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SINGULARITIES AND HIGHER TORSION IN SYMPLECTIC COBORDISM

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ABSTRACT. In this paper we construct higher two-torsion elements of all orders in the symplectic cobordism ring. We begin by constructing higher torsion elements in the symplectic cobordism ring with singularities using a geometric approach to the Adams-Novikov spectral sequence in terms of cobordism with singularities. Then we show how these elements determine particular elements of higher torsion in the symplectic cobordism ring.

1. **Introduction.** The symplectic cobordism ring MSp_* is the homotopy of the Thom spectrum MSp and classifies up to cobordism the ring of smooth manifolds with an Sp-structure on their stable normal bundles. Although MSp_* only has two-torsion, its ring structure is far more complicated than any of the other cobordism rings MG_* for the classical Lie groups $G = O$, SO, Spin, U, SU which have been completely computed. Over the past thirty years, these cobordism rings MG_* have had a major impact on differential topology and homotopy theory. On the other hand, if the complexity of the ring MSp_* were understood, then symplectic cobordism theory $\mathrm{MSp}^*(\cdot)$ would have the potential to become a powerful tool in algebraic topology.

The symplectic cobordism ring MSp_* is still far from being computed and understood despite much research on the subject over the past twenty years. It seems beyond present methods to completely compute MSp_* in the near future. Nevertheless, we can try to determine some general *structural* properties of this ring. The most striking example of such a result is the application of the Nilpotence Theorem [5] to MSp_* which says that all of its torsion elements are nilpotent. Another basic *structural* question is:

(1) *Do there exist elements of order 2^k in the ring MSp_* for all $k \geq 1$?*

Note that the corresponding structural property is well-known for all other classical cobordism rings as well as for framed cobordism, the stable homotopy groups of spheres. This paper gives an affirmative answer to (1).

We begin by describing the background of our research. In the torsion of MSp_* there are the fundamental Ray elements [13]: $\phi_0 = \eta \in \mathrm{MSp}_1$ which comes from framed cobordism, and $\phi_i \in \mathrm{MSp}_{8i-3}$ for $i \geq 1$. These are nonzero indecomposable elements of order two, and all torsion elements of MSp_* can be constructed from these Ray

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elements by using Toda brackets. These ϕ_i determine basic patterns in all approaches to understanding the structure of symplectic cobordism. In particular, projections of these elements to the Adams and Adams-Novikov spectral sequences for MSp have had a major impact on the description of their structure.

The approach based on the Adams spectral sequence (ASS) $\mathbb{E}_r^{*,*} \Longrightarrow \mathrm{MSp}_*$ was developed in North America. Computations through the 29 stem were made by D. Segal [14] in 1970. Subsequently the second author in [6], [7], [8] computed the $\mathbb{E}_2$ and $\mathbb{E}_3$-terms, showed that the spectral sequence does not collapse, computed the image of MSp_* in $\mathfrak{N}^*$, found elements of order four beginning in degree 111 and computed the first 100 stems.

The other approach based on the Adams-Novikov spectral sequence (ANSS) $E_r^{*,*} \Longrightarrow \mathrm{MSp}_*$ was developed in the former Soviet Union. In particular, V. Vershinin [16] computed the ANSS through the 52 stem and showed that the first element of order four in MSp_* occurs in degree 103 (unpublished).

It became apparent from both approaches that if there was torsion of order greater than four in MSp_* then it would occur in such a high degree that it would not be reasonable to try to discover it through stem by stem computations. In addition, there were no candidates for elements in E_2 of either the ASS or ANSS which might represent elements of higher torsion. (The only such family of candidates in the ASS was shown in [7] to be the image of higher differentials.) The determination of elements of higher torsion required new *geometric* ideas.

V. Vershinin's paper [17] provided new perspectives for viewing the symplectic cobordism ring. He constructed a sequence

$$\mathrm{MSp}_* \longrightarrow \mathrm{MSp}_*^{\Sigma_1} \longrightarrow \mathrm{MSp}_*^{\Sigma_2} \longrightarrow \cdots \longrightarrow \mathrm{MSp}_*^{\Sigma_n} \longrightarrow \cdots \longrightarrow \mathrm{MSp}_*^{\Sigma}$$

of cobordism rings $\mathrm{MSp}_*^{\Sigma_n}$ of symplectic manifolds with singularities which starts with the ring MSp_* and ends with MSp_*^{Σ}, a polynomial ring over the integers. In the two-local category, the spectrum MSp^{Σ} splits as a wedge of suspensions of the spectrum BP. Here $\Sigma = (P_1, \ldots, P_n, \ldots)$ and $\Sigma_n = (P_1, \ldots, P_n)$ are sequences of closed Sp-manifolds which represent the Ray elements $[P_1] = \eta$ and $[P_i] = \phi_{2^{i-2}}$ for $i \geq 2$. This led to the description of the Adams-Novikov spectral sequences for the spectra MSp^{Σ_n} in terms of cobordism with singularities [2], and, in particular, to a precise formula for the Adams-Novikov differential d_1 that reduces the computation of the E_2-term to elementary algebraic manipulations.

The opportunity that we have had to work together at York University has led to the understanding that the geometry of manifolds with singularities can be used to uncover the deep interaction between the Adams and Adams-Novikov spectral sequences for MSp thereby constructing torsion elements of all orders 2^k in MSp_*.

First we construct higher torsion elements in $\mathrm{MSp}_*^{\Sigma_n}$ for $n \geq 3$. The keys to this construction are that the cobordism ring $\mathrm{MSp}_*^{\Sigma_n}$ has new elements $w_1, \ldots, w_n$ that have the same degrees and behavior as the elements $v_1, \ldots, v_n \in \mathrm{BP}_*$ and that the Toda brackets $\langle \phi_i, w_k, \phi_j \rangle$ are defined for $k \leq n$. In Section 5, we prove the following theorem.

THEOREM A. *For each $n \geq 3$ and $n < i_1 < \cdots < i_s$, there exist indecomposable elements $\tau_n(i_1, \ldots, i_s) \in \mathrm{MSp}^{\Sigma_n}_{4*+1}$ with the following properties:*

(i) $\tau_n(i_1) = \phi_{2^{i_1}-2}$;

(ii) $\tau_n(i_1, \ldots, i_s) \in \langle \phi_{2^{i_s}-2}, w_{n-1}, \tau_n(i_1, \ldots, i_{s-1}) \rangle$;

(iii) $\tau_n(i_1, \ldots, i_s)$ *has order at least* $2^{[(s+1)/2]}$ *for* $s \geq 1$.

Using Bockstein long exact sequences we deduce Theorem B which gives a positive answer to question (1).

THEOREM B. *For each $k \geq 1$ there exist elements of order 2^k in the symplectic cobordism ring* MSp_*.

Theorem B does not give a particular way to construct higher torsion elements in MSp_*. The remainder of this paper is devoted to the construction of elements $\alpha(i_1, \ldots, i_s) \in \mathrm{MSp}_*$ from the elements $\tau_3(i_1, \ldots, i_s) \in \mathrm{MSp}^{\Sigma_3}_*$. Consider the diagram below

$$\begin{array}{ccccccc} \mathrm{MSp}^{\Sigma_3}_* & \xrightarrow{\beta_3} & \mathrm{MSp}^{\Sigma_2}_* & \xrightarrow{\beta_2} & \mathrm{MSp}^{\Sigma_1}_* & \xrightarrow{\beta_1} & \mathrm{MSp}_* \\ & & & & & \nwarrow \pi & \\ & & & & & & \mathrm{MSp}_* \end{array}$$

We define the elements $\alpha'(i_1, \ldots, i_s) = \beta_2\Big(\beta_3\big(\tau_3(i_1, \ldots, i_s)\big)\Big)$ in the ring $\mathrm{MSp}^{\Sigma_1}_*$. Then we construct elements $\alpha(i_1, \ldots, i_s) \in \mathrm{MSp}_{4*+1}$ such that $\pi\big(\alpha(i_1, \ldots, i_s)\big)$ and $2\alpha'(i_1, \ldots, i_s)$ in $\mathrm{MSp}^{\Sigma_1}_*$ project to the same element of $E_2^{3,4*+1}(\mathrm{MSp}^{\Sigma_1})$ in the Adams-Novikov spectral sequence. Finally we prove that the $\alpha(i_1, \ldots, i_s) \in \mathrm{MSp}_{4*+1}$ are elements of higher order in MSp_*.

THEOREM C. *The element $\alpha(i_1, \ldots, i_s) \in \mathrm{MSp}_{4*+1}$ has order at least $2^{[(s+1)/2]-3}$ for $s \geq 7$ and $3 \leq i_1 < \cdots < i_s$.*

We describe the contents of this paper in more detail. In Section 2, we summarize the basic facts about the spectra MSp^{Σ_n} which we will be using. In Section 3, we give the definition and basic properties of three-fold Toda brackets of manifolds with singularities. These Toda brackets will be used to inductively define the elements we construct. In Section 4, we study the Adams-Novikov spectral sequence for the MSp^{Σ_n}. The key technical and conceptual fact we use is that the Adams-Novikov spectral sequence for the spectrum MSp_* coincides with the Σ-singularities spectral sequence which is defined in terms of cobordism with singularities [2]. The Σ-singularities spectral sequence gives us a specific resolution $\mathcal{M}\langle n\rangle$ for computing E_2 of the ANSS for MSp^{Σ_n}. In particular, we identify torsion elements $\mathfrak{t}_n(\mathfrak{i})$ of all orders 2^k in the first line of the ANSS

$$E_2^{1,*} = \mathrm{Ext}^{1,*}_{A^{\mathrm{BP}}}\big(\mathrm{BP}^*(\mathrm{MSp}^{\Sigma_n}), \mathrm{BP}^*\big).$$

Let $\mathfrak{i} = (i_1, \ldots, i_s)$. In Section 5, we use Toda brackets to construct elements $\tau_n(\mathfrak{i})$ in $\mathrm{MSp}^{\Sigma_n}_*$ and prove Theorems A and B. In particular, the element $\tau_n(\mathfrak{i})$ has order at least

$2^{[(s+1)/2]}$ because it projects in the Adams-Novikov spectral sequence to the infinite cycle $t_n(\mathrm{i})$ that has order $2^{[(s+1)/2]}$ in $E_2^{1,*}(\mathrm{MSp}^{\Sigma_n})$. In Section 6, we study the elements

$$\gamma(\mathrm{i}) = \beta_3\big(\tau_3(\mathrm{i})\big) \in \mathrm{MSp}^{\Sigma_2}_{4*+3} \quad \text{and} \quad \alpha'(\mathrm{i}) = \beta_2\big(\gamma(\mathrm{i})\big) \in \mathrm{MSp}^{\Sigma_1}_{4*+1}$$

and identify their projections to the third line of the ANSS. In Section 7, we prove Theorem C by projecting the elements $\alpha'(i_1, \ldots, i_s)$ to the third line $E_2^{3,*}(\mathrm{MSp}^{\Sigma_1})$ of the ANSS. We use chromatic arguments to compute the order of these projections in $E_2^{3,*}$, and we show that they can not be killed by d_3-differentials.

In the paper [3], we show that the elements $\tau_3(\mathrm{i}) \in \mathrm{MSp}^{\Sigma_3}_*$ constructed here lift to elements $t[\mathrm{i}] \in \mathrm{MSp}^{\hat{\Sigma}_3}_*$ where $\mathrm{MSp}^{\hat{\Sigma}_n}_*$ is the cobordism ring with singularities $\hat{\Sigma}_n = (P_2, \ldots, P_n)$. We prove that the resulting elements $a[\mathrm{i}] = \hat{\beta}_3\big(\hat{\beta}_2(t[\mathrm{i}])\big) \in \mathrm{MSp}_*$ have order at least $2^{[(s+1)/2]-2}$, an improvement upon Theorem C. The paper [3] concludes with four conjectures on likely generalizations of our results.

All groups, rings and spectra are two-local throughout this paper.

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2. **Symplectic cobordism with singularities.** In this section we collect basic constructions and theorems concerning the spectra MSp^{Σ}_*, $\mathrm{MSp}^{\Sigma_n}_*$ and $\mathrm{MSp}^{\hat{\Sigma}_n}_*$ of symplectic cobordism with singularities. In particular, we determine formulas for computing the Bockstein operators which will be used in Sections 4 and 7 to make computations in the ANSS for MSp^{Σ_n}.

Let $\Sigma = (P_1, \ldots, P_n, \ldots)$ be a sequence of closed Sp-manifolds representing the Ray elements such that $[P_1] = \eta$ and $[P_i] = \phi_{2^{i-2}}$ for $i \geq 2$. Let Σ_n denote the sequence $(P_1, \ldots, P_n)$ for $n \geq 1$, and let $\hat{\Sigma}_n$ denote the sequence $(P_2, \ldots, P_n)$ for $n \geq 2$. The bordism theory of Sp-manifolds with Σ-, Σ_n-, $\hat{\Sigma}_n$-singularities is denoted by $\mathrm{MSp}^{\Sigma}_*(\cdot)$, $\mathrm{MSp}^{\Sigma_n}_*(\cdot)$, $\mathrm{MSp}^{\hat{\Sigma}_n}_*(\cdot)$, respectively. By [2], [17], [18] the theories $\mathrm{MSp}^{\Sigma_n}_*(\cdot)$, $\mathrm{MSp}^{\hat{\Sigma}_n}_*(\cdot)$ have admissible product structures, and the coefficient ring $\mathrm{MSp}^{\Sigma_n}_*$ is polynomial up to dimension $2^{n+2} - 3$. (See [17], [2, Theorem 3.3.3].) The following theorem describes the structure of the ring MSp^{Σ}_*.

THEOREM 2.1 (V.VERSHININ [17]). *There exists an admissible product structure in the theory* $\mathrm{MSp}^{\Sigma}_*(\cdot)$ *such that its coefficient ring* MSp^{Σ}_* *is isomorphic to the polynomial ring*

$$\mathrm{MSp}^{\Sigma}_* \cong \mathbb{Z}_{(2)}[w_1, \ldots, w_j, \ldots, x_2, x_4, x_5, \ldots, x_m, \ldots]$$

where $\deg w_j = 2(2^j - 1)$ *for* $j = 1, 2, \ldots$ *and* $\deg x_m = 4m$ *for* $m = 2, 3, 5, \ldots, m \neq 2^s - 1$. *The generators* w_j *are represented by* Sp-*manifolds* W_j *such that* $\partial W_j = 2P_j$.

In fact, the cobordism theory $\mathrm{MSp}^*_{\Sigma}(\cdot)$ splits as a sum of the theories $\mathrm{BP}^*(\cdot)$.

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THEOREM 2.2 ([2, COROLLARY 3.5.3]). *The ring spectrum* MSp^{Σ} *splits as*

$$\mathrm{MSp}^{\Sigma} = \mathrm{BP} \wedge M(G)$$

where $G = \mathbb{Z}_{(2)}[x_2, \ldots, x_m, \ldots]$, $m = 2, 4, 5, \ldots,\ m \neq 2^l - 1$, $\deg x_m = 4m$ *and* $M(G)$ *is a graded Moore spectrum.*

NOTE 2.1. Theorem 2.2 implies that the ANSS based on the cohomology theories $\mathrm{BP}^*(\cdot)$ and $\mathrm{MSp}^*_{\Sigma}(\cdot)$ are isomorphic.

There are Bockstein operators in the theory $\mathrm{MSp}^{\Sigma}_*(\cdot)$ for $i \geq 1$:

$$\beta_i\colon \mathrm{MSp}^{\Sigma}_*(\cdot) \longrightarrow \mathrm{MSp}^{\Sigma}_*(\cdot).$$

They have the following properties:

$$\beta_i \circ \beta_i = 0, \quad \text{and} \quad \beta_i \circ \beta_j = \beta_j \circ \beta_i.$$

In general a product formula for Bockstein operators acting on a bordism theory with singularities is too complicated to write down. However in our case this formula has the following simple form.

THEOREM 2.3 ([2, THEOREM 4.2.4]). *The product structure and the elements* w_i *in Theorem 2.1 may be chosen in such a way that the Bockstein operators* β_i, $i \geq 1$, *satisfy the product formula:*

$$\beta_i(x \cdot y) = (\beta_i x) \cdot y + x \cdot (\beta_i y) - w_i \cdot (\beta_i x) \cdot (\beta_i y) \tag{2}$$

where $x, y \in \mathrm{MSp}^{\Sigma}_*$.

To describe the action of the Bockstein operators on the polynomial generators of MSp^{Σ}_* we introduce the following notation. Let $m + 1 = 2^{i_1 - 1} + \cdots + 2^{i_s - 1}$ be a binary decomposition of the integer $m + 1$ where $1 \leq i_1 < i_2 < \cdots < i_s$. If m is odd, the generator x_m is denoted by $x_{i_1,\ldots,i_s}$. If $m = 2^{i-1}$ with $1 \leq i$ then the generator x_m is denoted by $x_{1,i}$.

THEOREM 2.4 ([2, THEOREM 4.5.1]). *There are generators* x_m *of the ring* MSp^{Σ}_* *such that the formulas below describe the action of Bockstein operators* β_k *on* x_m *for* $k \geq 2$.

1. If $m = 2^{i-1} + 2^{j-1} - 1$, $1 \leq i < j$ *then*

$$\beta_i x_{i,j} = w_j,\ \beta_j x_{i,j} = w_i,\ \text{and}\ \beta_k x_{i,j} = 0 \quad \text{if}\ k \neq i, j. \tag{3}$$

2. If $m = 2^{i_1 - 1} + \cdots + 2^{i_s - 1} - 1$, $2 \leq i_1 < i_2 < \cdots < i_s$ *and* $s \geq 3$, *then*

$$\beta_k x_{i_1,\ldots,i_s} = \begin{cases} w_1 \cdot x_{i_1,\ldots,\hat{\imath}_t,\ldots,i_s}, & \text{if}\ k = i_t \\ 0 & \text{if}\ k \neq i_1, \ldots, i_s \end{cases} \tag{4}$$

3. If m *is even and not a power of two then*

$$\beta_k x_m = 0. \tag{5}$$

■

Formulas (2)–(5) are the ones we will use in Sections 4, 5 and 6 to make computations in the ANSS for the spectra MSp^{Σ_n}. Note that (2) and (3) are invariant under permutations π of the subscripts where π is a permutation of the set of integers greater than one.

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3. **Toda brackets.** In this section we extend the construction of Alexander [1] to define triple Toda brackets in the ring $MSp_*^{\Sigma_n}$. Note that we do not claim that all of the usual properties of Massey products [10] and Toda brackets [15] generalize to cobordism with singularities. These Toda brackets will be used in the constructions of Sections 5 and 6.

Let P_n be closed Sp-manifolds, $[P_1]_{Sp} = \eta$ and $[P_n]_{Sp} = \phi_{2^{n-2}}$ for $n \geq 2$. Consider the manifold $P'_n = P_n^{(1)} \times P_n^{(2)} \times I$. Here $P_n^{(1)}, P_n^{(2)}$ are two copies of the Σ_n-manifold P_n such that: $\partial P'_n = \beta_n P'_n \times P_n$ and $\beta_n P'_n = P_n^{(1)} \times \{0\} \cup P_n^{(2)} \times \{1\}$. The cobordism class $[P'_n]_{\Sigma_n}$ is the obstruction to the existence of a product structure on MSp^{Σ_n}. In our case, $[P'_n]_{\Sigma_n} = 0$, and we let Q_n denote a Σ_n-manifold such that $\delta Q_n = P'_n$ as in [2, Theorem 4.2.4]. Thus, we have the following product construction of [2, Theorem 2.2.2].

A product $\mathfrak{m}_n(A^a, B^b)$ of two Σ_n-manifolds is defined by induction on n as follows:

$$\mathfrak{m}_1(A^a, B^b) = A^a \times B^b \cup (-1)^b \beta_1 A^a \times \beta_1 B^b \times Q_1$$

and for $n \geq 2$

$$\mathfrak{m}_n(A^a, B^b) = \mathfrak{m}_{n-1}(A^a, B^b) \cup (-1)^b \mathfrak{m}_{n-1}\Big(\mathfrak{m}_{n-1}(\beta_n A^a, \beta_n B^b), Q_n\Big).$$

In particular, if C is an Sp-manifold without singularities then $\mathfrak{m}_n(X, C) = X \times C$ and $\mathfrak{m}_n(C, X) = C \times X$ and we have the following diffeomorphism of Σ_n-manifolds:

$$\delta \mathfrak{m}_n(A^a, B^b) = \mathfrak{m}_n(\delta A, B) \cup (-1)^a \mathfrak{m}_n(A, \delta B). \tag{6}$$

By [2, Theorem 3.3.3], this *product of* Σ_n*-manifolds* $\mathfrak{m}_n$ determines an admissible product structure μ_n in the theory $MSp_*^{\Sigma_n}(\cdot)$ which is commutative and associative. At the level of Σ_n-manifolds *commutativity of* μ_n means (see [2, Definition 2.1.3]) that for all Σ_n-manifolds A^a, B^b there exists a Σ_n-manifold $\mathfrak{K}_n(A^a, B^b)$ called the *canonical commutativity construction* which is functorial in the category of Σ_n-manifolds and satisfies the formula

$$\delta \mathfrak{K}_n(A, B) = \mathfrak{m}_n(A, B) \cup (-1)^{ab} \mathfrak{m}_n(B, A) \cup -\mathfrak{K}_n(\delta A, B) \cup (-1)^{a+1} \mathfrak{K}_n(A, \delta B). \tag{7}$$

Associativity of μ_n at the level of Σ_n-manifolds means (see [2, Definition 2.1.3]) that for all Σ_n-manifolds A^a, B^b, C^c there exists a Σ_n-manifold $\mathfrak{A}_n(A^a, B^b, C^c)$ called the *canonical associativity construction* which is functorial in the category of Σ_n-manifolds and satisfies the formula

$$\begin{aligned} \delta \mathfrak{A}_n(A^a, B^b, C^c) = {} & \mathfrak{m}_n\big(A, \mathfrak{m}_n(B, C)\big) \cup -\mathfrak{m}_n\big(\mathfrak{m}_n(A, B), C\big) \cup \\ & - \mathfrak{A}_n(\delta A, B, C) \cup (-1)^{a+1} \mathfrak{A}_n(A, \delta B, C) \cup (-1)^{a+b+1} \mathfrak{A}_n(A, B, \delta C). \end{aligned} \tag{8}$$

Let A^a and B^b be Sp-manifolds without singularities. Then $\mathfrak{K}_n(A, B)$ can be taken to be the cylinder $\mathfrak{K}_n(A, B) = I \times A \times B$ with an Sp-structure such that there is a diffeomorphism preserving Sp-structures:

$$\partial (I \times A \times B) = A \times B \cup -(-1)^{ab} B \times A \cup -I \times \partial A \times B \cup (-1)^{a+1} I \times A \times \partial B.$$

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In this case $\mathfrak{K}_n(A, B)$ has been described [7, Section 10] in the special category of manifolds as a "cup-one product of manifolds". It projects in $\mathbb{E}_1$ of the ASS to an algebraic cup-one product. Moreover, the Sp-structure on $\mathfrak{K}_n(A, B)$ can be chosen so that $\mathfrak{K}_n$ satisfies the Hirsch formula [9]. Using the definition of $\mathfrak{K}_n(A, B)$, this property generalizes to the two cases of Σ_n-manifolds given in the following lemma which suffice for the constructions of this paper. Statement (b) is nontrivial; it is essential that the Σ_n-manifolds A^a and C^c have only one common singularity, *i.e.* there is only one $i \leq n$, such that both $\beta_i A \neq \emptyset$ and $\beta_i C \neq \emptyset$.

LEMMA 3.1. *(a) (Hirsch Formula) If C^c is an Σ_n-manifold and one of the Σ_n-manifolds A^a, B^b is an* Sp-*manifold without singularities then we have a diffeomorphism of Σ_n-manifolds preserving* Sp-*structures:*

$$\mathfrak{K}_n(A \times B, C) = (-1)^a \mathfrak{m}_n\big(A, \mathfrak{K}_n(B, C)\big) \cup (-1)^{bc} \mathfrak{m}_n\big(\mathfrak{K}_n(A, C), B\big). \tag{9}$$

(b) (Generalized Hirsch Formula) If A^a and C^c are closed Σ_n-manifolds that have only one nonempty common singularity and c is even then there is a Σ_n-cobordism between the manifolds

$$\mathfrak{K}_n\big(\mathfrak{m}_n(C, A), C\big) \cup \mathfrak{A}_n(C, A, C) \quad \text{and} \quad \mathfrak{m}_n\big(C, \mathfrak{K}_n(A, C)\big) \cup \mathfrak{m}_n\big(\mathfrak{K}_n(C, C), A\big). \tag{10}$$

COMMENTS ON THE PROOF. Part (a) is proved by comparison of the constructions on the left and right sides of equation (9). In part (b), the obstruction to the associativity of the product structure μ_n has order three in the group $\mathrm{MSp}_*^{\Sigma_n}$; see [2, Lemma 2.4.2]. Since the group $\mathrm{MSp}_*^{\Sigma_n}$ does not have any odd torsion, the associativity construction $\mathfrak{A}_n$ may be taken to be a cylinder. This gives a way to construct a cobordism between the Σ_n-manifolds in (10). The construction of this cobordism is straightforward when the manifolds A, C have only one common singularity. ∎

Now we are ready to define the Toda bracket $\langle a, b, c\rangle$, where $a, b, c \in \mathrm{MSp}_*^{\Sigma_n}$. Since the product of Σ_n-manifolds is not associative, we need to use an associativity construction (8) to glue together the two usual pieces which define such a bracket in an associative context. We use the standard sign conventions of [10].

DEFINITION 3.2. *Let a, b, $c \in \mathrm{MSp}_*^{\Sigma_n}$ such that $ab = 0$ and $bc = 0$. Let A, B, C be a Σ_n-manifold which represents a, b, c, respectively. Let X, Y be Σ_n-manifolds such that $\delta X = \mathfrak{m}_n(A, B)$ and $\delta Y = \mathfrak{m}_n(B, C)$. Then*

$$\delta \mathfrak{m}_n(X, C) = \mathfrak{m}_n\big(\mathfrak{m}_n(A, B), C\big), \quad \text{and} \quad \delta \mathfrak{m}_n(A, Y) = (-1)^{\deg A} \mathfrak{m}_n\big(A, \mathfrak{m}_n(B, C)\big).$$

Let the Toda bracket $\langle a, b, c\rangle$ be the set of all cobordism classes of Σ_n-manifolds

$$Z = (-1)^{1+\deg B} \mathfrak{m}_n(X, C) \cup (-1)^{1+\deg B} \mathfrak{A}_n(A, B, C) \cup (-1)^{\deg A + \deg B} \mathfrak{m}_n(A, Y).$$

NOTE 3.1. Let the Toda bracket $\langle a, b, c\rangle$ be defined where the element a is represented by a closed Sp-manifold A. In this case the Σ_n-manifold $\mathfrak{A}_n(A, B, C)$ is just the cylinder

$$\mathfrak{A}_n(A, B, C) = I \times A \times \mathfrak{m}_n(B, C);$$

see [2, Theorem 2.5.1]. Thus, the following Σ_n-manifold Z represents an element of $\langle a, b, c\rangle$:

$$Z = (-1)^{1+\deg B}\mathfrak{m}_n(X, C) \cup (-1)^{1+\deg B} I \times A \times \mathfrak{m}_n(B, C) \cup (-1)^{\deg A+\deg B}\mathfrak{m}_n(A, Y).$$

By properties of $\mathfrak{m}_n, \mathfrak{A}_n$ (see [2, Section 2.2]), the Σ_n-manifold Z depends only on the cobordism classes a, b, c and on the choice of the Σ_n-manifolds X and Y with $\delta X = \mathfrak{m}_n(A, B)$ and $\delta Y = \mathfrak{m}_n(B, C)$. Therefore we have the usual indeterminacy: *in* $\langle a, b, c\rangle = \{ay + xc \mid x, y \in \mathrm{MSp}_*^{\Sigma_n}\}$.

We define a generalized *quadratic construction* which we use in the next lemma to identify Toda brackets of the form $\langle a, b, a\rangle$. Suppose M is a Σ_n-manifold of dimension $2k$. Define a closed Σ_n-manifold $\Delta(M)$ as follows:

$$\Delta(M) = \mathfrak{m}_n(M^{(1)}, M^{(2)}) \times I \cup -\mathfrak{K}_n(M^{(2)}, M^{(1)}) \tag{11}$$

where we identify the following manifolds:

$$\mathfrak{m}_n(M^{(1)}, M^{(2)}) \times \{0\} \supset \mathfrak{m}_n(M^{(1)}, M^{(2)}) \subset -\delta\mathfrak{K}_n(M^{(2)}, M^{(1)}),$$
$$\mathfrak{m}_n(M^{(1)}, M^{(2)}) \times \{1\} \supset -\mathfrak{m}_n(M^{(2)}, M^{(1)}) \subset -\delta\mathfrak{K}_n(M^{(2)}, M^{(1)}).$$

Note that in the case where M is a manifold without singularities, the manifold $\Delta(M)$ is just the quadratic construction.

LEMMA 3.3. *Let the Toda bracket* $\langle a, b, a\rangle$ *be defined in* $\mathrm{MSp}_*^{\Sigma_n}$, *where* $a = [M]$ *and* $b = [R]$ *for* M *a* Σ_n*-manifold of dimension* $2k$ *and* R *a* Sp*-manifold. Then*

$$\langle a, b, a\rangle = \{(-1)^{1+\deg b} b[\Delta(M)] + ax \mid x \in \mathrm{MSp}_*^{\Sigma_n}\}. \tag{12}$$

PROOF. Throughout this proof we ignore trivial associativity constructions in which one of the three entries has empty singularities. Let Y be a Σ_n-manifold such that $\delta Y = R \times M$. Then

$$\delta\Big(Y \cup (R \times M \times I) \cup -\mathfrak{K}_n(R, M)\Big) = M \times R,$$

and the Σ_n-manifold

$$C = m_n(Y, M) \cup m_n(R \times M, M) \times I \cup -\mathfrak{m}_n\Big(\mathfrak{K}_n(R, M), M\Big) \cup -\mathfrak{m}_n(M, Y)$$

is a representative of the Toda bracket $(-1)^{1+\deg b}\langle M, R, M\rangle$. Glue the Σ_n-manifold $\mathfrak{K}_n(Y, M)$ to the cylinder $C \times I$ by identifying the following manifolds:

$$C \times \{1\} \supset m_n(Y, M) \times \{1\} = m_n(Y, M) \subset \delta\mathfrak{K}_n(Y, M),$$
$$C \times \{1\} \supset -m_n(M, Y) \times \{1\} = -m_n(M, Y) \subset \delta\mathfrak{K}_n(Y, M),$$
$$C \times \{1\} \supset m_n\Big(\mathfrak{K}(R, M), M\Big) \times \{1\} = m_n\Big(\mathfrak{K}(R, M), M\Big) \subset \delta\mathfrak{K}_n(Y, M).$$

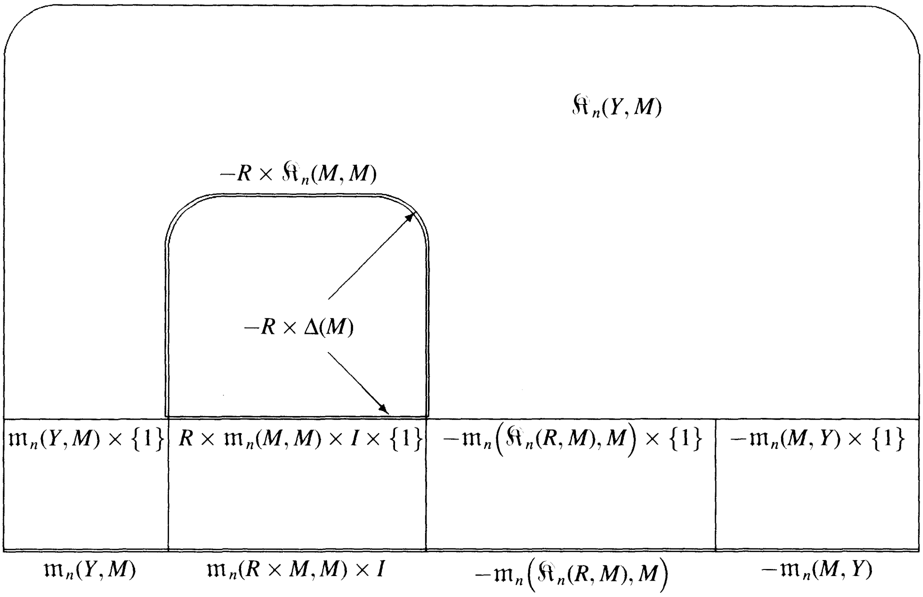

FIGURE 1: The Σ_n-manifold Z.

The boundary of the resulting Σ_n-manifold Z is given by

$$\delta Z = -R \times \Delta(M) \cup C \times \{0\}$$

since, using the Hirsch formula,

$$\delta \mathfrak{K}_n(Y, M) \cap \delta Z = -\mathfrak{K}_n(\delta Y, M) \cap \delta Z = -\mathfrak{K}_n(R \times M, M) \cap \delta Z = -R \times \mathfrak{K}_n(M, M).$$

See Figure 1. Thus, $(-1)^{1+\deg b}\langle a, b, a\rangle$ contains $[\Delta(M)]b$ and $in\langle a, b, a\rangle = a\,\mathrm{MSp}_*^{\Sigma_n}$. Therefore, (12) holds. ∎

The next property is well known for manifolds without singularities. See [1, Definition 2.1(5)].

LEMMA 3.4. *Let $\langle a, b, c\rangle$ be a Toda bracket which is defined in the ring* $\mathrm{MSp}_*^{\Sigma_n}$. *Assume that $a = [A]$ is represented by a closed* Sp*-manifold, and $b = [B]$, $c = [C]$ are represented by Σ_n-manifolds. Then the following inclusion holds in the ring* $\mathrm{MSp}_*^{\Sigma_n}$*:*

$$b\langle a, b, c\rangle \subset (-1)^{\deg a}\langle b, a, b\rangle c.$$

PROOF. Let X, Y be Σ_n-manifolds such that $\delta X = A \times B$ and $\delta Y = \mathfrak{m}_n(B, C)$. Then the following Σ_n-manifold Z represents an element of $(-1)^{1+\deg b}\langle a, b, c\rangle$:

$$Z = \mathfrak{m}_n(X, C) \cup -I \times A \times \mathfrak{m}_n(B, C) \cup (-1)^{1+\deg a} A \times Y.$$

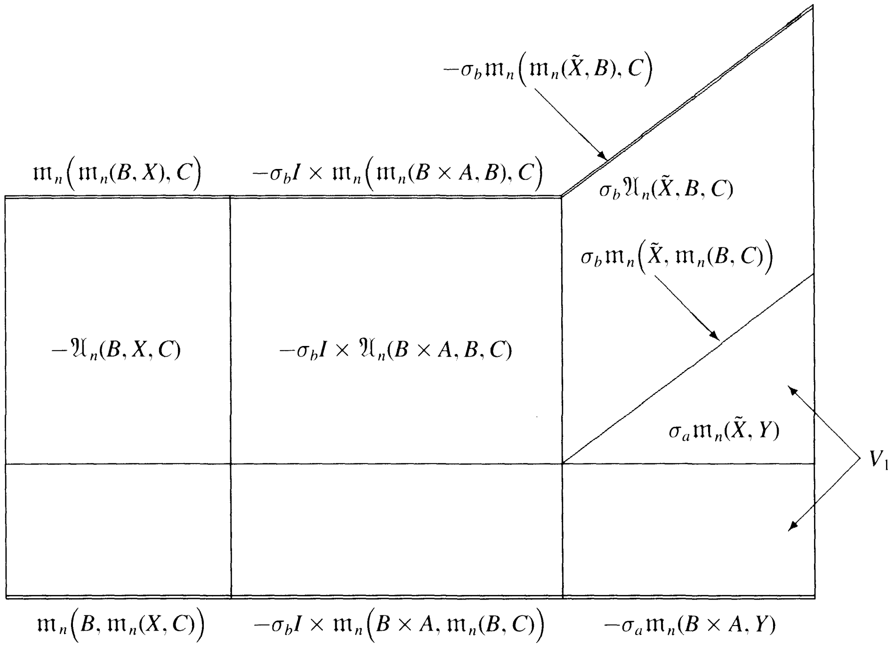

FIGURE 2: Σ_n-manifold V_2, where $\sigma_a = (-1)^{\deg a}$, $\sigma_b = (-1)^{\deg b}$.

Thus, the Σ_n-manifold $\mathfrak{m}_n(B, Z)$ represents an element of $(-1)^{1+\deg b} b\langle a, b, c\rangle$, and using (6):

$$\begin{aligned}
\mathfrak{m}_n(B, Z) &= \mathfrak{m}_n\big(B, \mathfrak{m}_n(X, C)\big) \cup \\
&\qquad - \mathfrak{m}_n\big(B, I \times A \times \mathfrak{m}_n(B, C)\big) \cup (-1)^{1+\deg a} \mathfrak{m}_n(B, A \times Y) \\
&= \mathfrak{m}_n\big(B, \mathfrak{m}_n(X, C)\big) \cup (-1)^{1+\deg b} I \times \mathfrak{m}_n\big(B \times A, \mathfrak{m}_n(B, C)\big) \\
&\qquad \cup (-1)^{1+\deg a} \mathfrak{m}_n(B \times A, Y).
\end{aligned}$$

Let $\tilde{X}$ be a Σ_n-manifold such that $\delta \tilde{X} = B \times A$. Glue $(-1)^{\deg a} \mathfrak{m}_n(\tilde{X}, Y)$ to the cylinder $\mathfrak{m}_n(B, Z) \times I$ along their common boundary:

$$\delta \mathfrak{m}_n(B, Z) \times \{1\} \supset \mathfrak{m}_n(B \times A, Y) \subset \delta \mathfrak{m}_n(\tilde{X}, Y).$$

Denote the resulting Σ_n-manifold as V_1; see Figure 2. Now consider the Σ_n-manifolds

$$\mathfrak{A}_n(B, X, C), \quad I \times \mathfrak{A}_n(B \times A, B, C), \quad \mathfrak{A}_n(\tilde{X}, B, C)$$

with boundaries:

$$\delta \mathfrak{A}_n(B, X, C) = \mathfrak{m}_n\big(B, \mathfrak{m}_n(X, C)\big) \cup -\mathfrak{m}_n\big(\mathfrak{m}_n(B, X), C\big) \cup (-1)^{\deg b + 1} \mathfrak{A}_n(B, \delta X, C)$$

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$$= \mathfrak{m}_n\big(B, \mathfrak{m}_n(X, C)\big) \cup -\mathfrak{m}_n\big(\mathfrak{m}_n(B, X), C\big) \cup (-1)^{\deg b+1}\mathfrak{A}_n(B, A \times B, C)$$
$$= \mathfrak{m}_n\big(B, \mathfrak{m}_n(X, C)\big) \cup -\mathfrak{m}_n\big(\mathfrak{m}_n(B, X), C\big) \cup (-1)^{\deg b+1}\mathfrak{A}_n(B \times A, B, C);$$

$$\delta\big(I \times \mathfrak{A}_n(B \times A, B, C)\big)$$
$$= I \times \big\{-\mathfrak{m}_n\big(B \times A, \mathfrak{m}_n(B, C)\big) \cup \mathfrak{m}_n\big(\mathfrak{m}_n(B \times A, B), C\big)\big\} \cup \partial I \times \mathfrak{A}_n(B \times A, B, C);$$

$$\delta\mathfrak{A}_n(\tilde{X}, B, C) = \mathfrak{m}_n\big(\tilde{X}, \mathfrak{m}_n(B, C)\big) \cup -\mathfrak{m}_n\big(\mathfrak{m}_n(\tilde{X}, B), C\big) \cup -\mathfrak{A}_n(\delta\tilde{X}, B, C)$$
$$= \mathfrak{m}_n\big(\tilde{X}, \mathfrak{m}_n(B, C)\big) \cup -\mathfrak{m}_n\big(\mathfrak{m}_n(\tilde{X}, B), C\big) \cup -\mathfrak{A}_n(B \times A, B, C).$$

Now we glue together the Σ_n-manifolds

$$V_1, \quad -\mathfrak{A}_n(B, X, C), \quad (-1)^{1+\deg b} I \times \mathfrak{A}_n(B \times A, B, C), \quad (-1)^{\deg b}\mathfrak{A}_n(\tilde{X}, B, C).$$

The resulting Σ_n-manifold V_2 gives a Σ_n-bordism between $\mathfrak{m}_n(B, Z) \times \{0\}$ and

$$\mathfrak{m}_n\big(-\mathfrak{m}_n(B, X) \cup (-1)^{\deg b} I \times \mathfrak{m}_n(B \times A, B) \cup (-1)^{\deg b}\mathfrak{m}_n(\tilde{X}, B), C\big).$$

The latter Σ_n-manifold represents an element of $(-1)^{1+\deg a+\deg b}\langle b, a, b\rangle c$. Thus,

$$(-1)^{\deg b+1} b\langle a, b, c\rangle \subset (-1)^{\deg a+\deg b+1}\langle b, a, b\rangle c$$

and $b\langle a, b, c\rangle \subset (-1)^{\deg a}\langle b, a, b\rangle c$, as required. ∎

NOTE 3.2. Let $\mathcal{B}$ be any (B,f)-structure as in [11], and let $M\mathcal{B}_*^{\Sigma}$ be the cobordism ring of $\mathcal{B}$-manifolds with singularities Σ. Assume that the theory $M\mathcal{B}_*^{\Sigma}(\cdot)$ has an admissible commutative and associative product structure. Then, as in Definition 3.2, Toda brackets can be defined in $M\mathcal{B}_*^{\Sigma}$. Moreover, if $M\mathcal{B}_*^{\Sigma}$ does not have 3-torsion then all the results of this section are valid for these Toda brackets. In particular, all the results of this section are true for the theories $\mathrm{MSp}_*^{\hat{\Sigma}_n}(\cdot)$ where $\hat{\Sigma}_n = (P_2, P_3, \ldots, P_n)$ for $n \geq 2$.

4. **Adams-Novikov spectral sequence for** MSp^{Σ_n}. Recall that our plan for determining elements of higher order in the ring MSp_* is to construct Σ_n-manifolds which project to infinite cycles in the E_2-terms of the ANSS and ASS of $\mathrm{MSp}_*^{\Sigma_n}$ for $n \geq 3$. Then we determine the order of these projections in E_2 of the ANSS and bring back these Σ_n-manifolds to MSp_*. In this section, we accomplish the first part of our program by describing particular torsion elements $\mathfrak{t}(i_1, \ldots, i_{2s})$ of higher order in the first line

$$E_2^{1,4*+1}(\mathrm{MSp}^{\Sigma_n}) = \mathrm{Ext}_{A^{\mathrm{BP}}}^{1,4*+1}\big(\mathrm{BP}^*(\mathrm{MSp}^{\Sigma_n}), \mathrm{BP}^*\big)$$

of the ANSS which are the projections of the Σ_n-manifolds which we will construct in Section 5.

Throughout this section, let $n \geq 0$ be a fixed integer, and let MSp^{Σ_0} denote MSp. In [2, Section 1.6], the ANSS for each of the spectra MSp^{Σ_n} is described in terms of geometrical constructions on manifolds with singularities. In particular, the ANSS for the spectrum

MSp^{Σ_n} is identified with the Σ-*singularities spectral sequence* (Σ-SSS) associated with the exact couple:

$$(13)\qquad \begin{array}{ccccccccc} \mathrm{MSp}_*^{\Sigma_n} & \xleftarrow{\gamma(1)_n} & & \mathrm{MSp}_*^{\Sigma\Gamma(1)_n} & \xleftarrow{\gamma(2)_n} & & \mathrm{MSp}_*^{\Sigma\Gamma(2)_n} & \xleftarrow{\gamma(3)_n} & \cdots \\ & {\scriptstyle\pi(0)_n}\searrow & \nearrow{\scriptstyle\partial(1)_n} & & {\scriptstyle\pi(1)_n}\searrow & \nearrow{\scriptstyle\partial(2)_n} & & {\scriptstyle\pi(2)_n}\searrow & \\ & & \mathrm{MSp}_*^{\Sigma(1)_n} & \xrightarrow{\beta(1)_n} & & \mathrm{MSp}_*^{\Sigma(2)_n} & \xrightarrow{\beta(2)_n} & & \cdots \end{array}$$

Here $\mathrm{MSp}_*^{\Sigma\Gamma(k)_n}$, $\mathrm{MSp}_*^{\Sigma(k)_n}$ are the coefficient groups of particular bordism theories closely related to the theories $\mathrm{MSp}_*^{\Sigma_n}(\cdot)$, $\mathrm{MSp}_*^{\Sigma}(\cdot)$; see [2, Section 1.4]. The $E_2^{*,*}$-term of the Σ-SSS (or the ANSS) is described as follows. Consider the bigraded commutative algebra $\mathcal{M}\langle n\rangle = \mathrm{MSp}_*^{\Sigma}[u_{n+1}, u_{n+2}, \ldots u_{n+k}, \ldots]$, where $|u_{n+k}| = \big(1, 2(2^{n+k}-1)\big)$, and $|x| = (0, \deg x)$ for $x \in \mathrm{MSp}_*^{\Sigma}$. Let $\mathcal{M}\langle n\rangle_s = \big\{z \in \mathcal{M}\langle n\rangle \;\big|\; |z| = (s,*)\big\}$. As we shall see, $\mathcal{M}\langle n\rangle_s$ is isomorphic to the s-th line $E_1^{s,*}$ of the ANSS for MSp^{Σ_n}. We have the following complex:

$$(14)\ \mathcal{M}\langle n\rangle_0 \xrightarrow{\mathcal{D}\langle n\rangle} \mathcal{M}\langle n\rangle_1 \xrightarrow{\mathcal{D}\langle n\rangle} \mathcal{M}\langle n\rangle_2 \xrightarrow{\mathcal{D}\langle n\rangle} \cdots \xrightarrow{\mathcal{D}\langle n\rangle} \mathcal{M}\langle n\rangle_k \xrightarrow{\mathcal{D}\langle n\rangle} \cdots.$$

The differential $\mathcal{D}\langle n\rangle$ is defined as

$$\mathcal{D}\langle n\rangle(x u_{i_1}^{a_1}\cdots u_{i_j}^{a_j}) = \sum_{t=1}^{j}(-1)^{\epsilon_t(\alpha)}\Big((\beta_{i_t}x)u_{i_1}^{a_1}\cdots u_{i_t}^{a_t+1}\cdots u_{i_j}^{a_j}\Big)$$

where $n < i_1 < \cdots < i_j$, $\alpha = (a_1, \ldots, a_j)$ is a sequence of nonnegative integers and $\epsilon_t(\alpha) = \sum_{i=1}^{t} a_i$. It follows from the product formula (2) for Bockstein operators, that the subalgebra of cycles of the algebra $\mathcal{M}\langle n\rangle$ is a DGA. Therefore, the homology $H_*(\mathcal{M}\langle n\rangle)$ of the complex $\mathcal{M}\langle n\rangle$ has an induced algebra structure from $\mathcal{M}\langle n\rangle$. The elements u_i, $i = n+1, n+2, \ldots$ are the projections of the "basic Ray elements" $u_1 = \eta$, $u_i = \phi_{2^{i-2}}$ for $i \geq 2$. We use the same notation for these elements and their projections to the ANSS.

THEOREM 4.1 ([2, THEOREMS 3.4.1, 4.4.5]). *(i) The exact couple (13) is an Adams resolution of the spectrum* MSp^{Σ_n} *in the theory* $\mathrm{BP}^*(\cdot)$.

(ii) There is an isomorphism of algebras

$$E_2(\mathrm{MSp}^{\Sigma_n}) = \mathrm{Ext}_{A^{\mathrm{BP}}}^{*,*}\big(\mathrm{BP}^*(\mathrm{MSp}^{\Sigma_n}), \mathrm{BP}^*\big) \cong H_*(\mathcal{M}\langle n\rangle).$$

In particular, there is a ring isomorphism

$$E_2^{0,*} = \mathrm{Hom}_{A^{\mathrm{BP}}}^*\big(\mathrm{BP}^*(\mathrm{MSp}^{\Sigma_n}), \mathrm{BP}^*\big) \cong H_0(\mathcal{M}\langle n\rangle). \qquad \blacksquare$$

NOTE 4.1. The complex $\mathcal{M}\langle n\rangle$ in (14) is the bottom line of the diagram (13). In particular the first Adams-Novikov differential $\mathcal{D}\langle n\rangle\colon \mathcal{M}\langle n\rangle_0 \longrightarrow \mathcal{M}\langle n\rangle_1$ is given by $\mathcal{D}\langle n\rangle = \beta(1)_n = \oplus_{k=n+1}^{\infty}\beta_k$.

Now we are ready to use the above results to set up the environment in which we will do our chromatic calculations of $E_2(\mathrm{MSp}_*^{\Sigma_n})$. Since the ring MSp_*^{Σ} is polynomial and w_m is one of its generators, we have the following exact sequence:

$$(15)\qquad 0 \longrightarrow \mathrm{MSp}_*^{\Sigma} \xrightarrow{\cdot w_m^k} \mathrm{MSp}_*^{\Sigma} \xrightarrow{\pi^{(k)}} \mathrm{MSp}_*^{\Sigma}/(w_m^k) \longrightarrow 0$$

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where (w_m^k) is the principal ideal generated by w_m^k and $\pi^{(k)}$ is the natural projection. Since $\mathcal{D}\langle n\rangle(w_m) = 0$, the exact sequence (15) induces the exact sequence of complexes:

$$0 \longrightarrow \mathcal{M}\langle n\rangle \xrightarrow{\cdot w_m^k} \mathcal{M}\langle n\rangle \xrightarrow{\pi^{(k)}} \mathcal{M}\langle n\rangle/(w_m^k) \longrightarrow 0. \tag{16}$$

We paste the sequences (16) together to obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}\langle n\rangle & \xrightarrow{\cdot w_m} & \mathcal{M}\langle n\rangle & \xrightarrow{\pi^{(1)}} & \mathcal{M}\langle n\rangle/(w_m) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \cdot w_m} & & \Big\downarrow{\scriptstyle \cdot w_m} & & \\
0 & \longrightarrow & \mathcal{M}\langle n\rangle & \xrightarrow{\cdot w_m^2} & \mathcal{M}\langle n\rangle & \xrightarrow{\pi^{(2)}} & \mathcal{M}\langle n\rangle/(w_m^2) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \cdot w_m} & & \Big\downarrow{\scriptstyle \cdot w_m} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \cdot w_m} & & \Big\downarrow{\scriptstyle \cdot w_m} & & \\
0 & \longrightarrow & \mathcal{M}\langle n\rangle & \xrightarrow{\cdot w_m^k} & \mathcal{M}\langle n\rangle & \xrightarrow{\pi^{(k)}} & \mathcal{M}\langle n\rangle/(w_m^k) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \cdot w_m} & & \Big\downarrow{\scriptstyle \cdot w_m} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{17}$$

Taking the direct limit of the rows of (17), we obtain the following short exact sequence of complexes:

$$0 \longrightarrow \mathcal{M}\langle n\rangle \longrightarrow w_m^{-1}\mathcal{M}\langle n\rangle \xrightarrow{\pi} \mathcal{M}\langle n\rangle/(w_m^\infty) \longrightarrow 0 \tag{18}$$

where

$$w_m^{-1}\mathcal{M}\langle n\rangle = \varinjlim(\mathcal{M}\langle n\rangle \xrightarrow{\cdot w_m} \mathcal{M}\langle n\rangle \xrightarrow{\cdot w_m} \cdots) \quad \text{and}$$

$$\mathcal{M}\langle n\rangle/(w_m^\infty) = \varinjlim\big(\mathcal{M}\langle n\rangle/(w_m) \xrightarrow{\cdot w_m} \mathcal{M}\langle n\rangle/(w_m^2) \xrightarrow{\cdot w_m} \cdots\big).$$

The short exact sequence of complexes (18) induces the following long exact sequence in homology:

$$\begin{aligned}
&0 \longrightarrow H_0(\mathcal{M}\langle n\rangle) \longrightarrow H_0(w_m^{-1}\mathcal{M}\langle n\rangle) \longrightarrow H_0(\mathcal{M}\langle n\rangle/w_m^\infty) \xrightarrow{\delta_n} H_1(\mathcal{M}\langle n\rangle)\\
&\longrightarrow H_1(w_m^{-1}\mathcal{M}\langle n\rangle) \longrightarrow H_1(\mathcal{M}\langle n\rangle/w_m^\infty) \xrightarrow{\delta_n} H_2(\mathcal{M}\langle n\rangle) \longrightarrow \cdots
\end{aligned} \tag{19}$$

The key point about (19) is that the complex $w_m^{-1}\mathcal{M}\langle n\rangle$ is acyclic.

LEMMA 4.2. *For $n \geq 1$, $n \geq m \geq 0$ and $s \geq 1$:*

$$H_s(w_m^{-1}\mathcal{M}\langle n\rangle) = 0. \tag{20}$$

It follows that we have the exact sequence

$$0 \longrightarrow H_0(\mathcal{M}\langle n\rangle) \longrightarrow H_0(w_m^{-1}\mathcal{M}\langle n\rangle) \longrightarrow H_0(\mathcal{M}\langle n\rangle/w_m^\infty) \xrightarrow{\delta_m} H_1(\mathcal{M}\langle n\rangle) \longrightarrow 0, \tag{21}$$

and for $s \geq 1$ we have group isomorphisms

$$H_s(\mathcal{M}\langle n\rangle/w_m^\infty) \cong H_{s+1}(\mathcal{M}\langle n\rangle).$$

PROOF. Consider the following subalgebra of $w_m^{-1}\mathcal{M}\langle n\rangle$:

$$\mathcal{L}(i_1,\ldots,i_k) = w_m^{-1}\,\mathrm{MSp}_*[u_{i_1},\ldots,u_{i_k}],$$

where $n < i_1 < \cdots < i_k$. This subalgebra is closed under the differential $\mathcal{D}\langle n\rangle$, so it is also a subcomplex of $w_m^{-1}\mathcal{M}\langle n\rangle$. To prove (20) it is enough to show that

$$H_s\big(\mathcal{L}(i_1,\ldots,i_k)\big) \cong 0 \quad \text{for all } k \text{ and } s \geq 1. \tag{22}$$

We prove (22) by induction on k.

THE CASE $k = 1$. $H_s\big(\mathcal{L}(i)\big) \cong 0$ for $s \geq 1$. By Theorem 2.4, $\beta_i x_{m,i} = w_m$. In terms of the algebra $\mathcal{L}(i)$ this means that $\mathcal{D}\langle n\rangle(w_m^{-1}x_{m,i}) = u_i$. Let $xu_i^a \in \mathcal{L}(i), a \geq 1$, be any element such that $\mathcal{D}\langle n\rangle(xu_i^a) = (\beta_i x)u_i^{a+1} = 0$. Then $\beta_i x = 0$ and $\mathcal{D}\langle n\rangle(w_m^{-1}x_{m,i}xu_i^{a-1}) = \beta_i(w_m^{-1}x_{m,i}x)u_i^{a-1} = xu_i^a$.

THE INDUCTION STEP. $H_s\big(\mathcal{L}(i_1,\ldots,i_{k-1})\big) \cong 0 \Longrightarrow H_s\big(\mathcal{L}(i_1,\ldots,i_k)\big) \cong 0$. We have the following exact sequences of complexes:

$$0 \longrightarrow \mathcal{L}(i_1,\ldots,i_{k-1}) \longrightarrow \mathcal{L}(i_1,\ldots,i_k) \longrightarrow \mathcal{L}(i_k) \longrightarrow 0. \tag{23}$$

The long exact sequence determined by (23) implies that $H_s\big(\mathcal{L}(i_1,\ldots,i_k)\big) \cong 0$ for $s \geq 1$. ∎

Now let $n \geq 3$. We describe the structure of the subring

$$H_0(w_{n-1}^{-1}\mathcal{M}\langle n\rangle) \subset w_{n-1}^{-1}\,\mathrm{MSp}_*^{\Sigma}.$$

Let w_i, $x_{i,j}$, $x_{i_1,\ldots,i_s}$ be the polynomial generators of the ring MSp_*^{Σ} described in Theorem 2.4. Define the following polynomial generators of $w_{n-1}^{-1}\,\mathrm{MSp}_*^{\Sigma}$:

$$Z_j = x_{j,n-1}, \quad j = 1,2,\ldots,n-2, \quad Z_n = x_{n-1,n}; \tag{24}$$

$$X_i = \frac{2x_{n-1,i}}{w_{n-1}} - w_i, \; Y_i = \frac{x_{n-1,i}}{w_{n-1}^2}(x_{n-1,i} - w_i w_{n-1}), \quad i \geq n+1. \tag{25}$$

Note that $X_i^2 = 4Y_i + w_i^2$. Let $1 \leq i < j$; $i,j \neq n-1$. Then define

$$X_{i,j} = x_{i,j} - \frac{w_i x_{n-1,j} - w_j x_{n-1,i}}{w_{n-1}} + \frac{2x_{n-1,i}x_{n-1,j}}{w_{n-1}^2}. \tag{26}$$

Note that if $1 \leq i < n-1$, $j \neq n-1$, then we could have chosen the polynomial generators

$$X'_{i,j} = x_{i,j} - \frac{w_i x_{n-1,j}}{w_{n-1}}. \tag{27}$$

We can also choose polynomial generators $X_{i_1,\ldots,i_s}$ of $w_{n-1}^{-1}\,\mathrm{MSp}_*^{\Sigma}$ for $s \geq 3$ as $x_{i_1,\ldots,i_s} + \cdots$. We only need their existence. Their exact definition, is not necessary for our computations. However, for completeness, we define them as follows. If $1 < i_1 < \cdots < i_s$, $i_1, \ldots, i_s \neq n-1$ then define

$$X_{i_1,\ldots,i_s} = x_{i_1,\ldots,i_s} + \sum_{k=1}^{s-2}\Big((-1)^k \frac{w_1}{w_{n-1}^k} \sum_{1\le t_1<\cdots<t_k\le s} x_{n-1,i_{t_1}} \cdots x_{n-1,i_{t_k}} x_{i_1,\ldots,\hat{\imath}_{t_1},\ldots,\hat{\imath}_{t_k},\ldots,i_s}\Big) + (-1)^{s-1} w_1 \Big(\frac{\sum_{t=1}^{s} w_{i_t} x_{n-1,i_1} \cdots \hat{x}_{n-1,i_t} \cdots x_{n-1,i_s}}{w_{n-1}^{s-1}} - \frac{2x_{n-1,i_1} \cdots x_{n-1,i_s}}{w_{n-1}^{s}}\Big).$$

If $i_1 < \cdots < i_s$ and $i_m = n-1$ then define

$$\begin{aligned} X_{i_1,\ldots,i_s} = x_{i_1,\ldots,i_s} &+ \sum_{k=1}^{s-m-2}\Big((-1)^k \frac{w_1}{w_{n-1}^k} \sum_{m+1\le t_1<\cdots<t_k\le s} x_{n-1,i_{t_1}} \cdots x_{n-1,i_{t_k}} x_{i_1,\ldots,\hat{\imath}_{t_1},\ldots,\hat{\imath}_{t_k},\ldots,i_s}\Big) \\ &+ (-1)^{s-m-1}\Big(\frac{w_1}{w_{n-1}^{s-1}} \sum_{t=m+1}^{s} x_{n-1,i_1} \cdots \hat{x}_{n-1,i_t} \cdots x_{n-1,i_s} x_{i_1,\ldots,i_m,i_t}\Big) \\ &+ (-1)^{s-1} \frac{\delta_m^1 w_1}{w_{n-1}^s} x_{n-1,i_2} \cdots x_{n-1,i_s} \\ &+ (-1)^{s-m} \frac{(1-\delta_m^1) w_1}{w_{n-1}^{s-1}} x_{n-1,i_{m+1}} \cdots x_{n-1,i_s} x_{i_1,\ldots,i_m} \end{aligned}$$

where δ_m^1 is the Kronecker delta.

We have the following two polynomial subrings of the polynomial ring MSp_*^Σ:

$$\mathcal{P}_* = \mathbb{Z}_{(2)}[x_m \mid m \text{ even and } m \neq 2^t], \quad W\langle n\rangle_* = \mathbb{Z}_{(2)}[w_1, \ldots, w_{n-2}].$$

We will also need the following polynomial subrings of $w_{n-1}^{-1}\,\mathrm{MSp}_*^\Sigma$:

$$\begin{aligned} R\langle n\rangle_1 &= \mathbb{Z}_{(2)}[Z_j, X_i, Y_i \mid j = 1, \ldots, n-2,\ n, i \ge n+1], \\ R\langle n\rangle_2 &= \mathbb{Z}_{(2)}[X_{i,j} \mid 1 < i < j,\ i,j \neq n-1], \\ R\langle n\rangle_3 &= \mathbb{Z}_{(2)}[X_{i_1,\ldots,i_s} \mid s \ge 3, 1 < i_1 < \cdots < i_s], \\ R\langle n\rangle &= R\langle n\rangle_1 \otimes R\langle n\rangle_2 \otimes R\langle n\rangle_3. \end{aligned}$$

LEMMA 4.3. *There is a ring isomorphism:*

$$H_0(w_{n-1}^{-1}\mathcal{M}\langle n\rangle) \cong \mathbb{Z}_{(2)}[w_{n-1}, w_{n-1}^{-1}] \otimes \mathcal{W}\langle n\rangle_* \otimes \mathcal{P}_* \otimes R\langle n\rangle.$$

PROOF. There is a ring isomorphism:

$$w_{n-1}^{-1}\,\mathrm{MSp}_*^\Sigma \cong \mathbb{Z}_{(2)}[w_{n-1}, w_{n-1}^{-1}] \otimes \mathcal{W}\langle n\rangle_* \otimes \mathcal{P}_* \otimes T \otimes R\langle n\rangle_2 \otimes R\langle n\rangle_3,$$

where

$$T = \mathbb{Z}_{(2)}[w_{n-1}, \ldots, w_{n+k}, \ldots, x_{1,n-1}, \ldots, x_{n-2,n-1}, x_{n-1,n}, \ldots, x_{n-1,n+k}, \ldots].$$

Since the subring T of MSp_*^Σ is closed under the action of the Bockstein operators β_j for $j \ge n+1$, T generates the subcomplex

$$\mathcal{T} = T[u_{n+1}, \ldots, u_{n+k}, \ldots] \tag{28}$$

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of $\mathcal{M}\langle n\rangle$. To prove this lemma, it suffices to establish the isomorphism:

$$H_0(\mathcal{T}) \cong w_{n-1}^{-1}R\langle n\rangle_1. \tag{29}$$

For $k \geq 1$, define the following subrings of $w_{n-1}^{-1}\,\mathrm{MSp}_*^{\Sigma}$:

$$\begin{aligned}
T^{(k)} &= \mathbb{Z}_{(2)}[w_{n-1}, \ldots, w_{n+k}, x_{1,n-1}, \ldots, x_{n-2,n-1}, x_{n-1,n}, \ldots, x_{n-1,n+k}],\\
T_0^{(k)} &= \mathbb{Z}_{(2)}[w_{n-1}, w_{n+k}, x_{n-1,n+k}],\\
R^{(k)} &= \mathbb{Z}_{(2)}[w_{n-1}, Z_j, X_i, Y_i \mid j = 1, \ldots, n-2, n,\ i = n+1, \ldots, n+k],\\
R_0^{(k)} &= \mathbb{Z}_{(2)}[w_{n-1}, X_{n+k}, Y_{n+k}].
\end{aligned}$$

Since the rings $T^{(k)}$, $T_0^{(k)}$ are also closed under the action of the Bockstein operators β_j for $j \geq n+1$, we can define the subcomplexes $\mathcal{T}^{(k)}$ and $\mathcal{T}_0^{(k)}$ of $\mathcal{M}\langle n\rangle$ as in (28). Direct computation shows that

$$H_0(w_n^{-1}\mathcal{T}_0^{(k)}) \cong w_{n-1}^{-1}R_0^{(k)}. \tag{30}$$

In particular, we have the isomorphisms:

$$\begin{aligned}
H_0(w_{n-1}^{-1}\mathcal{T}^{(1)}) &\cong H_0(w_{n-1}^{-1}\mathcal{T}_0^{(1)}) \otimes \mathbb{Z}_{(2)}[x_{1,n-1}, \ldots, x_{n-2,n-1}, x_{n-1,n}]\\
&\cong w_{n-1}^{-1}R_0^{(1)} \otimes \mathbb{Z}_{(2)}[x_{1,n-1}, \ldots, x_{n-2,n-1}, x_{n-1,n}] = w_{n-1}^{-1}R^{(1)}.
\end{aligned}$$

By induction on $k \geq 1$, we have the a homomorphism of short exact sequences of complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & w_{n-1}^{-1}\mathcal{T}^{(k)} & \longrightarrow & w_{n-1}^{-1}\mathcal{T}^{(k+1)} & \longrightarrow & w_{n-1}^{-1}\mathcal{T}_0^{(k+1)}\\
 & & \uparrow & & \uparrow & & \uparrow\\
0 & \longrightarrow & w_{n-1}^{-1}R^{(k)} & \longrightarrow & w_{n-1}^{-1}R^{(k+1)} & \longrightarrow & w_{n-1}^{-1}R_0^{(k+1)}
\end{array}$$

where the complexes on the bottom line have zero differential. The left and right vertical maps induce isomorphisms in homology. Thus, the Five Lemma completes the induction proof of the isomorphisms (30). Taking the direct limit over k of the isomorphisms (30) establishes the isomorphism (29). ∎

Define

$$t_n(i_1, \ldots, i_{2s}) = \delta_n\left(\frac{x_{n-1,i_1}\cdots x_{n-1,i_{2s}}}{w_{n-1}}\right) \in H_1(\mathcal{M}\langle n\rangle) \tag{31}$$

where $3 \leq n < i_1 < \cdots i_s$, δ_n is the boundary homomorphism of (21) and

$$H_1(\mathcal{M}\langle n\rangle) \cong \mathrm{Ext}_{A^{\mathrm{BP}}}^{1,*}\big(\mathrm{BP}^*(\mathrm{MSp}^{\Sigma_n}), \mathrm{BP}^*\big).$$

These are the required elements of higher torsion in the first line of the ANSS.

PROPOSITION 4.4. *The element*

$$t_n(i_1, \ldots, i_s) \in \mathrm{Ext}_{A^{\mathrm{BP}}}^{1,4*+1}\big(\mathrm{BP}^*(\mathrm{MSp}^{\Sigma_n}), \mathrm{BP}^*\big)$$

has order $2^{[(s+1)/2]}$ *for any sequence* $(i_1, \ldots, i_s)$, $s \geq 1$, $3 \leq n < i_1 < \cdots < i_s$.

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For convenience we prove Proposition 4.4 assuming that s is even. The proof for the case when s odd is a slight modification of the even case. The following technical lemma will be used to show that $2^{s-1}\mathrm{t}_n(i_1,\ldots,i_{2s})\neq 0$.

LEMMA 4.5. *There does not exist an element $Y\in \mathrm{MSp}^{\Sigma}_*$, such that the element*

$$Z=2^{s-1}x_{n-1,i_1}\cdots x_{n-1,i_{2s}}-w_{n-1}Y \tag{32}$$

belongs to the ring $H_0(\mathcal{M}\langle n\rangle)$.

PROOF. By the definition of $X_{i,j}$ in (26), $\xi(i,j)=w_{n-1}^2X_{i,j}\in \mathrm{MSp}^{\Sigma}_*$. Thus, $\xi(i,j)\in H_0(\mathcal{M}\langle n\rangle)$, and there is an element $a\in \mathrm{MSp}^{\Sigma}_*$ such that

$$\xi(i_1,i_2)\cdots\xi(i_{2s-1},i_{2s})=2^sx_{n-1,i_1}\cdots x_{n-1,i_{2s}}-w_{n-1}a\in H_0(\mathcal{M}\langle n\rangle).$$

Suppose that Y exists such that $Z\in H_0(\mathcal{M}\langle n\rangle)$. Then $2Z=2^sx_{n-1,i_1}\cdots x_{n-1,i_{2s}}-2w_{n-1}Y$ or $\xi(i_1,i_2)\cdots\xi(i_{2s-1},i_{2s})-2Z=w_{n-1}(2Y-a)$. In particular, in the polynomial ring $H_0(w_{n-1}^{-1}\mathcal{M}\langle n\rangle)\otimes\mathbb{Z}/2$ we have the identity

$$\xi(i_1,i_2)\cdots\xi(i_{2s-1},i_{2s})=w_{n-1}^{2s}X_{i_1,i_2}\cdots X_{i_{2s-1},i_{2s}}=w_{n-1}(2Y-a).$$

Thus, $2Y-a=w_{n-1}^{2s-1}X_{i_1,i_2}\cdots X_{i_{2s-1},i_{2s}}$. It remains to observe that the element $w_{n-1}^{2s-1}X_{i_1,i_2}\cdots X_{i_{2s-1},i_{2s}}$ does not belong to the ring $H_0(\mathcal{M}\langle n\rangle)\otimes\mathbb{Z}/2$, while the element $2Y-a$ does. ∎

Now we can prove Proposition 4.4 from Lemma 4.5 and the exact sequence (21).

PROOF OF PROPOSITION 4.4. The element

$$w_{n-1}^{-1}x_{n-1,i_1}\cdots x_{n-1,i_{2s}}\in H_0(\mathcal{M}\langle n\rangle/w_{n-1}^{\infty})$$

is a $\mathcal{D}\langle n\rangle$-cycle. Suppose that $2^{s-1}\mathrm{t}_n(i_1,\ldots,i_{2s})=0$ in $H_1(\mathcal{M}\langle n\rangle)$. By the exact sequence (21), there is an element $Y\in \mathrm{MSp}^{\Sigma}_*$ such that

$$Y+\frac{2^{s-1}x_{n-1,i_1}\cdots x_{n-1,i_{2s}}}{w_{n-1}}$$

is a cycle in $H_0(w_{n-1}^{-1}\mathcal{M}\langle n\rangle)$. Then the element $w_{n-1}Y+2^{s-1}x_{n-1,i_1}\cdots x_{n-1,i_{2s}}\in \mathrm{MSp}^{\Sigma}_*$ is a cycle in $\mathcal{M}\langle n\rangle$, which contradicts Lemma 4.5. ∎

5. **Existence of higher torsion elements.** This section is devoted to the proof of Theorem A. In particular, we construct elements $\tau_n(i_1,\ldots,i_s)\in \mathrm{MSp}^{\Sigma_n}_*$ which project to the elements $\mathrm{t}_n(i_1,\ldots,i_s)$ of higher torsion in the one line of the ANSS which we studied in Section 4. The idea is to reproduce geometrically the inductive algebraic construction of the elements $\mathrm{t}_n(i_1,\ldots,i_s)$ as $\mathrm{t}_n(i_1)=u_{i_1}$ and $\mathrm{t}_n(i_1,\ldots,i_s)\in\langle u_{i_s},w_{n-1},\mathrm{t}_n(i_1,\ldots,i_{s-1})\rangle$. Recall the following result where $\mathrm{MSp}^{\Sigma_0}_*$ denotes MSp and ϕ_{-1} denotes $\phi_0=\eta$.

THEOREM 5.1. (V.Gorbunov, [2, Theorem 4.3.5]) *For $j > 2^{n-2}$ and $n \geq 1$ the Toda bracket $\langle \phi_{2^{n-2}}, 2, \phi_j \rangle$ contains zero in the ring* $\mathrm{MSp}_*^{\Sigma_{n-1}}$.

Recall from Section 2 that $\mathrm{MSp}_*^{\Sigma_n}$ is a polynomial ring in degrees less than or equal to $2^{n+2}-4$ with $w_1, \ldots, w_n$ as polynomial generators. P_1 is an Sp-manifold which represents $u_1 = \eta$ and P_k is an Sp-manifold which represents the basic Ray element $u_k = \phi_{2^{k-2}}$ for $k \geq 2$. We also chose Sp-manifolds W_i such that $\partial W_i = 2P_i$. Let $j \geq n \geq 2$. By Theorem 5.1, the Toda bracket $\langle \phi_{2^{n-3}}, 2, \phi_{2^{j-2}} \rangle$ contains zero in the ring $\mathrm{MSp}_*^{\Sigma_{n-2}}$. In other words, there exist Σ_{n-2}-manifolds $\tilde{X}_{n-1,j}$, $W_j^{(n-1)}$ and $W_{n-1}^{(j)}$ such that $\delta W_{n-1}^{(j)} = P_{n-1} \times 2$, $\delta W_j^{(n-1)} = 2 \times P_j$ and $\delta \tilde{X}_{n-1,j} = W_{n-1}^{(j)} \times P_j \cup P_{n-1} \times W_j^{(n-1)}$. As Σ_n-manifolds we have

$$\delta \tilde{X}_{n-1,j} = W_{n-1}^{(j)} \times P_j. \tag{33}$$

Note that cobordism classes of the manifolds $W_{n-1}^{(j)}$ depend, in general, on j.

LEMMA 5.2. *For $j \geq n \geq 2$, there exist Σ_n-manifolds W_{n-1} and $X_{n-1,j}$, such that*

$$\delta X_{n-1,j} = W_{n-1} \times P_j \tag{34}$$

where W_{n-1} does not depend on j.

PROOF. We prove this lemma by induction on $n \geq 2$. Let $n = 2$. For each $j \geq 2$ we have that $\beta_1 W_1^{(j)} = 2$ by construction. For $j \geq 2$, all the Σ_1-bordism classes $[W_1^{(j)}]$ equal the same element $w_1 \in \mathrm{MSp}_2^{\Sigma_1}$ since w_1 is the unique cobordism class such that $\beta_1 w_1 = 2$. Now assume that this lemma is true for $n-1$. Let W_{n-1} be any Sp-manifold such that $\partial W_{n-1} = 2P_{n-1}$. Let $j \geq n$ with $\tilde{X}_{n-1,j}$ and $W_{n-1}^{(j)}$ Σ_{n-2}-manifolds as above. We will define an Σ_n-manifold $X_{n-1,j}$ that satisfies (34). Since $\mathrm{MSp}_*^{\Sigma_n}$ is a polynomial ring in degrees less than or equal to $2^{n+2}-4$, we have that $\gamma = [W_{n-1}^{(j)}] - [W_{n-1}]$, is a polynomial in the generators $w_1, \ldots, w_{n-1}, x_r$, $r \leq 2^{n-3}$, $r \neq 2^l - 1$, $\deg x_r = 4r$, as in the statement of Theorem 2.1. Let γ be the sum of k monomials: $\gamma = \sum_{i=1}^k \gamma_i$. Since $\dim W_{n-1} = 2(2^{n-1}-1)$, each monomial γ_i contains at least one factor w_{m_i}, $m_i \leq n-2$; write $\gamma_i = \tilde{\gamma}_i w_{m_i}$. By induction, there is a Σ_{m_i}-manifold $X_{m_i,j}$, $m_i < n-1$, such that $\delta X_{m_i,j} = W_{m_i} \times P_j$. Let $\tilde{\Gamma}_i$ be a Σ_n-manifold which represents $\tilde{\gamma}_i$. Define a Σ_n-manifold $X_{n-1,j}$ as the disjoint union of the following Σ_n-manifolds:

$$X_{n-1,j} = \tilde{X}_{n-1,j} \cup \bigcup_{i=1}^{k} -\mathfrak{m}_n(\tilde{\Gamma}_i, X_{m_i,j} \times P_j)$$

where the Σ_n-manifolds $\tilde{X}_{n-1,j}$ are as in (33). Clearly $\delta X_{n-1,j} = W_{n-1} \times P_j$. ∎

LEMMA 5.3. *There exist elements $\tau_n(i_1, \ldots, i_s)$ in the ring $\mathrm{MSp}_{4*+1}^{\Sigma_n}$ for $s \geq 1$ and $n < i_1 < \cdots < i_s$ such that:*

(i) $\tau_n(i) = u_i$;

(ii) $\tau_n(i_1, \ldots, i_s) \in \langle u_{i_s}, w_{n-1}, \tau_n(i_1, \ldots, i_{s-1}) \rangle$ *for $s \geq 2$;*

(iii) $w_{n-1}\tau_n(i_1, \ldots, i_s) = 0$.

PROOF. We construct the elements $\tau_n(i_1,\ldots,i_s)$ by induction on $s \geq 1$. For $s = 1$, Theorem 5.1 gives that $w_{n-1}u_j = 0$. Assume that this lemma is true for $s-1$. Select any element $\tau_n(i_1,\ldots,i_s)$ of the Toda bracket $\langle u_{i_s}, w_{n-1}, \tau_n(i_1,\ldots,i_{s-1})\rangle$. We must show that $w_{n-1}\tau_n(i_1,\ldots,i_s) = 0$. Let W_{n-1} be a Σ_n-manifold as in Lemma 5.2 which represents w_{n-1} so that there exists a Σ_n-manifold X_{n-1,i_s} with $\delta X_{n-1,i_s} = W_{n-1} \times P_{i_s}$. By Lemma 3.4 we have that $w_{n-1}\tau_n(i_1,\ldots,i_s) \in w_{n-1}\langle u_{i_s}, w_{n-1}, \tau_n(i_1,\ldots,i_{s-1})\rangle \subset \langle w_{n-1}, u_{i_s}, w_{n-1}\rangle\tau_n(i_1,\ldots,i_{s-1})$. Note that the Σ_n-manifold $\Delta(W_{n-1})$ of (11) has dimension $2(2^{n-1}-1)-1 = \dim u_n$. The element u_n is the unique nontrivial element of this degree in $\mathrm{MSp}_*^{\Sigma_{n-1}}$. Thus,

$$[\Delta(W_{n-1})]_{\Sigma_{n-1}} = \kappa u_n, \quad \text{where } \kappa = 0 \text{ or } 1. \tag{35}$$

Therefore, in $\mathrm{MSp}_*^{\Sigma_n}$, $[\Delta(W_{n-1})] = 0$. Thus in $\mathrm{MSp}_*^{\Sigma_n}$, Lemma 3.3 and the induction hypothesis give

$$\langle w_{n-1}, u_{i_s}, w_{n-1}\rangle\tau_n(i_1,\ldots,i_{s-1}) = (\kappa u_n u_{i_s} + w_{n-1}a)\tau_n(i_1,\ldots,i_{s-1}) = 0. \qquad \blacksquare$$

Next we determine the projection of $\tau_n(i_1,\ldots,i_s)$ into the E_2-term of the ANSS for MSp^{Σ_n} to be $\mathfrak{t}_n(i_1,\ldots,i_s)$ which we defined in (31) and studied in Proposition 4.4. We do this by constructing a Σ_n-manifold which represents $\tau_n(i_1,\ldots,i_s)$ and a Σ_n-manifold whose boundary equals $w_{n-1}\tau_n(i_1,\ldots,i_s)$ modulo the Adams-Novikov filtration. These manifolds will be used in the constructions of the next section. We denote as $\mathfrak{m}$, $\mathfrak{K}$, $\mathfrak{A}$ the constructions $\mathfrak{m}_n$, $\mathfrak{K}_n$, $\mathfrak{A}_n$ from Section 3.

LEMMA 5.4. *There exists an element $\tau_n(i_1,\ldots,i_s)$ of the Toda bracket*

$$\langle u_{i_s}, w_{n-1}, \tau_n(i_1,\ldots,i_{s-1})\rangle$$

such that the projection of $\tau_n(i_1,\ldots,i_s)$ into $E_2^{1,4*+1}(\mathrm{MSp}^{\Sigma_n})$ of the ANSS *equals*

$$\mathfrak{t}_n(i_1,\ldots,i_s) = \sum_{r=1}^{s} u_{i_r} x_{n-1,i_1}\cdots\hat{x}_{n-1,i_r}\cdots x_{n-1,i_s}.$$

PROOF. Lemma 5.3 gives us a Σ_n-manifold $T(i_1,\ldots,i_s)$ which represents the element $\tau_n(i_1,\ldots,i_s)$ and a Σ_n-manifold $H(i_1,\ldots,i_s)$ with

$$\delta H(i_1,\ldots,i_s) = \mathfrak{m}\big(W_{n-1}, T(i_1,\ldots,i_s)\big).$$

We use induction on $s \geq 1$ to define specific Σ_n-manifolds T_s and H_s such that:

(i) $\delta H_s = \mathfrak{m}(W_{n-1}, T_s)$;

(ii) T_s projects in the one line of the ANSS to $\mathfrak{t}_n(i_1,\ldots,i_s) \in H_1(\mathcal{M}\langle n\rangle)$;

(iii) H_s projects in the zero line of the ANSS to $x_{n-1,i_1}\cdots x_{n-1,i_s} \in H_0(\mathcal{M}\langle n\rangle)$;

(iv) T_s represents an element of $\langle u_{i_s}, w_{n-1}, \tau_n(i_1,\ldots,i_{s-1})\rangle$.

If $s = 1$ let $T_1 = P_{i_1}$ and $H_1 = X_{n-1,i_1}$. By induction suppose that we have Σ_n-manifolds T_{s-1} and H_{s-1} which satisfy the above four conditions. Consider the Σ_n-manifold $H^{(0)} = \mathfrak{m}(X_{n-1,i_s}, H_{s-1})$ with

$$\delta(H^{(0)}) = \mathfrak{m}\big(X_{n-1,i_s}, \mathfrak{m}(W_{n-1}, T_{s-1})\big) \cup \mathfrak{m}(W_{n-1} \times P_{i_s}, H_{s-1}).$$

Glue the Σ_n-manifolds $H^{(0)}$ and $-\mathfrak{A}(X_{n-1,i_s}, W_{n-1}, T_{s-1})$ together along their common boundary $\mathfrak{m}\big(X_{n-1,i_s}, \mathfrak{m}(W_{n-1}, T_{s-1})\big)$ to obtain the Σ_n-manifold $H^{(1)} = H^{(0)} \cup -\mathfrak{A}(X_{n-1,i_s}, W_{n-1}, T_{s-1})$ with

$$\begin{aligned}\delta(H^{(1)}) = \mathfrak{m}(W_{n-1} \times P_{i_s}, H_{s-1}) \cup \mathfrak{m}\big(\mathfrak{m}(X_{n-1,i_s}, W_{n-1}), T_{s-1}\big)\\ \cup \mathfrak{A}(W_{n-1} \times P_{i_s}, W_{n-1}, T_{s-1}).\end{aligned}$$

Next glue the Σ_n-manifolds $H^{(1)}$ and $-\mathfrak{m}\big(\mathfrak{K}(X_{n-1,i_s}, W_{n-1}), T_{s-1}\big)$ together along their common boundary $\mathfrak{m}\big(\mathfrak{m}(X_{n-1,i_s}, W_{n-1}), T_{s-1}\big)$ to obtain the Σ_n-manifold $H^{(2)} = H^{(1)} \cup -\mathfrak{m}\big(\mathfrak{K}(X_{n-1,i_s}, W_{n-1}), T_{s-1}\big)$ with

$$\begin{aligned}\delta(H^{(2)}) &= \mathfrak{m}(W_{n-1} \times P_{i_s}, H_{s-1}) \cup \mathfrak{m}\big(\mathfrak{m}(W_{n-1}, X_{n-1,i_s}), T_{s-1}\big)\\ &\quad \cup \mathfrak{m}\big(\mathfrak{K}(W_{n-1} \times P_{i_s}, W_{n-1}), T_{s-1}\big) \cup \mathfrak{A}(W_{n-1} \times P_{i_s}, W_{n-1}, T_{s-1})\\ &= \mathfrak{m}(W_{n-1}, P_{i_s} \times H_{s-1}) \cup \mathfrak{m}\big(\mathfrak{m}(W_{n-1}, X_{n-1,i_s}), T_{s-1}\big)\\ &\quad \cup \mathfrak{m}\Big(\mathfrak{m}\big(W_{n-1}, \mathfrak{K}(P_{i_s}, W_{n-1})\big), T_{s-1}\Big) \cup \mathfrak{m}\big(\mathfrak{K}(W_{n-1}, W_{n-1}) \times P_{i_s}, T_{s-1}\big)\\ &\quad \cup \mathfrak{A}(W_{n-1} \times P_{i_s}, W_{n-1}, T_{s-1})\end{aligned}$$

by the Hirsch formula of Lemma 3.1(a). Observe that $\mathfrak{K}(W_{n-1}, W_{n-1})$ must be the boundary of a Σ_n-manifold V since $\mathrm{MSp}^{\Sigma_n}_{2^{n+1}-3} = 0$. Thus, glue the Σ_n-manifolds $H^{(2)}$ and $\mathfrak{m}(V \times P_{i_s}, T_{s-1})$ along $\mathfrak{m}\big(\mathfrak{K}(W_{n-1}, W_{n-1}) \times P_{i_s}, T_{s-1}\big)$ to obtain the Σ_n-manifold $H^{(3)} = H^{(2)} \cup \mathfrak{m}(V \times P_{i_s}, T_{s-1})$ with

$$\begin{aligned}\delta(H^{(3)}) = \mathfrak{m}(W_{n-1}, P_{i_s} \times H_{s-1}) \cup \mathfrak{m}\big(\mathfrak{m}(W_{n-1}, X_{n-1,i_s}), T_{s-1}\big)\\ \cup \mathfrak{m}\big(\mathfrak{m}(W_{n-1}, \mathfrak{K}(P_{i_s}, W_{n-1})), T_{s-1}\big) \cup \mathfrak{A}(W_{n-1} \times P_{i_s}, W_{n-1}, T_{s-1})\end{aligned}$$

Finally glue the Σ_n-manifolds $H^{(3)}$ and $\mathfrak{A}(W_{n-1}, X_{n-1,i_s} \cup \mathfrak{K}(P_{i_s}, W_{n-1}), T_{s-1})$ along their common boundary

$$\begin{aligned}\mathfrak{m}\big(\mathfrak{m}(W_{n-1}, X_{n-1,i_s}), T_{s-1}\big) \cup \mathfrak{m}\Big(\mathfrak{m}\big(W_{n-1}, \mathfrak{K}(P_{i_s}, W_{n-1})\big), T_{s-1}\Big)\\ \cup \mathfrak{A}(W_{n-1} \times P_{i_s}, W_{n-1}, T_{s-1})\end{aligned}$$

to obtain the Σ_n-manifold $H_s = H^{(3)} \cup \mathfrak{A}\big(W_{n-1}, X_{n-1,i_s} \cup \mathfrak{K}(P_{i_s}, W_{n-1}), T_{s-1}\big)$ with

$$\begin{aligned}\delta(H_s) &= \mathfrak{m}(W_{n-1}, P_{i_s} \times H_{s-1}) \cup \mathfrak{m}\big(W_{n-1}, \mathfrak{m}(X_{n-1,i_s}, T_{s-1})\big)\\ &\quad \cup \mathfrak{m}\Big(W_{n-1}, \mathfrak{m}\big(\mathfrak{K}(P_{i_s}, W_{n-1}), T_{s-1}\big)\Big)\\ &= \mathfrak{m}\Big(W_{n-1}, P_{i_s} \times H_{s-1} \cup \mathfrak{m}\big(X_{n-1,i_s} \cup \mathfrak{K}(P_{i_s}, W_{n-1}), T_{s-1}\big)\Big) = \mathfrak{m}(W_{n-1}, T_s)\end{aligned}$$

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where $T_s = P_{i_s} \times H_{s-1} \cup \mathfrak{m}\big(X_{n-1,i_s} \cup \mathfrak{K}(P_{i_s}, W_{n-1}), T_{s-1}\big)$. Clearly (i) is satisfied, and (iv) is satisfied because $\delta\big(X_{n-1,i_s} \cup \mathfrak{K}(P_{i_s}, W_{n-1})\big) = \mathfrak{m}(P_{i_s}, W_{n-1})$. By the induction hypothesis, the projection of T_s to the one line of the ANSS equals

$$u_{i_s} x_{n-1,i_1} \cdots x_{n-1,i_{s-1}} + x_{n-1,i_s} \mathfrak{t}_n(i_1, \ldots, i_{s-1}) = \mathfrak{t}_n(i_1, \ldots, i_s).$$

Since T_{s-1} and P_{i_s} have Adams-Novikov filtration degree one, the projections of $\mathfrak{A}(X_{n-1,i_s}, W_{n-1}, T_{s-1})$, $\mathfrak{m}\big(\mathfrak{K}(X_{n-1,i_s}, W_{n-1}), T_{s-1}\big)$, $\mathfrak{m}(V \times P_{i_s}, T_{s-1})$ and $\mathfrak{A}\big(W_{n-1}, X_{n-1,i_s} \cup \mathfrak{K}(P_{i_s}, W_{n-1}), T_{s-1}\big)$ to the zero line of the ANSS are trivial. Thus, the projection of H_s to the zero line of the ANSS equals the projection of $H^{(0)}$ which by the induction hypothesis is $x_{n-1,i_s} \cdot x_{n-1,i_1} \cdots x_{n-1,i_{s-1}}$. ∎

PROOF OF THEOREM A. By the previous lemma, the element $\tau_n(i_1, \ldots, i_s) \in \mathrm{MSp}_*^{\Sigma_n}$ can be defined as required so that it projects to

$$\mathfrak{t}_n(i_1, \ldots, i_s) \in \mathrm{Ext}^{1,*}_{A^{\mathrm{BP}}}\big(\mathrm{BP}^*(\mathrm{MSp}^{\Sigma_n}), \mathrm{BP}^*\big)$$

which has order $2^{[(s+1)/2]}$ by Lemma 4.4. Therefore, $\tau_n(i_1, \ldots, i_s)$ has order greater than or equal to $2^{[(s+1)/2]}$ in $\mathrm{MSp}_*^{\Sigma_n}$. Finally, note that $\tau_n(i_1, \ldots, i_s)$ is indecomposable in $\mathrm{MSp}_*^{\Sigma_n}$ because its projection $\mathfrak{t}_n(i_1, \ldots, i_s)$ into the algebra $E_2^{*,*}(\mathrm{MSp}^{\Sigma_n})$ is indecomposable.

PROOF OF THEOREM B. By Theorem A, there are torsion elements of order greater than or equal to 2^k for all $k \geq 1$ in the ring MSp^{Σ_3}. Consider the Bockstein-Sullivan exact sequences:

$$\cdots \longrightarrow \mathrm{MSp}_*^{\Sigma_2} \xrightarrow{\cdot\phi_2} \mathrm{MSp}_*^{\Sigma_2} \xrightarrow{\pi_2} \mathrm{MSp}_*^{\Sigma_3} \xrightarrow{\beta_3} \mathrm{MSp}_*^{\Sigma_2} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{MSp}_*^{\Sigma_1} \xrightarrow{\cdot\phi_1} \mathrm{MSp}_*^{\Sigma_1} \xrightarrow{\pi_1} \mathrm{MSp}_*^{\Sigma_2} \xrightarrow{\beta_2} \mathrm{MSp}_*^{\Sigma_1} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{MSp}_* \xrightarrow{\cdot\eta} \mathrm{MSp}_* \xrightarrow{\pi_0} \mathrm{MSp}_*^{\Sigma_1} \xrightarrow{\beta_1} \mathrm{MSp}_* \longrightarrow \cdots$$

We show that exponents of the groups $\mathrm{Tors}\,\mathrm{MSp}_*^{\Sigma_2}$, $\mathrm{Tors}\,\mathrm{MSp}_*^{\Sigma_1}$ and $\mathrm{Tors}\,\mathrm{MSp}_*$ must be infinite since the exponent of $\mathrm{Tors}\,\mathrm{MSp}_*^{\Sigma_3}$ is infinite. Assume, to the contrary, that all torsion of $\mathrm{MSp}_*^{\Sigma_2}$ has exponent 2^k. We take an element $a \in \mathrm{MSp}_*^{\Sigma_3}$ of order 2^{2k+1}. Then the element $a_1 = \beta_2(a)$ has order no more than 2^k. From the above Bockstein-Sullivan exact sequence,

$$2^k a \in \mathrm{Im}\{\mathrm{MSp}_*^{\Sigma_2} \xrightarrow{\pi_1} \mathrm{MSp}_*^{\Sigma_3}\} \subset \mathrm{MSp}_*^{\Sigma_3}.$$

Let $\pi_2(a_2) = 2^k a$. Then $2^{k+1} a_2 \in \mathrm{Ker}\,\pi_2 = \mathrm{Im}(\cdot\phi_2)$, so $2^{k+1} a_2 = \phi_2 x$. Consequently $2^{k+2} a_2 = 0$, and a_2 has finite order. Since $\pi_2(2^k a_2) = 2^{2k} a \neq 0$, the element $a_2 \in \mathrm{MSp}_*^{\Sigma_2}$ has order greater than or equal to 2^{k+1}, contradicting the assumption that $\mathrm{Tors}\,\mathrm{MSp}_*^{\Sigma_2}$ has exponent 2^k. Thus, the exponent of $\mathrm{Tors}\,\mathrm{MSp}_*^{\Sigma_2}$ is infinite. ∎

6. **Construction of elements in** $\mathrm{MSp}_*^{\Sigma_2}$ **and** $\mathrm{MSp}_*^{\Sigma_1}$**.** In Lemma 5.3 we constructed elements of higher torsion

$$\tau(i_1,\ldots,i_s)=\tau_3(i_1,\ldots,i_s)\in \mathrm{MSp}^{\Sigma_3}_{4*+1}.$$

In this section we study the elements:

$$\gamma(i_1,\ldots,i_s)=\beta_3\tau(i_1,\ldots,i_s)\in \mathrm{MSp}^{\Sigma_2}_{4*+3},$$
$$\alpha'(i_1,\ldots,i_s)=\beta_2\gamma(i_1,\ldots,i_s)\in \mathrm{MSp}^{\Sigma_1}_{4*+1}.$$

In particular we compute their projection to the three line of the ANSS. Throughout this section, let $\mathfrak{m}$ and $\mathfrak{K}$ denote the canonical constructions $\mathfrak{m}_2$ and $\mathfrak{K}_2$ of Section 3.

We begin by interpreting $\beta_3\big(\tau_3(i_1,\ldots,i_s)\big)$ in terms of manifolds with singularities. Recall that by Lemma 5.3 there is a representative Σ_3-manifold $T(i_1,\ldots,i_s)$ of $\tau_3(i_1,\ldots,i_s)$ and a Σ_3-manifold $H(i_1,\ldots,i_s)$ such that

$$\delta H(i_1,\ldots,i_s)=\mathfrak{m}\big(W_2,T(i_1,\ldots,i_s)\big).$$

We can consider the manifold $T(i_1,\ldots,i_s)$, $H(i_1,\ldots,i_s)$ as a Σ_2-manifold $\tilde{T}(i_1,\ldots,i_s)$, $\tilde{H}(i_1,\ldots,i_s)$, respectively, with

$$\begin{aligned}\delta\tilde{H}(i_1,\ldots,i_s)&=\mathfrak{m}\big(W_2,\tilde{T}(i_1,\ldots,i_s)\big)\cup P_3\times E(i_1,\ldots,i_s),\\ \delta\tilde{T}(i_1,\ldots,i_s)&=P_3\times G(i_1,\ldots,i_s)\end{aligned}\tag{36}$$

where $G(i_1,\ldots,i_s)=\beta_3T(i_1,\ldots,i_s)$ represents the Σ_2-cobordism class $\gamma(i_1,\ldots,i_s)$. Note that $\delta E(i_1,\ldots,i_s)=\mathfrak{m}\big(W_2,G(i_1,\ldots,i_s)\big)$. To determine the projection of $E(i_1,\ldots,i_s)$ to the ANSS we need to identify the quadratic construction $\Delta(W_2)$ which was defined in Section 3. The following lemma is an easy computation in the ASS for $\mathrm{MSp}_*^{\Sigma_2}$. We defer its proof to [3, Corollary 4.6].

LEMMA 6.1. *The cobordism class* $[\Delta(W_2)]$ *equals* $[P_3]=\phi_2$ *in* $\mathrm{MSp}_*^{\Sigma_2}$. ∎

We are now ready to compute the projection of $E(i_1,\ldots,i_s)$ into the two line of the ANSS for MSp^{Σ_2}. This will lead directly to the identification of the projection of the $\gamma(i_1,\ldots,i_s)$ into the three line of the ANSS for MSp^{Σ_2}.

LEMMA 6.2. *(a)* $E(i_1)=\emptyset$.
(b) For $s\geq 2$, $E(i_1,\ldots,i_s)$ *projects in* $E_1^{2,4*+2}$ *of the* ANSS *for* MSp^{Σ_2} *to*

$$\mathfrak{e}(i_1,\ldots,i_s)=\sum_{1\leq t_1<t_2\leq s}u_{i_{t_1}}u_{i_{t_2}}x_{2,i_1}\cdots\hat{x}_{2,i_{t_1}}\cdots\hat{x}_{2,i_{t_2}}\cdots x_{2,i_s}.$$

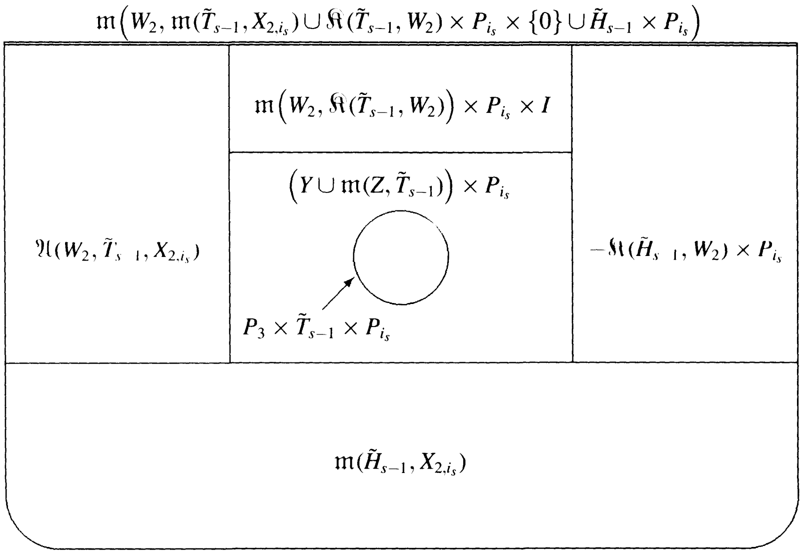

FIGURE 3: The Σ_n-manifold $\tilde{H}_s$.

PROOF. (a) Since $T(i) = P_i$, $H(i)$ can be taken to be the Σ_2-manifold $X_{2,i}$ of Lemma 5.2 with boundary $\delta X_{2,i} = W_2 \times P_i$ where W_2 does not depend on i. Thus, $E(i) = \emptyset$.

(b) By induction on s, we construct Σ_2-manifolds $\tilde{T}_s$, $\tilde{H}_s$, E_s and $\tilde{L}_s$ such that:

(i) $\tilde{T}_s$ represents $\tau(i_1, \ldots, i_s)$;

(ii) $\delta\tilde{H}_s = \mathfrak{m}(W_2, \tilde{T}_s) \cup P_3 \times E_s \cup \tilde{L}_s$;

(iii) $\tilde{T}_s$ projects to $\mathfrak{t}(i_1, \ldots, i_s)$ in the one line of the ANSS;

(iv) $\tilde{H}_s$ projects to $x_{2,i_1} \cdots x_{2,i_s}$ in the zero line of the ANSS;

(v) E_s projects to $\mathfrak{e}(i_1, \ldots, i_s)$ in the two line of the ANSS;

(vi) $\tilde{L}_s$ has Adams-Novikov filtration degree four.

The case $s = 2$ will be proved as a special case of the induction step with $\tilde{T}_1 = P_{i_1}$, $\tilde{H}_1 = X_{2,i_1}$, $E_1 = \emptyset$ and $\tilde{L}_1 = \emptyset$. Clearly statements (i)–(vi) are valid for $s = 1$. Thus, assume that $s \geq 2$ and that our six assertions are true in the case $s - 1$. Define the Σ_2-manifold

$$\tilde{H}'_s = \mathfrak{m}(\tilde{H}_{s-1}, X_{2,i_s}) \cup \mathfrak{A}(W_2, \tilde{T}_{s-1}, X_{2,i_s}) \cup \\ - \mathfrak{K}(\tilde{H}_{s-1}, W_2) \times P_{i_s} \cup \mathfrak{m}\big(W_2, \mathfrak{K}(\tilde{T}_{s-1}, W_2)\big) \times P_{i_s} \times I.$$

See Figure 3. Let F^p denote the set of Σ_2-manifolds of Adams-Novikov filtration degree p. Then, as in the proof of Lemma 5.4, we have modulo F^4 that

$$\delta\tilde{H}'_s \equiv \mathfrak{m}\big(W_2, \mathfrak{m}(\tilde{T}_{s-1}, X_{2,i_s}) \cup \tilde{H}_{s-1} \times P_{i_s} \cup \mathfrak{K}(\tilde{T}_{s-1}, W_2) \times P_{i_s} \times \{0\}\big) \\ \cup P_3 \times \mathfrak{m}(E_{s-1}, X_{2,i_s}) \cup \mathfrak{K}\big(\mathfrak{m}(W_2, \tilde{T}_{s-1}), W_2\big) \times P_{i_s}$$

$$\cup \mathfrak{A}(W_2, \tilde{T}_{s-1}, W_2) \times P_{i_s} \cup -\mathfrak{m}\big(W_2, \mathfrak{K}(\tilde{T}_{s-1}, W_2)\big) \times P_{i_s} \times \{1\}.$$

Observe that $\delta T_{s-1} \equiv \delta W_2 \equiv \emptyset$ module F^3. Thus by the generalized Hirsch formula (10), we have modulo F^3 that there is a Σ_2-manifold Y such that

$$\delta Y \equiv -\mathfrak{K}\big(\mathfrak{m}(W_2, \tilde{T}_{s-1}), W_2\big) \cup -\mathfrak{A}(W_2, \tilde{T}_{s-1}, W_2)$$
$$\cup\, \mathfrak{m}\big(W_2, \mathfrak{K}(\tilde{T}_{s-1}, W_2)\big) \cup \mathfrak{m}\big(\mathfrak{K}(W_2, W_2), \tilde{T}_{s-1}\big).$$

By Lemma 6.1 there is a Σ_2-manifold Z such that $\delta Z = -\mathfrak{K}(W_2, W_2) \cup P_3$. Let $\tilde{H}_s = \tilde{H}'_s \cup Y \times P_{i_s} \cup \mathfrak{m}(Z, \tilde{T}_{s-1}) \times P_{i_s}$. Then

$$\begin{aligned} \delta\tilde{H}_s &= \mathfrak{m}(W_2, \tilde{T}_s) \cup P_3 \times E_s \cup \tilde{L}_s \quad \text{where} \\ \tilde{T}_s &= \mathfrak{m}(\tilde{T}_{s-1}, X_{2.i_s}) \cup \tilde{H}_{s-1} \times P_{i_s} \cup \mathfrak{K}(\tilde{T}_{s-1}, W_2) \times P_{i_s}, \\ E_s &= \mathfrak{m}(E_{s-1}, X_{2.i_s}) \cup \tilde{T}_{s-1} \times P_{i_s} \end{aligned}$$

and $\tilde{L}_s$ has Adams-Novikov filtration degree four. By the induction hypothesis, $\tilde{T}_s$ projects in the one line of the ANSS to

$$\mathfrak{t}(i_1, \ldots, i_{s-1})x_{2,i_s} + x_{2,i_1} \cdots x_{2,i_{s-1}} \cdot u_{i_s} = \mathfrak{t}(i_1, \ldots, i_s),$$

$\tilde{H}_s$ projects in the zero line of the ANSS to $x_{2,i_1} \cdots x_{2,i_{s-1}} \cdot x_{2,i_s}$, and E_s projects in the two line of the ANSS to

$$\gamma(i_1, \ldots, i_{s-1})x_{2,i_s} + \mathfrak{t}(i_1, \ldots, i_{s-1})u_{i_s} = \gamma(i_1, \ldots, i_s).$$

This completes the induction step. Observe that E_s and $E(i_1, \ldots, i_s)$ differ by a Σ_2-manifold of Adams-Novikov filtration degree five. Therefore, they have the same projection to the three line of the ANSS. ∎

We can now determine the basic properties of the $\gamma(i_1, \ldots, i_s)$.

PROPOSITION 6.3. *The elements* $\gamma(i_1, \ldots, i_s) = \beta_3\tau(i_1, \ldots, i_s) \in \mathrm{MSp}^{\Sigma_2}_{4*+3}$ *satisfy the following conditions:*

(a) $\gamma(i_1) = \gamma(i_1, i_2) = 0$;
(b) $\gamma(i_1, i_2, i_3) = u_{i_1}u_{i_2}u_{i_3}$;
(c) For $s \geq 4$, $\gamma(i_1, \ldots, i_s) \in \langle u_{i_s}, w_2, \gamma(i_1, \ldots, i_{s-1})\rangle$;
(d) For $s \geq 3$, $\gamma(i_1, \ldots, i_s)$ *projects in* $E_1^{3,*}(\mathrm{MSp}^{\Sigma_2})$ *of the* ANSS *to*

$$\mathfrak{g}(i_1, \ldots, i_s) = \sum_{1 \leq t_1 < t_2 < t_3 \leq s} u_{i_{t_1}} u_{i_{t_2}} u_{i_{t_3}} x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots \hat{x}_{2,i_{t_3}} \cdots x_{2,i_s}.$$

PROOF. (a) Since $\tau(i_1) = u_{i_1}$ and $\tau(i_1, i_2)$ is represented by

$$P_{i_1} \times X_{2,i_2} \cup \mathfrak{K}(P_{i_1}, W_2) \times P_{i_2} \cup X_{2,i_1} \times P_{i_2}$$

it follows that $G(i_1) = G(i_1, i_2) = \emptyset$.

(b), (c) Let $s \geq 3$. By Lemma 5.3(ii) and (36), we have the boundary of Σ_2-manifolds

$$\delta\tilde{T}(i_1,\ldots,i_s) = \mathfrak{m}\big(X_{i_s,2}, P_3 \times G(i_1,\ldots,i_{s-1})\big) \cup P_{i_s} \times P_3 \times E(i_1,\ldots,i_{s-1}).$$

Therefore, $\gamma(i_1,\ldots,i_s) = \beta_3\tau(i_1,\ldots,i_s)$ is represented by the Σ_2-manifold

$$G(i_1,\ldots,i_s) = \mathfrak{m}\big(X_{i_s,2}, G(i_1,\ldots,i_{s-1})\big) \cup P_{i_s} \times E(i_1,\ldots,i_{s-1}). \tag{37}$$

If $s = 3$ then the first unionand is vacuous. Observe that the constructions in the proof of Lemma 6.2 give $E_2 = P_{i_1} \times P_{i_2}$ and $\tilde{L}_2 = \emptyset$. Thus, $\gamma(i_1,i_2,i_3)$ is represented by $P_{i_1} \times P_{i_2} \times P_{i_3}$. If $s \geq 4$ then the element in (37) is an element of the Toda bracket $\langle u_{i_s}, w_2, \gamma(i_1,\ldots,i_{s-1})\rangle$.

(d) We use induction on $s \geq 3$. The case $s = 3$ follows from (b). Assume the case $s-1$. By the description of the Σ_2-manifold $G(i_1,\ldots,i_s)$ in the proof of (c), $\gamma(i_1,\ldots,i_s)$ projects in the three line of the ANSS to

$$\mathfrak{g}(i_1,\ldots,i_s) = x_{i_s,2}\mathfrak{g}(i_1,\ldots,i_{s-1}) \cup u_{i_s} \times \mathfrak{e}(i_1,\ldots,i_{s-1}).$$

By the induction hypothesis and the previous lemma,

$$\mathfrak{g}(i_1,\ldots,i_s) = x_{i_s,2} \sum_{1\leq t_1<t_2<t_3\leq s-1} u_{i_{t_1}}u_{i_{t_2}}u_{i_{t_3}}x_{2,i_1}\cdots\hat{x}_{2,i_{t_1}}\cdots\hat{x}_{2,i_{t_2}}\cdots\hat{x}_{2,i_{t_3}}\cdots x_{2,i_{s-1}}$$
$$+ u_{i_s} \sum_{1\leq t_1<t_2\leq s-1} u_{i_{t_1}}u_{i_{t_2}}x_{2,i_1}\cdots\hat{x}_{2,i_{t_1}}\cdots\hat{x}_{2,i_{t_2}}\cdots x_{2,i_{s-1}}.$$

This is the asserted value of $\mathfrak{g}(i_1,\ldots,i_s)$ in (d). ∎

We complete this section by computing the projection of the element

$$\alpha'(i_1,\ldots,i_s) = \beta_2\gamma(i_1,\ldots,i_s) \in \mathrm{MSp}_*^{\Sigma_1}$$

to the ANSS $E_1^{3,*}(\mathrm{MSp}^{\Sigma_1})$ where $\beta_2\colon \mathrm{MSp}_*^{\Sigma_2} \longrightarrow \mathrm{MSp}_*^{\Sigma_1}$ is the Bockstein operator. To describe this projection we introduce the notation

$$\mathfrak{p}(j_1,\ldots,j_q) = \sum_{1\leq t_1<t_2<t_3\leq q} u_{j_{t_1}}u_{j_{t_2}}u_{j_{t_3}}w_{j_1}\cdots\hat{w}_{j_{t_1}}\cdots\hat{w}_{j_{t_2}}\cdots\hat{w}_{j_{t_3}}\cdots w_{j_q}$$

in $E_1^{3,*}(\mathrm{MSp}^{\Sigma_1})$ for $q \geq 3$.

PROPOSITION 6.4. *The elements* $\alpha'(i_1,\ldots,i_s) = \beta_2\gamma(i_1,\ldots,i_s) \in \mathrm{MSp}_{4*+1}^{\Sigma_1}$ *satisfy the following conditions.*

(a) $\alpha'(i_1) = \alpha'(i_1,i_2) = \alpha'(i_1,i_2,i_3) = 0$.

(b) For $s \geq 4$, $\alpha'(i_1,\ldots,i_s)$ *projects in* $E_1^{3,*}(MSP^{\Sigma_1})$ *of the* ANSS *to*

$$\mathfrak{a}'(i_1,\ldots,i_s) = \sum_{k=4}^{s}(-1)^k w_2^{k-4} \sum_{1\leq t_1<\cdots<t_k\leq s} \mathfrak{p}(i_{t_1},\ldots,i_{t_k})x_{2,i_1}\cdots\hat{x}_{2,i_{t_1}}\cdots\hat{x}_{2,i_{t_k}}\cdots x_{2,i_s}.$$

(c) For $s \geq 4$, *there are elements* $\alpha(i_1,\ldots,i_s) \in \mathrm{MSp}_*$ *such that the element* $\pi_*\big(\alpha(i_1,\ldots,i_s)\big) \in \mathrm{MSp}_*^{\Sigma_1}$ *projects to* $2\mathfrak{a}'(i_1,\ldots,i_s) \in E_2^{3,4*+1}(\mathrm{MSp}^{\Sigma_1})$ *where* $\pi\colon \mathrm{MSp} \longrightarrow \mathrm{MSp}^{\Sigma_1}$ *is the natural map.*

(d) The projection $\mathfrak{a}(i_1,\ldots,i_s) \in E_2^{3,4*+1}(\mathrm{MSp})$ *of* $\alpha(i_1,\ldots,i_s) \in \mathrm{MSp}_*$ *is a nonzero infinite cycle.*

PROOF. (a) This follows from Proposition 6.3(a),(b).

(b) The Bockstein operator $\beta_2\colon \mathrm{MSp}_*^{\Sigma_2} \longrightarrow \mathrm{MSp}_*^{\Sigma_1}$ induces the homomorphism

$$\mathfrak{b}_2\colon E_2^{*,*}(\mathrm{MSp}^{\Sigma_2}) \longrightarrow E_2^{*,*}(\mathrm{MSp}^{\Sigma_1}).$$

Then $\alpha'(i_1,\dots i_s) = \beta_2\big(\gamma(i_1,\dots,i_s)\big)$ projects in $E_1^{3,*}(\mathrm{MSp}^{\Sigma_1})$ to $\mathfrak{b}_2$ of the projection of $\gamma(i_1,\dots,i_s)$ to $E_1^{3,*}(\mathrm{MSp}^{\Sigma_2})$. In Proposition 6.3(d) we determined the latter projection $\mathfrak{g}(i_1,\dots i_s)$. Direct computations establishes formula (6.4) for $\mathfrak{a}'(i_1,\dots,i_s)$.

(c), (d) Since $2\eta = 0$, the element $2\alpha'(\mathrm{i}) - w_1\beta_1\big(\alpha'(\mathrm{i})\big)$ is in the kernel of the Bockstein operator $\beta_1\colon \mathrm{MSp}_*^{\Sigma_1} \longrightarrow \mathrm{MSp}_*$ which equals the image of $\pi_*\colon \mathrm{MSp}_* \longrightarrow \mathrm{MSp}_*^{\Sigma_1}$. Let $\pi_*\big(\alpha(\mathrm{i})\big) = 2\alpha'(\mathrm{i}) - w_1\beta_1\big(\alpha'(\mathrm{i})\big)$. Observe that the element $\beta_1\big(\alpha'(\mathrm{i})\big)$ has Adams-Novikov filtration at least four since $\beta_1\mathfrak{a}'(\mathrm{i}) = 0$ in $E_2^{3,*}(\mathrm{MSp}^{\Sigma_1})$ from (b). Therefore, the element $2\alpha'(\mathrm{i}) - w_1\beta_1\big(\alpha'(\mathrm{i})\big)$ projects to $2\mathfrak{a}'(i_1,\dots,i_s) \in E_2^{3,4*+1}(\mathrm{MSp}^{\Sigma_1})$. ∎

7. **Proof of Theorem C.** In this section, we prove Theorem C. Let $\mathrm{i} = (i_1,\dots,i_s)$. In Sections 5 and 6, we constructed the elements $\tau(\mathrm{i}) \in \mathrm{MSp}_*^{\Sigma_3}$, $\gamma(\mathrm{i}) = \beta_3\sigma(\mathrm{i}) \in \mathrm{MSp}_*^{\Sigma_2}$, $\alpha'(\mathrm{i}) = \beta_2\gamma(\mathrm{i}) \in \mathrm{MSp}_*^{\Sigma_1}$ and $\alpha(\mathrm{i}) \in \mathrm{MSp}_*$ such that the element $\pi_*\big(\alpha(\mathrm{i})\big) = 2\alpha'(\mathrm{i}) - w_1\beta_1\big(\alpha'(\mathrm{i})\big)$ projects to $\mathfrak{a}'(\mathrm{i})$ in $E_2^{3,4*+1}$ of the ANSS for MSp^{Σ_1}. Note that the lowest degree element of order at least eight which we have identified in MSp_* is $\alpha(\mathrm{i}_0)$ in degree 32,769, where $\mathrm{i}_0 = (3,4,5,6,7,8,9,10,11,12,13)$.

Let $\mathfrak{t}(\mathrm{i})$, $\mathfrak{g}(\mathrm{i})$, $\mathfrak{a}'(\mathrm{i})$ denote the projection of $\tau(\mathrm{i})$, $\gamma(\mathrm{i})$, $\alpha'(\mathrm{i})$ to $E_2^{*,*}(\mathrm{MSp}^{\Sigma_3})$, $E_2^{*,*}(\mathrm{MSp}^{\Sigma_2})$, $E_2^{*,*}(\mathrm{MSp}^{\Sigma_1})$, respectively. The chromatic technique was developed to make computations in the E_2-term of the ANSS for spheres. See [12]. In this section, we use a chromatic argument to prove the following proposition.

PROPOSITION 7.1. *Let $\mathrm{i} = (i_1,\dots,i_s)$ with $3 \le i_1 < \cdots < i_s$ and $s \ge 6$.*

(i) The element $\mathfrak{g}(\mathrm{i}) \in E_2^{3,*}(\mathrm{MSp}^{\Sigma_2})$ has order at least $2^{[(s+1)/2]-1}$.

(ii) The element $\mathfrak{a}'(\mathrm{i}) \in E_2^{3,*}(\mathrm{MSp}^{\Sigma_1})$ has order at least $2^{[(s+1)/2]-2}$.

PROOF OF THEOREM C USING PROPOSITION 7.1. Recall from Proposition 6.4(d) that the infinite cycle $\mathfrak{a}(\mathrm{i}) \in E_2^{3,4*+1}(\mathrm{MSp})$ is the projection of $\alpha(\mathrm{i}) \in \mathrm{MSp}_*$. By Proposition 6.4(c), $\pi_*\big(\mathfrak{a}(\mathrm{i})\big) = 2\mathfrak{a}(\mathrm{i})'$ where $\pi_*\colon E_1^{3,*}(\mathrm{MSp}) \longrightarrow E_1^{3,*}(\mathrm{MSp}^{\Sigma_1})$. By Proposition 7.1, the element $2\mathfrak{a}'(\mathrm{i}) = \pi_*\big(\mathfrak{a}(\mathrm{i})\big)$ has order at least $2^{[(s+1)/2]-3}$ in $E_2^{3,4*+1}(\mathrm{MSp}^{\Sigma_1})$. Thus $\mathfrak{a}(\mathrm{i})$ has order at least $2^{[(s+1)/2]-3}$ in $E_2^{3,4*+1}(\mathrm{MSp})$. Since $E_2^{1,4*+2}(\mathrm{MSp}) = E_2^{0,4*+2}(\mathrm{MSp}) = 0$, the element $2^t\mathfrak{a}(\mathrm{i})$ cannot be killed by differentials for $t = 1,\dots,[(s+1)/2]-4$. Thus, $2^{[(s+1)/2]-4}\alpha(\mathrm{i})$ projects to a nonzero element of $E_\infty^{3,*}(\mathrm{MSp})$ and must be nonzero. ∎

NOTE 7.1. This argument can not be used to prove directly that $2^{[(s+1)/2]-3}\mathfrak{a}'(\mathrm{i})$ is nonzero in $E_\infty(\mathrm{MSp}^{\Sigma_1})$ because $E_2^{0,4*+2}(\mathrm{MSp}^{\Sigma_1})$ is nonzero which raises the possibility of hitting $2^{[(s+1)/2]-3}\mathfrak{a}'(\mathrm{i})$ by a d_3-differential.

The following lemma shows that we can assume that $i_1 \ge 4$ in proving Proposition 7.1(i).

LEMMA 7.2. *Let $s \ge 6$. If $\mathfrak{g}'(i_1,i_2,\dots,i_s) \in E_2^{3,*}(\mathrm{MSp}^{\Sigma_1})$ has order $2^{[(s+1)/2]-1}$ for all $4 \le i_1 < \cdots < i_s$ then $\mathfrak{g}'(3,i_2,\dots,i_s) \in E_2^{3,*}(\mathrm{MSp}^{\Sigma_1})$ has order $2^{[(s+1)/2]-1}$ for all $3 < i_2 < \cdots < i_s$.*

PROOF. Let $t = [(s+1)/2] - 2$. Suppose there is $\mathfrak{x} \in E_1^{2,*}(\mathrm{MSp}^{\Sigma_1})$ such that $d_1(\mathfrak{x}) = 2^t \mathfrak{g}(3, i_2, \ldots, i_s)$. Choose $n > i_s$. The element $\mathfrak{x}$ depends on the generators $x_{j_1,\ldots,j_k}$, w_j and u_j. In particular, the formula for the first differential is invariant under the transposition $(3, n)$ in all entries of the elements $\mathfrak{x}$ and $\mathfrak{g}(3, i_2, \ldots, i_s)$. Applying this permutation we obtain an element $\mathfrak{x}'$, such that $d_1(\mathfrak{x}') = 2^t \mathfrak{g}(i_2, \ldots, i_s, n)$, a contradiction. ∎

We give the proof of Proposition 7.1 in the case s even and $4 \le i_1 < \cdots < i_s$. The proof for the case s odd is obtained by a slight modification. Thus, $\mathfrak{i}$ will denote $i_1, \ldots, i_{2s}$ for the remainder of this section. We prove Proposition 7.1(i) by showing that $\mathfrak{g}(\mathfrak{i})$ has order at least 2^{s-1}. Let $\pi\colon \mathrm{MSp}^{\Sigma_2} \longrightarrow \mathrm{MSp}^{\Sigma_3}$ denote the canonical map which induces $\pi_*\colon E_2^{*,*}(\mathrm{MSp}^{\Sigma_2}) \longrightarrow E_2^{*,*}(\mathrm{MSp}^{\Sigma_3})$. Let $\tilde{\mathfrak{g}}(\mathfrak{i}) = \pi_*\big(\mathfrak{g}(\mathfrak{i})\big) \in E_2^{3,*}(\mathrm{MSp}^{\Sigma_3})$. By Proposition 6.3(d),

$$\tilde{\mathfrak{g}}(\mathfrak{i}) = \sum_{1 \le t_1 < t_2 < t_3 \le 2s} u_{i_{t_1}} u_{i_{t_2}} u_{i_{t_3}} x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots \hat{x}_{2,i_{t_3}} \cdots x_{2,i_{2s}}.$$

We use chromatic methods to determine the order of $\tilde{\mathfrak{g}}(\mathfrak{i})$. Consider the following exact sequences of complexes:

$$0 \longrightarrow \mathcal{M}\langle 3\rangle \longrightarrow w_2^{-1}\mathcal{M}\langle 3\rangle \longrightarrow \mathcal{M}\langle 3\rangle/(w_2^\infty) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{M}\langle 3\rangle/(w_2^\infty) \longrightarrow w_1^{-1}\mathcal{M}\langle 3\rangle/(w_2^\infty) \longrightarrow \mathcal{M}\langle 3\rangle/(w_2^\infty, w_1^\infty) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{M}\langle 3\rangle/(w_2^\infty, w_1^\infty) \longrightarrow w_3^{-1}\mathcal{M}\langle 3\rangle/(w_2^\infty, w_1^\infty) \longrightarrow \mathcal{M}\langle 3\rangle/(w_3^\infty, w_2^\infty, w_1^\infty) \longrightarrow 0.$$

Recall that by Theorem 5.1 there exist elements $x_{1,k} \in \mathrm{MSp}_*^{\Sigma_1}$, $x_{2,k} \in \mathrm{MSp}_*^{\Sigma_2}$, and $x_{3,k} \in \mathrm{MSp}_*^{\Sigma_3}$ such that $\beta_k(x_{1,k}) = w_1$, $\beta_k(x_{2,k}) = w_2$, $\beta_k(x_{3,k}) = w_3$. The arguments used to prove Lemma 4.2 can be used to prove the following lemma.

LEMMA 7.3. *The complexes $w_2^{-1}\mathcal{M}\langle 3\rangle$, $w_1^{-1}w_3^{-1}\mathcal{M}\langle 3\rangle$, $w_1^{-1}\mathcal{M}\langle 3\rangle/(w_2^\infty)$, $w_3^{-1}\mathcal{M}\langle 3\rangle/(w_2^\infty)$ and $w_3^{-1}\mathcal{M}\langle 3\rangle/(w_2^\infty, w_1^\infty)$ are acyclic, i.e. their n-th homology groups are zero for $n \ge 1$.* ∎

Consider the following composition of boundary homomorphisms:

$$H_0\big(\mathcal{M}\langle 3\rangle/(w_2^\infty, w_3^\infty, w_1^\infty)\big) \xrightarrow{\delta(0)} H_1\big(\mathcal{M}\langle 3\rangle/(w_2^\infty, w_1^\infty)\big) \xrightarrow{\delta(1)} H_2\big(\mathcal{M}\langle 3\rangle/(w_2^\infty)\big) \xrightarrow{\delta(2)} H_3(\mathcal{M}\langle 3\rangle).$$

By Lemma 7.3, $\delta(0)$ is an epimorphism and $\delta(1)$, $\delta(2)$ are isomorphisms.

Define the element

$$\tilde{\mathfrak{g}}_2(\mathfrak{i}) = w_2^{-1} \sum_{1 \le t_1 < t_2 \le 2s} u_{i_{t_1}} u_{i_{t_2}} x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots x_{2,i_{2s}},$$
$$\tilde{\mathfrak{g}}_1(\mathfrak{i}) = w_2^{-1} w_1^{-1} \sum_{1 \le t_1 < t_2 \le 2s} x_{1,i_{t_1}} u_{i_{t_2}} x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots x_{2,i_{2s}},$$
$$\tilde{\mathfrak{g}}_0(\mathfrak{i}) = w_2^{-1} w_1^{-1} w_3^{-1} \sum_{1 \le t_1 < t_2 \le 2s} x_{1,i_{t_1}} x_{3,i_{t_2}} x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots x_{2,i_{2s}}.$$

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in the homology group $H_2\big(\mathcal{M}\langle 3\rangle/(w_2^\infty)\big)$, $H_1\big(\mathcal{M}\langle 3\rangle/(w_2^\infty, w_1^\infty)\big)$, $H_0\big(\mathcal{M}\langle 3\rangle/(w_2^\infty, w_3^\infty, w_1^\infty)\big)$, respectively. A direct calculation gives that $\delta(0)\big(\tilde{\mathfrak{g}}_0(\mathfrak{i})\big) = \tilde{\mathfrak{g}}_1(\mathfrak{i})$, $\delta(1)\big(\tilde{\mathfrak{g}}_1(\mathfrak{i})\big) = \tilde{\mathfrak{g}}_2(\mathfrak{i})$ and $\delta(2)\big(\tilde{\mathfrak{g}}_2(\mathfrak{i})\big) = \tilde{\mathfrak{g}}(\mathfrak{i})$. Proposition 7.1(i) is a consequence of the following lemma.

LEMMA 7.4. *For $s \geq 3$, the element $\tilde{\mathfrak{g}}_1(\mathfrak{i}) \in H_1\big(\mathcal{M}\langle 3\rangle/(w_2^\infty, w_1^\infty)\big)$ has order at least 2^{s-1}.*

PROOF. Assume that $2^{s-2}\tilde{\mathfrak{g}}_1(\mathfrak{i}) = 0$. Then $\delta(0)\big(2^{s-2}\tilde{\mathfrak{g}}_0(\mathfrak{i})\big) = 2^{s-2}\tilde{\mathfrak{g}}_1(\mathfrak{i}) = 0$. Therefore, there is an element $Z_1 \in H_0\big(w_3^{-1}\mathcal{M}\langle 3\rangle/(w_1^\infty, w_2^\infty)\big)$ such that $\pi_1(Z_1) = 2^{s-2}\tilde{\mathfrak{g}}_0(\mathfrak{i})$ where π_1 is the homomorphism in the diagram below. In the following diagram the row and both columns are exact.

$$
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & 0 & & H_0\big(\mathcal{M}\langle 3\rangle/(w_1^\infty, w_2^\infty)\big) & & \\
 & & \uparrow & & \downarrow & & \\
H_0\big(w_3^{-1}\mathcal{M}\langle 3\rangle/w_2^\infty\big) & \longrightarrow & H_0(w_1^{-1}w_3^{-1}\mathcal{M}\langle 3\rangle/w_2^\infty) & \xrightarrow{\pi_2} & H_0(w_3^{-1}\mathcal{M}\langle 3\rangle/(w_1^\infty, w_2^\infty)) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \pi_3} & & {\scriptstyle \pi_1}\downarrow & & \\
 & & H_0(w_2^{-1}w_1^{-1}w_3^{-1}\mathcal{M}\langle 3\rangle) & & H_0\big(\mathcal{M}\langle 3\rangle/(w_1^\infty, w_2^\infty, w_3^\infty)\big) & & \\
 & & \uparrow & & {\scriptstyle \delta(0)}\downarrow & & \\
 & & H_0(w_1^{-1}w_3^{-1}\mathcal{M}\langle 3\rangle) & & H_1\big(\mathcal{M}\langle 3\rangle/(w_1^\infty, w_2^\infty)\big) & & \\
 & & \uparrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

From the diagram above, we see that there exist $Z_2 \in H_0(w_1^{-1}w_3^{-1}\mathcal{M}\langle 3\rangle/w_2^\infty)$ and $Z_3 \in H_0(w_2^{-1}w_1^{-1}w_3^{-1}\mathcal{M}\langle 3\rangle)$ such that $\pi_3(Z_3) = Z_2$ and $\pi_2(Z_2) = Z_1$. Let

$$S = 2^{s-2}\Big(\sum_{1\leq t_1<t_2\leq 2s} x_{1,i_{t_1}} x_{3,i_{t_2}} x_{2,i_1}\cdots \hat{x}_{2,i_{t_1}}\cdots \hat{x}_{2,i_{t_2}}\cdots x_{2,i_{2s}}\Big).$$

By definition of the elements Z_1, Z_2, Z_3 we have that

$$
\begin{aligned}
Z_1 &= w_1^{-1}w_2^{-1}w_3^{-1}S + w_1^{-p}w_2^{-b_1}Y_1, \\
Z_2 &= w_1^{-1}w_2^{-1}w_3^{-1}S + w_1^{-p}w_2^{-b_1}Y_1 + w_2^{-b_2}Y_2, \qquad (38)\\
Z_3 &= w_1^{-1}w_2^{-1}w_3^{-1}S + w_1^{-p}w_2^{-b_1}Y_1 + w_2^{-b_2}Y_2 + Y_3
\end{aligned}
$$

where $Y_1, Y_2, Y_3 \in \mathrm{MSp}_*^\Sigma$ and $p, b_1, b_2 \geq 1$. Let $q = \max\{b_1, b_2\}$. Note that we can define Y_1 and Y_2 so that $q \geq 3$. It follows from (38) that

$$w_1^p w_2^q w_3 Z_3 = w_1^{p-1}w_2^{q-1}S + w_3(w_2^{q-b_1}Y_1) + w_1^p(w_2^{q-b_2}w_3Y_2) + w_2^q(w_1^p w_3 Y_3).$$

Since $w_1^p w_2^q w_3 Z_3 \in H_0(\mathcal{M}\langle 3\rangle)$, the following lemma produces the contradiction which proves our lemma.

LEMMA 7.5. *For $p \geq 1$ and $q, s \geq 3$, there are no elements $B_1, B_2, B_3 \in \mathrm{MSp}_*^{\Sigma}$ such that the element*

$$C = w_1^{p-1} w_2^{q-1} S + w_1^p B_1 + w_2^q B_2 + w_3 B_3$$

belongs to the ring $H_0(\mathcal{M}\langle 3\rangle)$.

PROOF. Recall from Lemma 4.3 that $H_0(w_2^{-1}\mathcal{M}\langle 3\rangle)$ is a polynomial ring and that there is a ring monomorphism: $\pi_*: H_0(\mathcal{M}\langle 3\rangle) \longrightarrow H_0(w_2^{-1}\mathcal{M}\langle 3\rangle)$. Assume that we have chosen Y_1 and Y_2 in (38) to make q so large that the cycle C can be written as a polynomial in the polynomial generators of $H_0(w_2^{-1}\mathcal{M}\langle 3\rangle)$. Also, we can always choose integral polynomial generators for $H_0(w_2^{-1}\mathcal{M}\langle 3\rangle)$, *i.e.* from the image $\pi_*\big(H_0(\mathcal{M}\langle 3\rangle)\big)$, which include

$$\begin{aligned} \xi_{1,j} = w_2 x_{1,j} - w_1 x_{2,j} = w_2 X'_{1,j}, \quad \xi_{2,j} = 2x_{2,j} - w_2 w_j = w_2 X_j, \\ \xi_{3,j} = w_2 x_{3,j} - w_3 x_{2,j} = w_2 X'_{3,j}, \\ \xi_{i,j} = 2x_{2,i} x_{2,j} - w_2(w_i x_{2,j} - w_j x_{2,i}) + w_2^2 x_{i,j} = w_2^2 X_{i,j} \end{aligned} \tag{39}$$

where $1 \leq i < j$, $i, j \notin \{1, 2, 3\}$ and $X'_{1,j}$, $X'_{3,j}$, $X_{i,j}$, X_j are the polynomial generators of $H_0(w_2^{-1}\mathcal{M}\langle 3\rangle)$ defined in (25), (26), (27). Define $X(j_1, \ldots, j_{2t}) = X_{j_1,j_2} \cdots X_{j_{2t-1},j_{2t}} \in H_0(w_2^{-1}\mathcal{M}\langle 3\rangle)$ and $\xi(j_1, \ldots, j_{2t}) = \xi_{j_1,j_2} \cdots \xi_{j_{2t-1},j_{2t}} = w_2^{2t} X(j_1, \ldots, j_{2t}) \in H_0(\mathcal{M}\langle 3\rangle)$. It follows from (39) that

$$\begin{aligned} w_2^2 x_{1,i_{t_1}} x_{3,i_{t_2}} &= \xi_{1,i_{t_1}} \xi_{3,i_{t_2}} + w_1 x_{2,i_{t_1}} \xi_{3,i_{t_2}} + w_3 x_{2,i_{t_2}} \xi_{1,i_{t_1}} + w_1 w_3 x_{2,i_{t_1}} x_{2,i_{t_2}} \\ &= \xi_{1,i_{t_1}} \xi_{3,i_{t_2}} + w_1 a_1 + w_3 a_2 \end{aligned} \tag{40}$$

where $a_1, a_2 \in \mathrm{MSp}_*^{\Sigma}$. Consider the monomial

$$\mathfrak{I} = 2^{s-1} x_{1,i_{t_1}} x_{3,i_{t_2}} x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots x_{2,i_{2s}}.$$

It follows from (39) and (40) that

$$\begin{aligned} w_2^2 \mathfrak{I} &= (w_2^2 x_{1,i_{t_1}} x_{3,i_{t_2}})(2^{s-1} x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots x_{2,i_{2s}}) \\ &= (\xi_{1,i_{t_1}} \xi_{3,i_{t_2}} + w_1 a_1 + w_3 a_2)\big(\xi(i_1, \ldots, \hat{\imath}_{t_1}, \ldots, \hat{\imath}_{t_2}, \ldots, i_{2s}) + w_2 D\big) \\ &= \xi_{1,i_{t_1}} \xi_{3,i_{t_2}} \xi(i_1, \ldots, \hat{\imath}_{t_1}, \ldots, \hat{\imath}_{t_2}, \ldots, i_{2s}) \\ &\quad + \big(w_1 a_1(i_{t_1}, i_{t_2}) + w_3 a_3(i_{t_1}, i_{t_2})\big)\xi(i_1, \ldots, \hat{\imath}_{t_1}, \ldots, \hat{\imath}_{t_2}, \ldots, i_{2s}) \\ &\quad + w_2^3 x_{1,i_{t_1}} x_{3,i_{t_2}} D(i_1, \ldots, \hat{\imath}_{t_1}, \ldots, \hat{\imath}_{t_2}, \ldots, i_{2s}). \end{aligned}$$

Assume that B_1, B_2, B_3 exist which make C a cycle in $\mathcal{M}\langle 3\rangle$. Then

$$\begin{aligned} 2C &= w_1^{p-1} w_2^{q-3}(2w_2^2 S) + 2(w_1^p B_1 + w_2^q B_2 + w_3 B_3) \\ &= w_1^{p-1} w_2^{q-3} \sum_{1 \leq t_1 < t_2 \leq 2s} \xi_{1,i_{t_1}} \xi_{3,i_{t_2}} \xi(i_1, \ldots, \hat{\imath}_{t_1}, \ldots, \hat{\imath}_{t_2}, \ldots, i_{2s}) \\ &\quad + w_1^p w_2^{q-3} A_1 + w_3 w_1^{p-1} A_3 + w_1^{p-1} w_2^q D \end{aligned}$$

$$
\begin{aligned}
&+ w_1^p 2B_1 + w_2^q 2B_2 + w_3 2B_3 \\
= {}& w_1^{p-1} w_2^{q-3} \sum_{1 \le t_1 < t_2 \le 2s} \xi_{1.i_{t_1}} \xi_{3.i_{t_2}} \xi(i_1, \ldots, \hat{\imath}_{t_1}, \ldots, \hat{\imath}_{t_2}, \ldots, i_{2s}) \\
&+ w_1^p (w_2^{q-3} A_1 + 2B_1) + w_3 (w_1^{p-1} A_3 + 2B_3) + 2 w_2^q B_2 \\
&+ w_1^{p-1} w_2^q D.
\end{aligned}
$$

Let

(41) $$\Xi = \sum_{1 \le t_1 < t_2 \le 2s} \xi_{1.i_{t_1}} \xi_{3.i_{t_2}} \xi(i_1, \ldots, \hat{\imath}_{t_1}, \ldots, \hat{\imath}_{t_2}, \ldots, i_{2s}).$$

Then in $H_0(\mathcal{M}\langle 3 \rangle)$:

(42) $$2C - w_1^{p-1} w_2^{q-3} \Xi = w_1^p K_1 + w_2^q K_2 + w_3 K_3 = K.$$

Let $\beta_{l_1,\ldots,l_k} X$ denote the composition $\beta_{l_1}\Big(\beta_{l_2}\big(\cdots(\beta_{l_k} X)\cdots\big)\Big)$ for an element X of MSp_*^Σ. Recall that $\beta_i(\beta_j X) = \beta_j(\beta_i X)$. Thus $\beta_{l_1,\ldots,l_k} X$ does not depend on the order of $l_1, \ldots, l_k$. The proof of the following lemma is straightforward.

LEMMA 7.6. *Let* $X \in \mathrm{MSp}_*^\Sigma$ *with* $\deg X < 2(2^n - 1)$. *Then*

$$2^n X + \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \le l_1 < \cdots < l_k \le n} w_{l_1} \cdots w_{l_k} \beta_{l_1,\ldots,l_k} X$$

belongs to the ring $H_0(\mathcal{M}\langle 3 \rangle) \subset \mathrm{MSp}_*^\Sigma$. ∎

PROOF OF LEMMA 7.5 CONTINUED. Choose n so that $2(2^n - 1) > \deg K$. Then

$$S_i = 2^n K_i + \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \le l_1 < \cdots < l_k \le n} w_{l_1} \cdots w_{l_k} \beta_{l_1,\ldots,l_k} K_i$$

are cycles for $i = 1, 2, 3$. Since K is a cycle,

$$\beta_{l_1,\ldots,l_k} K = w_1^p \beta_{l_1,\ldots,l_k} K_1 + w_2^q \beta_{l_1,\ldots,l_k} K_2 + w_3 \beta_{l_1,\ldots,l_k} K_3 = 0.$$

Therefore,

$$
\begin{aligned}
2^n K = {}& 2^n K + \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \le l_1 < \cdots < l_k \le n} w_{l_1} \cdots w_{l_k} \beta_{l_1,\ldots,l_k} K \\
&- \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \le l_1 < \cdots < l_k \le n} w_{l_1} \cdots w_{l_k} \beta_{l_1,\ldots,l_k} K \\
= {}& 2^n w_1^p K_1 + 2^n w_2^q K_2 + 2^n w_3 K_3 \\
&+ \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \le l_1 < \cdots < l_k \le n} w_{l_1} \cdots w_{l_k} \\
&(w_1^p \beta_{l_1,\ldots,l_k} K_1 + w_2^q \beta_{l_1,\ldots,l_k} K_2 + w_3 \beta_{l_1,\ldots,l_k} K_3) \\
= {}& w_1^p S_1 + w_2^q S_2 + w_3 S_3.
\end{aligned}
$$

Multiplying (42) by 2^n we get the following equality in the ring $H_0(\mathcal{M}\langle 3\rangle)$:

$$(43) \qquad 2^{n+1}C - 2^n w_1^{p-1} w_2^{q-3}\Xi = w_1^p S_1 + w_2^q S_2 + w_3 S_3.$$

Observe that we can choose Y_1 and Y_2 in (38) to make q is so large that C, S_1, S_2, S_3 all belong to $H_0(w_2^{-1}\mathcal{M}\langle 3\rangle)$. Then the equality (43) occurs in the polynomial ring $H_0(w_2^{-1}\mathcal{M}\langle 3\rangle)$. By the definition of Ξ in (41), $w_1^{p-1}w_2^{q-3}\Xi$ is a linear combination of monomials in the canonical polynomial generators with odd coefficients. Moreover, these monomials do not use the generators w_1, w_3 and are not divisible by w_2 in $H_0(\mathcal{M}\langle 3\rangle)$. Thus, (43) is a nontrivial relation in the polynomial ring $H_0(w_2^{-1}\mathcal{M}\langle 3\rangle)$, a contradiction. This completes the proof of Lemmas 7.4, 7.5 and Proposition 7.1(i). ∎

Let $\mathfrak{i} = (i_1, \ldots, i_{2s})$ for $s \geq 6$ and $3 \leq i_1 \cdots < i_{2s}$. We turn our attention to the elements $\alpha'(\mathfrak{i}) = \beta_2\big(\gamma(\mathfrak{i})\big)$ and prove Proposition 7.1(ii). Recall that we already know the projection of $\alpha'(\mathfrak{i})$ into $E_2^{3,*}(\mathrm{MSp}^{\Sigma_1})$ from Proposition 6.4(b). In $E_2^{3,*}(\mathrm{MSp}^{\Sigma_3})$, for $4 \leq k \leq 2s$ define

$$\mathfrak{a}^{(k-4)}(\mathfrak{i}) = w_2^{k-4} \sum_{1\leq t_1<\cdots<t_k\leq 2s} \mathfrak{p}(i_{t_1}, \ldots, i_{t_k}) x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_k}} \cdots x_{2,i_{2s}}$$

where

$$\mathfrak{p}(j_1, \ldots, j_t) = \sum_{1\leq t_1<t_2<t_3\leq t} u_{j_{t_1}} u_{j_{t_2}} u_{j_{t_3}} w_{j_1} \cdots \hat{w}_{j_{t_1}} \cdots \hat{w}_{j_{t_2}} \cdots \hat{w}_{j_{t_3}} \cdots w_{j_t}.$$

Then we can rewrite the projection $\tilde{\mathfrak{a}}'(\mathfrak{i})$ of $\alpha'(\mathfrak{i})$ into $E_2^{3,*}(\mathrm{MSp}^{\Sigma_3})$ as

$$\tilde{\mathfrak{a}}'(\mathfrak{i}) = \sum_{k=4}^{2s}(-1)^k \mathfrak{a}^{(k-4)}(\mathfrak{i}).$$

Direct computation shows that the elements $\mathfrak{a}^{(k-4)}(\mathfrak{i})$ are d_1-cycles. In the following lemma, we obtain a convenient description of $\mathfrak{a}'(\mathfrak{i})$.

LEMMA 7.7. *In the algebra $E_2^{*,*}(\mathrm{MSp}^{\Sigma_3})$ for $4 \leq k \leq 2s-1$ and $s \geq 6$:*

$$2\mathfrak{a}^{(k-4)}(\mathfrak{i}) = \mathfrak{a}^{(k-3)}(\mathfrak{i}).$$

In particular, there is an odd number λ such that $\tilde{\mathfrak{a}}'(\mathfrak{i}) = \lambda\, \mathfrak{a}^{(0)}(\mathfrak{i})$.

PROOF. In $E_1^{*,*}(\mathrm{MSp}^{\Sigma_3})$ for $t \geq 3$, define

$$\mathfrak{h}(j_1, \ldots, j_t) = \sum_{1\leq q_1<q_2\leq t} u_{j_{q_1}} u_{j_{q_2}} w_{j_1} \cdots \hat{w}_{j_{q_1}} \cdots \hat{w}_{j_{q_2}} \cdots w_{j_t} \quad \text{and}$$

$$\mathfrak{c}^{(k-4)}(\mathfrak{i}) = w_2^{k-4} \sum_{1\leq t_1<\cdots<t_k\leq 2s} \mathfrak{h}(i_{t_1}, \ldots, i_{t_k}) x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_k}} \cdots x_{2,i_{2s}}.$$

Then $d_1\mathfrak{c}^{(k-4)}(\mathfrak{i}) = 2\mathfrak{a}^{(k-4)}(\mathfrak{i}) - \mathfrak{a}^{(k-3)}(\mathfrak{i})$. Thus, $\tilde{\mathfrak{a}}'(\mathfrak{i}) = \lambda\, \mathfrak{a}^{(0)}(\mathfrak{i})$ where $\lambda = 1 - 2 + 4 + \cdots + 2^{(2s-4)}$ is odd. ∎

The following lemma indicates the relationship between $\tilde{\mathfrak{g}}(\mathfrak{i})$ and $\tilde{\mathfrak{a}}'(\mathfrak{i})$.

LEMMA 7.8. *In the algebra $E_2^{*,*}(\mathrm{MSp}^{\Sigma_3})$, $2\lambda\tilde{\mathfrak{g}}(\mathfrak{i}) = w_2\tilde{\mathfrak{a}}'(\mathfrak{i})$ for $s \geq 6$.*

PROOF. Define

$$\mathfrak{c}(\mathfrak{i}) = \sum_{1\le t_1<t_2<t_3\le 2s} \mathfrak{h}(i_{t_1}, i_{t_2}, i_{t_3}) x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots \hat{x}_{2,i_{t_3}} \cdots x_{2,i_{2s}}.$$

Then $d_1\big(\lambda\mathfrak{c}(\mathfrak{i})\big) = 2\lambda\tilde{\mathfrak{g}}(\mathfrak{i}) - w_2\lambda\mathfrak{a}^{(0)}(\mathfrak{i}) = 2\lambda\tilde{\mathfrak{g}}(\mathfrak{i}) - w_2\tilde{\mathfrak{a}}'(\mathfrak{i})$. ∎

PROOF OF PROPOSITION 7.1(ii). Suppose that $2^{s-3}\mathfrak{a}'(\mathfrak{i}) = 0$. By the previous lemma, $0 = 2^{s-3}w_2\tilde{\mathfrak{a}}'(\mathfrak{i}) = 2^{s-2}\tilde{\mathfrak{g}}(\mathfrak{i})$. This contradicts Proposition 7.1(i). ∎

This completes the proof of Proposition 7.1 and consequently of Theorem C.

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