng}},$$

i.e.,
$$q_g\left\{\left(1 - \frac{1}{2} q_n\right)\left(2 - q_{ng}\right) + \left(q_{ng} - q_n\right)\right\} = 2\left(q_{ng} - q_n\right)$$

Whence we get
$$q_g = \frac{q_{ng} - q_n}{1 - q_n + \frac{1}{4} q_n q_{ng}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{F}$$

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Equations D and F represent respectively the probability of dying and the rate of mortality, from war causes, as expressed in terms of the known quantities q_{ng} and q_n. Formula F agrees with that got by Prof. Cantelli except for the third term in the denominator, his result being $\frac{q_{ng} - q_n}{1 - q_n}$. The Professor's reasoning as quoted by Mr. Savory evidently omits certain considerations.

With regard to the statement credited to the Professor that his result follows from Karup's theorem

$$p_{ng} = p_n \times p_g$$

the probabilities p_n and p_g in this particular instance are not altogether subjects for multiplication as they do not relate to events that are quite independent. The true relation is

$$p_{ng} = (1 - q_{ng}) = 1 - (q'_n + q'_g)$$

Formulas D and E would furnish the means of expressing p_{ng} in terms of p_n and p_g.

As regards the method adopted by Prof. Hersch, this would have been quite correct had he been dealing with probabilities instead of rates, for $q'_g = q_{ng} - q'_n$, but as indicated above, it is not accurate to add or subtract rates.

Among our writers there is, so far as terminology is concerned, a want of precision in discriminating between a rate and a probability. In a pension fund for instance, we require the *probabilities of exit* by death, withdrawal, &c. Out of a number who attain a given age we require to know simply the proportion that will go off the fund by each of the modes of exit within a year, and the proportion remaining on the fund at the end of the year ; it is a simple splitting up of the given number, the results representing the mutually exclusive probabilities in respect of each of the possible ways in which the event can happen, and adding to unity. These proportions or probabilities are commonly and loosely described as rates of mortality, withdrawal, &c., but they are clearly not properly so described. Another term used with an appearance of greater precision is "the rate of mortality while on the active list", but this is also inexact. The rate of mortality while on the active list is in fact correctly represented by the following expression :

$$\left.\begin{array}{c}\text{Rate of mortality while}\\ \text{on the active list}\end{array}\right\} = \frac{q'_x}{1 - \frac{1}{2}(wq'_x + rq'_x)}$$

where q', wq' and rq' represent the probabilities of exit by death, withdrawal, and retirement. For if not, what is the correct designation of this expression ?

The cause of the confusion is evident. Insurance is effected

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against a probability, never against a rate; and when dealing with ordinary life insurance, actuaries could hardly help seeking for the probability of dying within a year, or the simple proportion of those dying in a year out of a number alive at the commencement. It was expedient to define the abstract function known as the rate of mortality so that it would coincide with this, rather than to define it in any other way, such for instance as that proposed by Dr. Farr; and therefore when we are dealing with a single force such as death or withdrawal, the probability and the rate have the same value. In short, actuaries obtained the probability and called it the rate. The term "rate", however, which a nice discrimination would have restricted to the abstract idea, was allowed to impose itself everywhere, even in respect of probabilities with quite a different value, to the extent that the idea of a probability with its resultant simplicity appears to have been banished to a precarious footing on the margin of consciousness.

It is curious to note how even some of the most eminent of our writers have paid homage to the usurper. The late Mr. Manly, for instance, obtained what were apparently true rates of mortality and withdrawal (q_x and wq_x), and used them as probabilities of exit, admittedly as an approximate measure, but without an appropriate change in designation, following in this respect the prevalent custom (*J.I.A.*, vol. xxxvi, pp. 211, 260, 261). The retention of the incorrect terminology is responsible for half the difficulties alluded to on p. 4 of vol. xlii, besides rendering the reasoning obscure, and leaving its mark on the formulas there given. On the other hand, in vol. xxxix, p. 133, Mr. George King lays down very clearly the correct procedure for deriving the probabilities of exit required in a pension fund, but his ultimate retention of the incorrect term "rate" has led to his explanation being unnecessarily cumbered by the fiction (doing violence to the facts) that "in getting out the rate of mortality we must therefore treat the withdrawals and retirements as at risk for the whole year", &c.: a relative, apparently, of the older fiction that in getting out the rate of mortality simpliciter, deaths were to be treated as at risk for the whole year.

In Insurance functions of all kinds it is always *the probability* that we require, and the expulsion of the term "rate" herefrom at once removes the accompanying obscurities and rationalizations, and clears the channel of thought.

A great deal would be gained if a special symbol were used, where appropriate, to designate a probability as distinct from a rate (as in this communication for example), and if at the same time the concrete functions employed in financial computations, particularly those used in respect of pension funds, were given their proper designations. This need not in any way interfere with the undoubted convenience gained from the use of the term "rate" in the purely general sense. We could very easily speak of the general characteristics of the rates of mortality, &c., for example, and at the same time take care to describe our algebraic and tabular functions as probabilities or proportions.

I should add that the above examples are not quoted by way of criticism of the eminent writers named but merely as illustrations.

I am, Dear Sirs,

Yours faithfully,

A. T. TRAVERSI.

Wellington,
New Zealand.
10 *August* 1918.

P.S.—Though in the problem dealt with by him Dr. Sprague very clearly differentiated between a rate and a probability, nevertheless his awkward expression "the annual marriage rate among bachelors who do not die in the year" (*J.I.A.*, vol. xxi, pp. 413 and 415) involves some confusion regarding the definition of a rate, and it is surprising that his terminology in this instance should apparently have remained unchallenged. If "rate" is defined as in the second paragraph of this letter (*mutatis mutandis*), it is clear that the function in question could be described with the most rigid accuracy and with greater simplicity as *the annual marriage rate among bachelors*, notwithstanding Dr. Sprague's deliberate rejection of the latter expression in favour of his own.

Dr. Sprague's phrase is in fact a distinct misdescription of the function, and a realization of this fact would be of considerable help to students. If we were really in pursuit of the marriage rate among bachelors who do not die in the year we should require to exclude altogether from the figures the bachelors who die in the year. True, it might at first sight appear that there should be no difference between the annual marriage rate among bachelors and that among bachelors who do not die in the year, seeing that the rate among bachelors who die in the year is nil: but as indicated above, we cannot add or subtract rates.

Similar remarks apply to the phrase "the annual death rate among bachelors who do not marry in the year."

A. T. T.

THE NATIONALITY OF TETENS.

To the Editors of the Journal of the Institute of Actuaries.

DEAR SIRS,—May I be permitted, as a Corresponding Member of the Institute, to call attention to an excusable error concerning one of my countrymen, which I happened to notice in that reliable standard work, the Institute *Text-Book*. I find that the first inventor of commutation-columns (J. N. Tetens) is called a German professor,