Canad. Math. Bull. Vol. **21** (1), 1978

ISOMORPHISMS OF FUNCTION ALGEBRAS AND ALGEBRAS OF ANALYTIC FUNCTIONS

BY
BRUCE LUND

ABSTRACT. Let R be a finite open Riemann surface with analytic boundary Γ. Set $\bar{R}=R\cup\Gamma$ and define $A(R)=\{f\in C(\bar{R}):f$ is analytic on $R\}$. Conditions are given on a function algebra A on a compact Hausdorff space X which imply that A is isomorphic to a subalgebra of $A(R)$ of finite codimension.

§1 **Introduction.** Let R be a connected finite open Riemann surface with analytic boundary Γ and set $\bar{R}=R\cup\Gamma$. Let $C(\bar{R})$ denote the complex-valued continuous functions on $\bar{R}$. We define $A(R)=\{f\in C(\bar{R}):f$ is analytic on $R\}$. Then $A(R)$ is a function algebra on $\bar{R}$. The goal of this paper is to give an abstract function algebraic characterization of the subalgebras of finite codimension of the function algebra $A(R)$. This work will employ the results of "f-regularity" theory introduced by E. Bishop. The main result of the paper will be related to a theorem of T. Gamelin on embedding a Riemann surface in the maximal ideal space of a function algebra.

§2 **Definitions and statement of results.** Throughout this paper A will be a function algebra on a compact Hausdorff space X. We list some definitions and facts about function algebras and refer the reader to the references [8] and [18] for details. We let M_A denote the maximal ideal space of A and give M_A the weak-star topology which is inherited from the dual space of A. If $f\in A$, then $\hat{f}$ will be the Gelfand transform of f. If f is a map from Y to Z, then $\#f^{-1}(z)$ denotes the cardinality of the set $f^{-1}(z)$. In particular, if $f\in A$, then $\#\hat{f}^{-1}(s)$ denotes the cardinality of the set $\{\sigma\in M_A:\hat{f}(\sigma)=s\}$.

We next define the notion of an analytic embedding Ψ from R into M_A. Let $H^\infty(R)$ be the set of bounded analytic functions on R. Let $W\subset M_A$. If there is a continuous mapping Ψ of R onto W (where W has the relative topology of M_A) such that $\tilde{f}(z)=\hat{f}\circ\Psi(z)\in H^\infty(R)$ for $f\in A$, then we will call Ψ an *analytic mapping from R onto W*. We will be concerned with analytic maps Ψ for which the set $\{(a,b):a\neq b, \Psi(a)=\Psi(b)\}$ is finite. In this case we will say that Ψ *identifies a finite number of points.*

AMS (MOS) Subject Classification, Primary 46J10
Key Words and Phrases: function algebra, Gleason part, f-regular
Supported in part by NRC (of Canada) grant A8763.
Received by the editors January 4, 1977 and, in revised form, July 29, 1977.

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

As further notation we use $T=\{z:|z|=1\}$ and $\Delta=\{z:|z|<1\}$. Our principal result is stated below:

THEOREM 1. *Let A be a function algebra on X. Assume that*

(i) *There is $W\subset M_A$ and an analytic map $\Psi:R\to W$ such that Ψ identifies at most a finite number of points.*

(ii) $\bar{W}\setminus W=X$

(iii) *The set $S=\{g\in A:|g|=1$ on $X\}$ separates points on X and there is $F\in S$, nonconstant on X, such that $E=\{x\in T:\#\hat{F}^{-1}(x)<\infty\}$ contains a set of positive Lebesgue measure in T.*

Then Ψ extends continuously to $\bar{R}$ and maps Γ onto X. Let $\tilde{\Psi}:A\to A(R)$ be defined by $\tilde{\Psi}(f)=\hat{f}\circ\Psi$ and define $A'=\tilde{\Psi}(A)$. Then $\tilde{\Psi}$ is an isomorphism of A onto A' and as a vector subspace A' has finite codimension in $A(R)$. Moreover, $M_A=\bar{W}$.

REMARK. Let f be analytic in a neighborhood of $\bar{R}$. If there is an integer n such that f maps $\bar{R}$ onto $\bar{\Delta}$ in an n-to-one manner counting multiplicity with $f(\Gamma)=T$, then we will call f a *unimodular function.* That such functions exist was proved by Ahlfors. Stout [17] showed that the unimodular functions separate points on $\bar{R}$. In light of these remarks, it is clear that $A(R)$ satisfies the hypotheses (i), (ii), (iii) of the theorem above.

For later reference we define the *order of a unimodular function* f to be n if f is an n-to-one (counting multiplicity) map of $\bar{R}$ onto $\bar{\Delta}$.

§3. **Preliminary lemmas.** Let the boundary Γ of R consist of simple closed analytic curves $\Gamma_1,\ldots,\Gamma_n$. We call $B\subset R$ a *boundary strip* if B consists of n pairwise disjoint annular domains $K_1,\ldots,K_n$ such that $\Gamma_j\subset\bar{K}_j$ and such that there is a coordinate map ϕ_j from an annulus $\{z:s<|z|<1\}$ onto K_j which extends to map T onto Γ_j.

The lemma below appears to be straightforward; however, a careful proof becomes somewhat detailed. This result does not seem to appear in standard references.

LEMMA 1. *Let $f:R\to\mathbb{C}$ be an analytic map. Suppose there is a positive integer m such that $\#f^{-1}(x)\le m$ for all $x\in f(R)$ and suppose $E=\{x\in f(R):\#f^{-1}(x)<m\}$ is a discrete set in $f(R)$. Then f is a proper map. (That is, if $L\subset f(R)$ is compact, then $f^{-1}(L)$ is compact in R.)*

Proof. It suffices to show that if $x_0\in f(R)$, then there is a compact neighborhood N of x_0 such that $f^{-1}(N)$ is compact in R. The case where $x_0\in f(R)\setminus E$ is straightforward and we consider the case where $x_0\in E$. Let G be an open set with compact closure in R such that $f^{-1}(x_0)\subset G$. Pick a disk $V(r)$ about x_0 of radius r so that $V(r)\subset f(G)$ and $V(r)\cap E=\{x_0\}$. We wish to show that r may be picked so small that $f^{-1}(\overline{V(r)})\subset\bar{G}$ which will prove the Lemma.

Given a boundary strip B such that $\bar{B}\cap\bar{G}=\varnothing$, it is easy to prove that r may

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

be chosen so small that $f^{-1}(V(r)) \subset G \cup B$. Suppose B and r are selected as above. We wish to show that $f^{-1}(V(r)) \cap B = \emptyset$. We suppose the contrary.

Let $V_0(r) = V(r) \setminus \{x_0\}$. Then f is an m-to one map of $f^{-1}(V_0(r))$ onto $V_0(r)$. It follows that if f is restricted to any component U of $f^{-1}(V_0(r))$, then f is a regular covering map of U onto $V_0(r)$ ([1] p. 29). Let U be a component which lies in B. Since U is connected, there is a single annular domain $K_j \subset B$ with $U \subset K_j$. In what follows we will identify K_j with $W = \{z : s < |z| < 1\}$ and Γ_j with T.

Given any $\varepsilon > 0$ we claim that we can find a neighborhood Ω of T such that if $a \,\varepsilon \Omega \cap W$, then $|f(a) - x_0| < \varepsilon$. Once we have shown this, then f can be extended continuously to T by setting $f(z) = x_0$ for $z \in T$. In this case f must reduce to a constant function giving us a contradiction.

We now proceed to find the desired neighborhood Ω. We may suppose that $\varepsilon < r$ and thus the conclusions above apply to the punctured disk $V_0(\varepsilon)$. Let $\gamma(t)$ be the circle about x_0 of radius $\varepsilon/2$. Since $f : U \to V_0(\varepsilon)$ is a finite-to-one covering, a suitable multiple of γ lifts to a closed curve $\tilde{\gamma}$ lying in U. Moreover, $\tilde{\gamma}$ is not null-homotopic in W since otherwise $f \circ \tilde{\gamma}$ would be null-homotopic in $\mathbb{C} \setminus \{x_0\}$. Thus $\tilde{\gamma}$ is homotopic to a multiple of a generator of the fundamental group of W.

Consequently, if Ω is the unbounded component of $\mathbb{C} \setminus \tilde{\gamma}$, then $T \subset \Omega$ and $\{z : |z| \leq s\} \cap \Omega = \emptyset$. We claim that Ω is our sought-after neighborhood of T. Take $a \,\varepsilon \Omega \cap W$. We must show that $|f(a) - x_0| < \varepsilon$.

Let $W' = \{z : s' < |z| < 1\}$ be chosen so that $a \notin W'$. By the arguments given earlier, there is an $\varepsilon' < \varepsilon$ such that $f^{-1}(V_0(\varepsilon')) \cap W \subset W'$. Furthermore, if γ' is the circle about x_0 of radius $\varepsilon'/2$, then a suitable multiple of γ' lifts to a closed curve $\tilde{\gamma}'$ lying in W' which is not null-homotopic in W'. Now let Ω' be the component of $\mathbb{C} \setminus \tilde{\gamma}'$ which contains the origin. Then $\Omega' \subset \Delta$. Thus, $\Omega \cap \Omega'$ is a subset of W with $\partial(\Omega \cap \Omega') \subset (\partial\Omega \cup \partial\Omega') \subset \tilde{\gamma} \cup \tilde{\gamma}'$. Since $|f(z) - x_0| < \varepsilon$ on $\tilde{\gamma} \cup \tilde{\gamma}'$ and $a \in \Omega \cap \Omega'$, it follows by the maximum principle that $|f(a) - x_0| < \varepsilon$. This completes the proof.

In the next lemma important use is made of the results of "f-regularity". See the references [3, 4] and [9] for an exposition of this theory.

Lemma 2. *Let A be a function algebra on X. Let $W \subset M_A$ satisfy $\bar{W} \setminus W \subset X$ and suppose $\Psi : R \to W$ is an analytic map of R onto W which identifies a finite number of points. Let $F \in A$ be nonconstant on W with $|F| = 1$ on X and suppose that the set $\{x \in T : \# \hat{F}^{-1}(x) < \infty\}$ contains a set of positive Legesgue measure. Then Ψ is a proper map of R onto W (where W has the relative topology of M_A) and $\tilde{F}(z) = \hat{F} \circ \Psi(z)$ is a unimodular function on $\bar{R}$. Furthermore, if Ψ is a 1–1 map, then Ψ is a homeomorphism.*

Proof. Since $\tilde{F}$ is analytic and nonconstant on R and since the topological boundary of $\hat{F}(M_A)$ is contained in $F(X)$, it follows that $F(X) = T$ and

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

$\hat{F}(M_A)=\bar{\Delta}$. From the analyticity of $\hat{F}$ on W we see that $W\cap X=\varnothing$ and $W\subset\hat{F}^{-1}(\Delta)$.

The hypotheses on F imply that Δ is F-regular of some multiplicity $n_0>0$ ([9] p. 62). Since Ψ identifies only a finite number of points, we may therefore conclude that there is some integer $M>0$ where $M=\max\{\#\tilde{F}^{-1}(z):z\in\tilde{F}(R)\}$. Let $V=\{z\in\tilde{F}(R):\#\tilde{F}^{-1}(z)=M\}$. Since $\tilde{F}$ is an open map, it follows that V is open.

Since Δ is F-regular of order n_0, the set $\{z\in\Delta:\hat{F}^{-1}(z)<n_0\}$ is discrete in Δ. Moreover, since Ψ identifies only a finite number of points, it then follows that there exists $z'\in V$ and $g\in A$ such that $\tilde{g}=\hat{g}\circ\Psi$ separates $\tilde{F}^{-1}(z')$. Then for $z\in V$, let $\tilde{F}^{-1}(z)=\{a_1,\ldots,a_M\}$ and define

$$\Phi(z)=\begin{cases}0 & z\in\tilde{E}(R)\setminus V\\ \prod_{i<j}(\tilde{g}(a_i)-\tilde{g}(a_j))^2 & z\in V\end{cases}$$

Then $\Phi(z')\neq 0$ and $\Phi(z)$ is analytic on V. We claim that Φ is analytic on all of $\tilde{F}(R)$. By Rado's theorem we need prove only that $\Phi(z)$ is continuous on $\tilde{F}(R)$ ([18] p. 302).

Let z_0 be a boundary point of V as a subset of $\tilde{F}(R)$. It suffices to prove that if $\{z_n\}$ is a sequence in V which converges to z_0, then $\Phi(z_n)$ converges to $\Phi(z_0)=0$. Let $\varepsilon>0$ be given. We will find an N such that if $n\geq N$, then $|\Phi(z_n)|<\varepsilon$.

Let U be a neighborhood in M_A of $\hat{F}^{-1}(z_0)$ such that if $\sigma\in U$, then

$$\min\{|\hat{g}(\sigma)-\hat{g}(\alpha)|:\alpha\in\hat{F}^{-1}(z_0)\}<\varepsilon'$$

where

$$\varepsilon'=\frac{1}{2}\sqrt{\frac{\varepsilon}{(2\|g\|)^{M^2-M-2}}}.$$

Now as a preliminary step we wish to show that there is an integer N' such that if $n\geq N'$, then $\Psi(\tilde{F}^{-1}(z_n))\subset U$.

We suppose the contrary. Then it follows that there is a subsequence $\{s_n\}$ of $\{z_n\}$ and a sequence $\{w_n\}$ in R with $w_n\in\tilde{F}^{-1}(s_n)$ such that $\Psi(w_n)\notin U$ for all n. Now $\{\Psi(w_n)\}$ is a net in M_A. Let $\Theta\in M_A$ be an accumulation point and let $\{\Psi(w_{j(\alpha)})\}$ be a subnet converging to Θ. The continuity of $\hat{F}$ implies that $\hat{F}(\Psi(w_{j(\alpha)}))$ converges to $\hat{F}(\Theta)$. Since $\hat{F}(\Psi(w_{j(\alpha)}))=\tilde{F}(w_{j(\alpha)})=s_{j(\alpha)}$, we may conclude that $\hat{F}(\Theta)=z_0$. Therefore, $\Theta\in\hat{F}^{-1}(z_0)$ and for some α' we have $\Psi(w_{j(\alpha')})\in U$ giving us a contradiction.

Finally, since Ψ identifies only a finite number of points, we can find $N\geq N'$ such that if $n\geq N$, then $\Psi(\tilde{F}^{-1}(z_n))\subset U$ and, additionally, so that Ψ is 1–1 when restricted to $\tilde{F}^{-1}(z_n)$.

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

Since $z_0 \notin V$, it follows that $\hat{F}^{-1}(z_0) \cap W$ contains less than M points. Meanwhile, for $n \geq N$ we have $\Psi(\tilde{F}^{-1}(z_n)) \subset U$ and $\Psi(\tilde{F}^{-1}(z_n))$ contains M points. Therefore, for each $n \geq N$, there exists some $\alpha^* \in \hat{F}^{-1}(z_0) \cap W$ such that at least two points α_n' and $\alpha_n'' \in \Psi(\tilde{F}^{-1}(z_n))$ satisfy the conditions:

$$|\hat{g}(\alpha_n') - \hat{g}(\alpha^*)| < \varepsilon'$$

$$|\hat{g}(\alpha_n'') - \hat{g}(\alpha^*)| < \varepsilon'$$

which together imply that $|\Phi(z_n)| < \varepsilon$. Now by Rado's theorem we conclude that $\Phi(z)$ is analytic on $\tilde{F}(R)$.

Since $\Phi(z)$ is not the zero function, the zero set of $\Phi(z)$ is discrete in $\tilde{F}(R)$. In particular, the complement of V is discrete in $\tilde{F}(R)$. Then as a consequence of Lemma 1 we conclude that $\tilde{F}: R \to \tilde{F}(R)$ is a proper map.

We next prove that $\Psi: R \to W$ is a proper map. Let $K \subset W$ be compact. Then $\hat{F}(K)$ is compact in $\tilde{F}(R)$. Since $\tilde{F}$ is proper, we conclude that $\tilde{F}^{-1}(\hat{F}(K))$ is compact in R. Then $\Psi^{-1}(K)$ is a closed subset of $\tilde{F}^{-1}(\hat{F}(K))$ and is therefore compact.

Let $\lambda \in \Gamma$ and let $\{z_n\}$ be a sequence in R converging to λ. Since Ψ is proper and $\bar{W} \setminus W \subset X$, we conclude that the accumulation points of $\{\Psi(z_n)\}$ are all contained in X. Consequently, $\text{limit}_{n\to\infty} |\tilde{F}(z_n)| = \text{limit}_{n\to\infty} |\hat{F} \circ \Psi(z_n)| = 1$. Applying the reflection principle, we see that $\tilde{F}$ extends as a meromorphic function on the doubled surface $\hat{R}$ of R. It follows that $\tilde{F}$ is unimodular on $\bar{R}$.

The set W is locally compact in the relative topology of M_A. To see this we note that $\tilde{F}^{-1}(\bar{\Delta}_r)$ is compact in R where $\Delta_r = \{z : |z| < r\}$ and $r < 1$. Hence $\Psi(\tilde{F}^{-1}(\bar{\Delta}_r))$ is compact in W. Since $\hat{F}^{-1}(\Delta_r) \cap W \subset \Psi(\tilde{F}^{-1}(\bar{\Delta}_r))$, it follows that W is locally compact. Consequently, if we assume in addition that Ψ is 1–1, then Ψ is a homeomorphism ([14] p. 285). This completes the proof.

The remarks above contain a proof of the following Lemma.

Lemma 3. *Let A be a function algebra on X. Let $W \subset M_A$ satisfy $\bar{W} \setminus W \subset X$ and suppose $\Psi: R \to W$ is a proper analytic map of R onto W. Then if $g \in A$ satisfies $|g| = 1$ on X, then $\tilde{g}(z) = \hat{g} \circ \Psi(z)$ is a unimodular function on $\bar{R}$ or $\tilde{g}$ is a constant function.*

The assumption of continuity for an analytic map Ψ, rather than assuming further that Ψ is also a proper map, is natural since various embedding theorems which give analytic structure to certain types of subsets W in M_A produce a map $\Psi: R \to W$ which need only be continuous. (See [8] p. 154–161 and [18] p. 170–187 for a discussion of analytic structure in Gleason parts. This topic will be developed in §4.) Lemma 2 then gives conditions on A and W which imply that an analytic mapping $\Psi: R \to W$ is proper. The papers [11] and [13] discuss this topic and related problems.

Lemma 4. *Let A be a function algebra on X. Let $W \subset M_A$ satisfy $\bar{W} \setminus W \subset X$ and suppose $\Psi: R \to W$ is an analytic mapping of R onto W which identifies a*

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

finite number of points. Assume that $S=\{g\in A:|g|=1$ *on* $X\}$ *separates points on* X *and that there is* $F\in S$*, nonconstant on* W*, such that* $\{x\in T:\#\hat{F}^{-1}(x)<\infty\}$ *contains a set of positive Lebesgue measure. Then*

(i) Ψ *extends continuously to* $\bar{R}$ *and maps* Γ *into* X.

(ii) *If* $f\in A$*, then* $\tilde{f}=\hat{f}\circ\Psi\in A(R)$.

(iii) *If* $L=\{(a,b)\in\bar{R}\times\bar{R}:a\neq b,\ \Psi(a)=\Psi(b)\}$*, then* $(a,b)\in L$ *if and only if* $\tilde{f}(a)=\tilde{f}(b)$ *for all* $f\in A$.

Proof. By the preceding lemmas it follows that Ψ is proper. We now prove (i). Let $\lambda\in\Gamma$ and let $\{a_n\}\subset R$ converge to λ. We may assume that $\{a_n\}$ does not meet the set of points identified by Ψ. Since Ψ is proper and $\bar{W}\setminus W\subset X$, it follows that the accumulation points of $\{\Psi(a_n)\}$ are contained in X. Suppose there exist distinct accumulation points x and y. By assumption there is $g\in S$ so that $g(x)\neq g(y)$.

Let $\{\alpha_\beta\}$ be a subnet of $\{\Psi(a_n)\}$ which converges to x. By continuity $\hat{g}(\alpha_\beta)$ converges to $\hat{g}(x)$. Since $\hat{g}(\alpha_\beta)=\tilde{g}(\Psi^{-1}(\alpha_\beta))$ and $\tilde{g}\in A(R)$ by Lemma 3, we may conclude that $\hat{g}(x)=\tilde{g}(\lambda)$. Likewise, we may show that $\hat{g}(y)=\tilde{g}(\lambda)$. Hence, we must have $x=y$. We let $\Psi(\lambda)$ denote the unique accumulation point of $\{\Psi(a_n)\}$. If $\{b_n\}\subset R$ is another sequence which converges to λ, a similar argument will show that $\Psi(\lambda)$ is the only accumulation point of $\{\Psi(b_n)\}$.

We use $\Psi(\lambda)$ to extend Ψ to Γ. We have $\Psi(\Gamma)\subset X$ and a straightforward argument shows that Ψ is continuous on $\bar{R}$. The proofs of (ii) and (iii) are now immediate.

Proof of Theorem 1. Since $\bar{W}\setminus W=X$, we conclude that $\Psi(\Gamma)=X$. Let $\tilde{\Psi}:A\to A(R)$ be defined by $\tilde{\Psi}(f)=\hat{f}\circ\Psi$. Then Ψ is an isomorphism of A onto $\tilde{\Psi}(A)$. Set $A'=\tilde{\Psi}(A)$.

Let $g\in S$ be nonconstant. Then $\tilde{g}$ is unimodular of some order n. Since Ψ identifies only finitely many points of R, we can find some $w_0\in\Delta$ and $f\in A$ such that $\tilde{f}$ separates $\tilde{g}^{-1}(w_0)$. Let $\tilde{g}^{-1}(w)=\{z_1,\ldots,z_n\}$ (including repetitions) for $w\in\mathbb{C}$. Set $w\in\bar{\Delta}$ and let $z_1=z$. Then $\Theta(z)=\prod_{i=2}^n(\tilde{f}(z)-\tilde{f}(z_i))\in A(R)$ and $\Theta(z)\not\equiv 0$. In particular, there is some $t\in T$ such that $\tilde{f}$ separates the n distinct points of $\tilde{g}^{-1}(t)$. By (iii) of Lemma 4 we conclude that Ψ is 1–1 on $\tilde{g}^{-1}(t)$.

Since S separates points on X, there is a polynomial Q in elements of S such that Q separates the points of $\Psi(\tilde{g}^{-1}(t))$. Note that both $\tilde{g}$ and $\tilde{Q}$ extend meromorphically to $\hat{R}$. It follows that $\Omega(z)=\prod_{i=2}^n(\tilde{Q}(z)-\tilde{Q}(z_i))$ is analytic on a neighborhood of $\bar{R}$ and $\Omega(z)\not\equiv 0$. In this case the singular part of $\Omega(z)$ as an element of $A(R)$ in the sense of [16] is zero. This property of $\Omega(z)$ is evident if $\Omega(z)$ has no zeros on Γ and can be established by localization and the use of the inner-outer factorization for $A(\Delta)$ if $\Omega(z)$ has zeros on Γ.

If $J=\{f\in A(R):fA(R)\subset A'\}$, then J is a closed ideal of $A(R)$ with $J\subset A'$ and $\Omega(z)\in J$([12]). Since $\Omega(z)$ has zero singular part and only a finite number

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

of zeros in R, it follows that J, and hence A', has finite codimension in $A(R)$ ([16], Theorem I and [15], Lemma 2.5).

We recall that $M_{A(R)}=\bar{R}$ ([2]). The maximal ideal space of A' is obtained by identifying the points of $\bar{R}$ which are identified by Ψ ([7], Theorem 9.8). We see then that $\tilde{\Psi}^*: M_{A'}\to M_A$ defined by $\tilde{\Psi}^*(\sigma)(f)=\sigma(\tilde{\Psi}(f))$ for $\sigma\in M_{A'}$ and $f\in A$ is a homeomorphism and $M_A=\tilde{\Psi}^*(M_{A'})=\Psi(\bar{R})=\bar{W}$. Therefore, $\bar{W}=M_A$ as required. That completes the proof.

Remark. In the proof of Theorem 1 we constructed two functions $\tilde{g}$ and $\tilde{Q}$ that separate all but a finite subset of $\bar{R}$. Hence, according to Lemma 4 (iii), we may conclude that the set $L=\{(a, b)\in\bar{R}\times\bar{R}: a\neq b, \Psi(a)=\Psi(b)\}$ is finite. This remark will be used in §4.

§4 **Analytic maps and Gleason parts.** In this section we apply Gamelin's theorem on embedding analytic structure in certain subsets of the maximal ideal space of a function algebra to Theorem 1 to obtain entirely "internal" conditions on A which imply that A is isomorphic to a finite codimensional subalgebra of $A(R)$. We will use standard definitions and results in function algebras which are found in the references [8] or [18].

Theorem (Gamelin [8] p. 161 and [7]). *Let A be a function algebra on X. Suppose the set of representing measures supported on X for $\phi\in M_A$ is finite dimensional and that ϕ has a unique logmodular measure on X. Suppose the Gleason part P which contains ϕ contains more than one point. Then there is a $1-1$ analytic mapping Ψ of R onto P.*

The mapping Ψ need not be a homeomorphism nor even a proper map (see [5] p. 251 and [13] p. 1095). By combining Theorem 1 and Gamelin's theorem we obtain the following result:

Theorem 2. *Let A be a function algebra on X. Assume that*

(i) *There is $\phi\in M_A$ such that the set of representing measures on X for ϕ is finite dimensional and that ϕ has a unique logmodualr measure m on X. Furthermore, suppose that the Gleason part P which contains ϕ satisfies $\bar{P}\setminus P=X$.*

(ii) *The set $S=\{g\in A: |g|=1$ on $X\}$ separates points on X and there is $F\in S$, nonconstant on X, such that $E=\{x\in T: \#\hat{F}^{-1}(x)<\infty\}$ contains a set of positive Lebesgue measure in T.*

Then there is a finite set L contained in $\{(a, b)\in\Gamma\times\Gamma: a\neq b\}$ such that A is isomorphic to the algebra $A'=\{f\in A(R): f(a)=f(b)$ for $(a, b)\in L\}$.

Before beginning the proof of Theorem 2, we will briefly discuss the analytic mapping of Gamelin's theorem as well as the necessary background from function algebra theory. If $\phi\in M_A$ we will let $M_\phi(X)$ denote the set of representing measures supported on X for ϕ. If $M_\phi(X)$ is finite dimensional

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

and if ϕ has a unique logmodular measure on X, then all the representing measures for ϕ are mutually absolutely continuous ([7] p. 129–130). Now let ϕ have finite dimensional $M_\phi(X)$ with unique logmodular measure m on X. If $\Theta \in M_A$ belongs to the same Gleason part as ϕ, then $M_\Theta(X)$ is also finite dimensional and each $\mu \in M_\Theta(X)$ is mutually absolutely continuous with m ([8] p. 143).

We now fix $\phi \in M_A$ such that $M_\phi(X)$ is finite dimensional and contains a unique logmodular measure m and we suppose that the part P containing ϕ contains more than one point. We let $H^\infty(m)$ denote the weak-star closure of A in $L^\infty(m)$. Let $\iota : A \to H^\infty(m)$ be the inclusion map and let the continuous map $\iota^* : M_{H^\infty(m)} \to M_A$ be given by $\iota^*(\Theta)(f) = \Theta(\iota(f))$ and $f \in A$ and $\Theta \in M_{H^\infty(M)}$.

Each $\Theta \in P$ extends to a homomorphism $\Theta' \in M_{H^\infty(m)}$ by setting $\Theta'(f) = \int f\,d\mu$ for $f \in H^\infty(m)$ where $\mu \in M_\Theta(X)$. Let $\prod$ be the part of $M_{H^\infty(m)}$ which contains ϕ'. If $\Theta' \in \iota^{*-1}(P)$, then the representing measures for Θ' can be identified with $M_\Theta(X)$ ([8], p. 102). By a short argument it follows that ι^* is a 1–1 map of $\widehat{\prod}$ onto P. Furthermore, if $f_n \in A$ converges weak-star to $f \in H^\infty(m)$, then $\widehat{\iota(f_n)}$ converges pointwise to $\hat{f}$ on $\prod$. (cf [18] p. 183–184).

Gamelin's theorem produces a $1-1$ analytic map χ from R onto $\prod$ with respect to the algebra $H^\infty(m)$. Then $\Psi = \iota^* \circ \chi$ is a 1–1 analytic map from R onto P for the algebra A. Notice that $\widehat{\iota(f)} \circ \chi = \hat{f} \circ \Psi$ for $f \in A$. We remark that if a bounded sequence $\{f_n\}$ in A converges weak-star to $f \in H^\infty(m)$, then $\widehat{\iota(f_n)} \circ \chi$ converges uniformly on compact subsets of R to $\hat{f} \circ \chi$.

Proof of Theorem 2. Gamelin's theorem supplies a 1–1 analytic map Ψ of R onto P, and this map extends continuously to $\bar{R}$ to map Γ onto X according to Theorem 1. The map $\tilde{\Psi} : A \to A(R)$ defined by $\tilde{\Psi}(f) = \hat{f} \circ \Psi$ is an isomorphism of A onto $\tilde{\Psi}(A)$. Set $A' = \tilde{\Psi}(A)$. Let $L = \{(a, b) \in \bar{R} \times \bar{R} : a \neq b$ and $\Psi(a) = \Psi(b)\}$. By the remark following the proof of Theorem 1 we know that L is finite. Also, since Ψ restricted to R is 1–1, we conclude that $L \subset \Gamma \times \Gamma$.

Define $B' = \{f \in A(R) : f(a) = f(b)$ if $(a, b) \in L\}$. Then $A' \subset B'$. To complete the proof we must show that $A' = B'$. Assume instead that A' is properly contained in B'. By Theorem 1 we know that A' has finite codimension in $A(R)$, and hence, in B'. Consequently, there is a nonzero continuous point derivation D on B' such that $A' \subset \ker D$ ([7], Theorem 9.8).

Since $M_{A(R)} = \bar{R}$, the maximal ideal space for B' is the quotient space obtained by identifying the points of $\bar{R}$ which are identified by Ψ. Each level set of the restriction of Ψ to Γ is a peak set for B' and such sets are peak points when B' is regarded as a function algebra on $M_{B'}$. It follows that D must be a point derivation at some point $s \in R$ ([5], Corollary 1.6.7). Consequently, $A' \subset \ker D = M_s^2 + \mathbb{C}$ where $M_s = \{f \in B' : f(s) = 0\}$ and $M_s^2 = \{fg : f$ and $g \in M_s\}$. As a result we conclude that if $h \in A$, then $d\tilde{h}(s) = 0$ where $\tilde{h} = \hat{h} \circ \Psi$ and $d\tilde{h}(s)$ is the differential of $\tilde{h}$ at s.

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

According to hypothesis there is a unique logmodular measure m on X for $\phi \in P$. Let $\Psi^{-1}(\phi) = z_0$. There is a logmodular measure m_0 for z_0 supported on Γ for the algebra $A(R)$ ([8] p. 33 and 110). We may define a Borel measure m' on X by setting $m'(K) = m_0(\Psi^{-1}(K))$ for Borel sets $K \subset X$. Then it follows that m' is a logmodular measure for ϕ. Consequently, $m' = m$.

Let $B = \{f \in C(M_A) : f \circ \Psi \in B'\}$. Then m is multiplicative with respect to B. Let $H^\infty(B)$ be the weak-star closure of B in $L^\infty(m)$. Then $H^\infty(m) \subset H^\infty(B)$. According to a maximality theorem ([8] p. 115) we have $H^\infty(m) = H^\infty(B)$.

If $f \in H^\infty(m)$, then there is $\{f_n\} \subset A$ such that $\|f_n\| \leq \|f\|$ and f_n converges to f a.e. $[m]$ ([10], p. 137 and p. 149). In this case $\widehat{\iota(f_n)} \circ \chi$ converges uniformly on compact subsets of R to $\hat{f} \circ \chi$. As a result $d(\hat{f} \circ \chi)(s) = 0$ for all $f \in H^\infty(m)$. However, it is easy to construct a $G \in B$ such that for $g = G \circ \Psi \in B'$ we have $dg(s) \neq 0$. Furthermore, $\iota(G) \in H^\infty(m)$ and $\widehat{\iota(G)} \circ \chi(z) = g(z)$. Consequently, $d(\widehat{\iota(G)} \circ \chi)(s) \neq 0$ giving a contradiction. This completes the proof.

Let A be a function algebra on X. We say that A is a Dirichlet algebra on X if $\{\mathrm{Re}(f) : \in A\}$ is uniformly dense in the set of real-valued continuous functions on X. An algebra A is a Dirichlet algebra on X if and only if there are no nonzero real annihilating measures of A supported on X.

Let $A^\perp = \{\mu : \mu$ is a finite regular Borel measure on X such that $\int f d\mu = 0$ for all $f \in A\}$.

Corollary 1. *In addition to the assumptions of Theorem 2 suppose that $\phi \in M_A$ has a unique representing measure supported on X. Then A is a Dirichlet algebra on X.*

Proof. The Riemann surface R of Gamelin's theorem is now the open unit disk Δ. ([8], p. 158) and Γ becomes T. Let $\mathcal{S} = \{K \subset T : \Psi(K)$ is Borel$\}$. It is straightforward to show that $\mathcal{S}$ is a σ-algebra which contains the Borel sets. Suppose that $\mu \in A^\perp$ is a nonzero real measure on X. Let $E = \{t \in T :$ there is $t' \neq t$ with $\Psi(t') = \Psi(t)\}$. By Theorem 2 we know that E is a finite set. Now for $t \in E$ we let $E_t = \{t' \in E : \Psi(t') = \Psi(t)\}$. Using Theorem 2 we see that $\Psi(E_t)$ is a peak point for the function algebra A and therefore $\mu(\Psi(E_t)) = 0$. Furthermore, we see that $|\mu|(\Psi(E)) = 0$. It now follows that we may define a finite regular Borel measure $\tilde{\mu}$ on T by setting $\tilde{\mu}(K) = \mu(\Psi(K))$ for Borel sets $K \subset T$. Then $\tilde{\mu}$ is real and $\tilde{\mu}$ annihilates A'.

Let $f \in A'$ be an outer function which vanishes precisely on E. Then $fA(\Delta) \subset A'$ and so $f\tilde{\mu} \in A(\Delta)^\perp$. By the F. and M. Riesz theorem there is $h \in H_0^1(\Delta)$ such that $f\tilde{\mu} = h\lambda$ where λ is Lebesgue measure on T. As a result we have $\tilde{\mu} = (h/f)\lambda + \tilde{\mu}\mid_E$. Since $|\tilde{\mu}|(E) = 0$, we obtain $\tilde{\mu} = (h/f)\lambda$. Now h/f is real a.e. $[\lambda]$ and $h/f \in L^1(T)$. But then $h/f \in H^1(\Delta)$ ([6] p. 28). But a function in $H^1(\Delta)$ which is real a.e. must be constant. Since $0 = \int d\tilde{\mu} = \int (h/f)\, d\lambda$, we see that $h/f = 0$ a.e. $[\lambda]$. Therefore, $\tilde{\mu}$ is the zero measure and it follows that μ is the zero measure on X. This proves the Corollary.

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

Remark. One may check that the algebra $A=\{f\in A(\Delta):f(1)=f(-1)\}$ regarded as a function algebra on the set X obtained from T by identifying the points 1 and -1 satisfies the hypotheses of Theorem 2. (A point $z\in\Delta$ has a unique representing measure on X since A is Dirichlet on X as shown by the proof of Corollary 1.) As a result we see that the hypotheses of Theorem 2 do not allow us to capture the entire algebra $A(R)$ through the isomorphism $\tilde{\Psi}$. A condition which implies that $\tilde{\Psi}$ maps A onto $A(R)$ is given in the Corollary below.

Corollary 2. *In addition to the assumptions of Theorem* 2 *suppose there is a positive integer n such that $F:X\to T$ is precisely an n-to-one map. Then A is isomorphic to $A(R)$.*

Proof. We will prove that $\Psi:\bar{R}\to M_A$ is a 1–1 map which will establish the Corollary. The order of the unimodular function $\tilde{F}=\hat{F}\circ\Psi$ is at most n ([9] p. 62). If $t\in T$, then $\Psi(\tilde{F}^{-1}(t))\subset\hat{F}^{-1}(t)$. Since Ψ maps Γ onto X, we may conclude that $\tilde{F}$ has order n and that Ψ is 1–1 on $\tilde{F}^{-1}(t)$. By Lemma 4 (iii), we conclude that Ψ is 1–1 on Γ. Consequently, using Theorem 2 we see that Ψ is 1–1 on all of $\bar{R}$. This completes the proof.

References

1. L. Ahlfors and L. Sario, *Riemann surfaces*, Princeton University Press, Princeton, N.J., 1960.
2. R. Arens, *The closed maximal ideals of algebras of functions holomorphic on a Riemann surface, Rend. Cir. Mat. di Palermo* (**2**) **7** (1958), 245–260.
3. J.-E. Björk, Analytic structures, *Papers from the Summer Gathering on Functional Algebras at Aarhus*, July (1969), 19–28.
4. J.-E. Björk, *On analytic structure in the maximal ideal space of a function algebra, Papers from the Summer Gathering on Function Algebras at Aarhus*, July (1969), 29–35.
5. A. Browder, *Introduction to function algebras*, W. A. Benjamin, New York, 1969.
6. P. L. Duren, *Theory of H^P spaces*, Academic Press, New York, 1970.
7. T. W. Gamelin, *Embedding Riemann surfaces in maximal ideal spaces, J. Functional Analysis* **2** (1968), 123–146.
8. T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.
9. T. W. Gamelin, *Polynomial approximation on thin sets, Symposium on Several Complex Variables*, Park City, Utah (1970), Lecture notes in mathematics 184, Springer Verlag, Berlin, 1971.
10. T. W. Gamelin and G. Lumer, *Abstract Hardy spaces and universal Hardy class, Advances in Math.* **2** (1968), 118–174.
11. K. Kiski, *Homeomorphism between the open unit disk and a Gleason part, J. Math. Soc. Japan* **27** (1975), 467–473.
12. B. Lund, *Subalgebras of finite codimension in the algebra on analytic functions on a Riemann surface, Pacific J. Math.* **51** (1974), 495–497.
13. B. Lund, *Analytic embeddings in logmodular algebras, Indiana Univ. Math. J.* **24** (1975), 1093–1098.
14. J. R. Munkres, *Topology*, Prentice-Hall, Englewood Cliffs, N.J., 1975.
15. A. Read, *A converse of Cauchy's theorem and applications to extremal problems, Acta Math.* **100** (1958), 1–22.
16. C. M. Stanton, *The closed ideals of a function algebra, Trans, Amer. Math. Soc.* **154** (1971), 289–300.

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press

17. E. L. Stout, *On some algebras of analytic functions on finite open Riemann surfaces, Math. Z.* **92** (1966), 366–379.

18. E. L. Stout, *The theory of uniform algebras,* Bogden and Quigley, Tarrytown-on-Hudson, N.Y., 1971.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NEW BRUNSWICK
FREDERICTON, N. B., CANADA

https://doi.org/10.4153/CMB-1978-010-1 Published online by Cambridge University Press