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HYPERBOLICITY OF HOMOCLINIC CLASSES OF C^1 VECTOR FIELDS

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Abstract

Let γ be a hyperbolic closed orbit of a C^1 vector field X on a compact C^∞ manifold M and let $H_X(\gamma)$ be the homoclinic class of X containing γ. In this paper, we prove that if a C^1-persistently expansive homoclinic class $H_X(\gamma)$ has the shadowing property, then $H_X(\gamma)$ is hyperbolic.

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1. Introduction

A main research topic in differentiable dynamical systems is to understand the influence of a persistent dynamical property (that is, a property that holds for a system and all C^1 nearby ones) on the underlying manifold would influence the behavior of the tangent map on the tangent bundle . For instance, Moriyasu *et al.* [9] showed that if a C^1 vector field X on a compact C^∞ manifold M is C^1-persistently expansive (that is, X and all C^1 nearby ones are expansive), then X is quasi-Anosov; that is, any nonzero vector grows exponentially in norm by positive or negative directions of the tangent maps DX_t.

In this paper, we study the case where the homoclinic class $H_X(\gamma)$ of X containing a hyperbolic closed orbit γ is C^1-persistently expansive. Let us be more precise. Let M be a compact C^∞ manifold and let $\mathfrak{X}^1(M)$ be the space of C^1 vector fields on M endowed with the C^1 topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM.

We know that every $X \in \mathfrak{X}^1(M)$ generates a C^1 flow $X_t : M \times \mathbb{R} \to M$. Throughout this paper, for $X, Y, \ldots \in \mathfrak{X}^1(M)$, we always denote the generated flows by $X_t, Y_t, \ldots,$

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respectively. For a compact X_t-invariant set $\Lambda \subset M$, we say that X is *expansive* on Λ (or Λ is *expansive* for X) if for any $\epsilon > 0$ there is $e > 0$ such that $d(X_t(x), X_{h(t)}(y)) \leq e$ for $x, y \in \Lambda$ and some $h \in \mathbf{Rep}(\mathbb{R})$, then $y \in X_{[-\epsilon,+\epsilon]}(x) = \{X_t(x) : -\epsilon \leq t \leq +\epsilon\}$, where $\mathbf{Rep}(\mathbb{R})$ denotes the set of all increasing homeomorphisms (reparametrizations) $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$. Here e is called an *expansive constant* of $X_t|_\Lambda$. Expansiveness is a property shared by a large class of dynamical systems exhibiting chaotic behavior. It is well known that there are only a finite number of singularities for an expansive flow and each is an isolated point of M. This reduces the study of expansive flows to those without singularities (for more details, see [3]).

Recall that a compact invariant set $\Lambda \subset M$ is called *hyperbolic* for X if the tangent bundle $T_\Lambda M$ has a DX_t-invariant splitting $E^s \oplus \langle X_\Lambda \rangle \oplus E^u$ and there exist constants $C > 0, 0 < \lambda < 1$ such that

$$\|DX_t|_{E^s(x)}\| \leq Ce^{-\lambda t} \quad \text{and} \quad \|DX_{-t}|_{E^u(x)}\| \leq Ce^{-\lambda t}$$

for all $t \geq 0$, where $\langle X_\Lambda \rangle$ is the subspace generated by the vector field X.

For any two hyperbolic closed orbits γ_1 and γ_2 of X, we say that γ_1 and γ_2 are *homoclinically related*, denoted by $\gamma_1 \sim \gamma_2$, if the stable manifold $W^s(\gamma_1)$ of γ_1 has a transversal intersection with the unstable manifold $W^u(\gamma_2)$ of γ_2 and the unstable manifold $W^u(\gamma_1)$ of γ_1 has a transversal intersection with the stable manifold $W^s(\gamma_2)$ of γ_2; that is,

$$W^s(\gamma_1) \pitchfork W^u(\gamma_2) \neq \emptyset \quad \text{and} \quad W^s(\gamma_2) \pitchfork W^u(\gamma_1) \neq \emptyset.$$

For any hyperbolic closed orbit γ of X, the set

$$H_X(\gamma) = \overline{\{\gamma' \in PO_h(X) : \gamma' \sim \gamma\}}$$

is called the *homoclinic class* of X containing γ, where $PO_h(X)$ denotes the set of hyperbolic closed orbits of X. It is clear that if $\gamma' \sim \gamma$, then $index(\gamma) = index(\gamma')$; that is, $\dim W^s(\gamma) = \dim W^s(\gamma')$. Note that if γ is a hyperbolic closed orbit of X, then there are a C^1 neighborhood $\mathcal{U}(X)$ of X and a neighborhood U of γ such that for any $Y \in \mathcal{U}(X)$, there exists a unique hyperbolic closed orbit γ_Y of Y in U with $index(\gamma) = index(\gamma_Y)$. Such hyperbolic closed orbit γ_Y is called the *continuation* of γ with respect to Y.

We say that the homoclinic class $H_X(\gamma)$ is C^1*-persistently expansive* if there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that for any $Y \in \mathcal{U}(X)$, $H_Y(\gamma_Y)$ is expansive for Y.

A series of works to show the hyperbolicity of (C^1-persistently expansive) homoclinic classes $H_f(p)$ of a diffeomorphism f containing a hyperbolic periodic orbit p has been done by Bonatti, Gan, Pacifico, Sambarino, Vieitez and others in [1, 7, 10–12, 14]. In this direction, the following problem is still open: *are the C^1-persistently expansive homoclinic classes $H_f(p)$ hyperbolic?* Recently, Wen *et al.* [13] showed that if a C^1-persistently expansive homoclinic class $H_f(p)$ has the shadowing property, then it is hyperbolic.

In this paper, we extend the above results, which are obtained for the case of diffeomorphisms, to the case of vector fields. Many results on dynamics for

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diffeomorphisms can be extended to the case of vector fields, but this is not always possible. In particular, the results involving the hyperbolic structure may not be extended to the case of vector fields. For example, it is well known that if a diffeomorphism f has a C^1 neighborhood $\mathcal{U}(f)$ such that every periodic point of $g \in \mathcal{U}(f)$ is hyperbolic, then the nonwandering set $\Omega(f)$ is hyperbolic. However, the result is not true for the case of vector fields (for more details, see [5]).

In attempting to solve the problem mentioned above, we were faced with several difficulties. For instance, the hyperbolic-like structures near singular points and periodic orbits of a vector field are qualitatively different, the time parameterization in the expansiveness of vector fields causes complexity of the calculations, what kinds of dominated splitting (for flow or for linear Poincaré flow) are suitable to get the hyperbolic structure?, etc.

A sequence $\{(x_i, t_i) : x_i \in M, t_i \geq 1, a < i < b\}$ (a can be $-\infty$ and b can be ∞) in $M \times \mathbb{R}$ is called a $(\delta, 1)$-*pseudo orbit* of X if $d(X_{t_i}(x_i), x_{i+1}) < \delta$ for any $a < i < b - 1$. Let Λ be a compact X_t-invariant set. We say that X has *the shadowing property* on Λ (or Λ is *shadowable* for X_t) if, for any $\epsilon > 0$, there is $\delta > 0$ satisfying the following property: given any $(\delta, 1)$-pseudo orbit $\{(x_i, t_i) : x_i \in \Lambda, t_i \geq 1, a < i < b\}$, there exist a point $y \in M$ and $h \in \mathbf{Rep}(\mathbb{R})$ such that

$$d(X_{t-T_i}(x_i), X_{h(t)}(y)) < \varepsilon \quad \text{for all } T_i \leq t < T_{i+1}.$$

The main purpose of this paper is to characterize the hyperbolicity of the homoclinic class $H_X(\gamma)$ containing a hyperbolic closed orbit γ by making use of the expansiveness under the C^1 open condition. More precisely, we have the following theorem.

MAIN THEOREM. *Every C^1-persistently expansive homoclinic class $H_X(\gamma)$ is hyperbolic if it is shadowable.*

2. Proof of main theorem

We assume that the exponential map $\exp_p : T_pM(1) \to M$ is well defined for all $p \in M$, where $T_pM(r)$ denotes the r-ball $\{v \in T_pM : \|v\| \leq r\}$ in T_pM. For every regular point $x \in M$, let

$$N_x = (\langle X(x) \rangle)^{\perp} \subset T_xM$$

and $N_x(r)$ be the r-ball in N_x. Let $\widetilde{N}_{x,r} = \exp_x(N_x(r))$. Given any regular point $x \in M$ and $t \in \mathbb{R}$, there are $r > 0$ and a C^1 map $\tau : \widetilde{N}_{x,r} \to \mathbb{R}$ with $\tau(x) = t$ such that $X_{\tau(y)}(y) \in \widetilde{N}_{X_t(x),1}$ for any $y \in \widetilde{N}_{x,r}$. We say that $\tau(y)$ is *the first time* of y. Then we define the *Poincaré map* $f_{x,t}$ by

$$f_{x,t} : \widetilde{N}_{x,r} \to \widetilde{N}_{X_t(x),1}; y \mapsto f_{x,t}(y) = X_{\tau(y)}(y).$$

Let $M_X = \{x \in M : X(x) \neq 0\}$. It is easy to check that for any fixed t, there exists a continuous map $r_0 : M_X \to (0, 1)$ such that for any $x \in M_X$, the Poincaré map $f_{x,t} : \widetilde{N}_{x,r_0(x)} \to \widetilde{N}_{X_t(x),1}$ is well defined and the respective time function τ satisfies $\frac{2}{3}t < \tau(y) < \frac{4}{3}t$ for $y \in \widetilde{N}_{x,r_0(x)}$.

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During the study of the structural stability conjecture, Liao [8] raised the notion of *linear Poincaré flow* (LPF) for a C^1 vector field. Let $\mathcal{N} = \bigcup_{x \in M_X} N_x$ be the normal bundle based on M_X. Then we can define a flow, called *linear Poincaré flow*, $\Psi_t : \mathcal{N} \to \mathcal{N}$ by

$$\Psi_t|_{N_x} = \pi_{N_x} \circ D_x X_t|_{N_x},$$

where $\pi_{N_x} : T_x M \to N_x$ is the projection along $X(x)$ and $D_x X_t : T_x M \to T_{X_t(x)} M$ is the derivative map of X_t.

Definition 2.1. Let Λ be an X_t-invariant subset of M which contains no singularity. We call an Ψ_t-invariant splitting $\mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u$ a l-dominated splitting for the linear Poincaré flow Ψ_t if

$$\|\Psi_t|_{\Delta^s(x)}\| \cdot \|\Psi_{-t}|_{\Delta^u(X_t(x))}\| \leq \tfrac{1}{2}$$

for any $x \in \Lambda$ and any $t \geq l$. We call an Ψ_t-splitting $\mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u$ *a hyperbolic splitting* if there are $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|\Psi_t|_{\Delta^s(x)}\| \leq C\lambda^t \quad \text{and} \quad \|\Psi_{-t}|_{\Delta^u(x)}\| \leq C\lambda^t$$

for any $x \in \Lambda$ and $t > 0$.

Note that the dominated splitting for the linear Poincaré flow Ψ_t and the dominated splitting for the corresponding flow need not be equivalent (for more details, see [2]). Unlike the case of dominated splitting, the hyperbolic splitting for the linear Poincaré flow and the hyperbolic splitting for the corresponding flow are equivalent, as we can see in the following lemma.

Lemma 2.2 [4]. *Let $\Lambda \subset M$ be a compact X_t-invariant set which contains no singularity. Then Λ is a hyperbolic set of X_t if and only if the linear Poincaré flow Ψ_t restricted on Λ has a hyperbolic splitting.*

The following two lemmas, which will be useful for the proof of our main theorem, were proved in [7].

Lemma 2.3. *Let $X \in \mathfrak{X}^1(M)$ and let γ be a hyperbolic closed orbit of X. If $H_X(\gamma)$ is C^1-persistently expansive, then $H_X(\gamma)$ admits a Ψ_t-dominated splitting $\mathcal{N}_{H_X(\gamma)} = \Delta^s \oplus \Delta^u$ with* $\dim(\Delta^s) = index(\gamma)$.

Lemma 2.4. *Let $X \in \mathfrak{X}^1(M)$ and let γ be a hyperbolic closed orbit of X. If $H_X(\gamma)$ is C^1-persistently expansive, then there exist constants $T \geq 1$, $\eta > 0$ and $\tilde{T} > 0$ such that for any $\gamma' \sim \gamma$, if the period τ of γ' is greater than $\tilde{T}$, then the following properties hold:*

(i) *for any $x \in \gamma'$ and $t \geq T$,*

$$\frac{1}{t}(\log \|\Psi_t|_{\Delta^s(x)}\| - \log m(\Psi_t|_{\Delta^u(x)})) < -2\eta;$$

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(ii) *let $x \in \gamma'$ and let $0 = T_0 < T_1 < \cdots < T_\iota = \tau$ be a partition with $T \le T_i - T_{i-1} < 2T$ for any $i = 1, 2, \ldots, \iota$. Then*

$$\frac{1}{\tau}\sum_{i=1}^{\iota} \log \|\Psi_{T_i - T_{i-1}}|_{\Delta^s(X_{T_{i-1}}(x))}\| < -\eta,$$

$$\frac{1}{\tau}\sum_{i=1}^{\iota} \log m(\Psi_{T_i - T_{i-1}}|_{\Delta^u(X_{T_{i-1}}(x))}) > \eta.$$

To prove our main theorem, it suffices to prove the following proposition.

Proposition 2.5. *Let $X \in \mathfrak{X}^1(M)$ and let γ be a hyperbolic closed orbit of X. Let $T \ge 1, \eta > 0$ and $\tilde{T} > 0$ be given. For any $\gamma' \sim \gamma$, if the period τ of γ' is greater than $\tilde{T}$, we assume that $H_X(\gamma)$ satisfies the following properties (P1)–(P3).*

(P1) *For any $x \in \gamma'$ and $t \ge T$,*

$$\frac{1}{t}(\log \|\Psi_t|_{\Delta^s(x)}\| - \log m(\Psi_t|_{\Delta^u(x)})) < -2\eta.$$

(P2) *Let $x \in \gamma'$ and $0 = T_0 < T_1 < \cdots < T_\iota = \tau$ be a partition with $T \le T_i - T_{i-1} < 2T$ for any $i = 1, \ldots, \iota$. Then*

$$\frac{1}{\tau}\sum_{i=1}^{i=\iota} \log \|\Psi_{T_i - T_{i-1}}|_{\Delta^s(X_{T_{i-1}}x)}\| < -\eta,$$

$$\frac{1}{\tau}\sum_{i=1}^{i=\iota} \log m(\Psi_{T_i - T_{i-1}}|_{\Delta^u(X_{T_{i-1}}x)}) > \eta.$$

(P3) *X has the shadowing property on $H_X(\gamma)$.*
Then $H_X(\gamma)$ is hyperbolic.

First of all, we introduce two technical lemmas which are necessary to control the time reparametrization for flows.

Lemma 2.6. *Let $X \in \mathfrak{X}^1(M)$ and let Λ be a closed invariant set of X_t containing no singularity. Then there exist a neighborhood U of Λ and a constant $T_0 > 0$ which satisfy the following:*

(1) *for $0 < \epsilon < T_0$, there is $\delta > 0$ such that for any $x \in U$ and $0 \le s, t \le T_0$, $d(X_s(x), X_t(x)) < \delta$ implies $|s - t| < \epsilon$;*
(2) *for $T \in (0, T_0)$, there is $\epsilon > 0$ such that $X_{[0,t]}(x) \subset B(x, \epsilon)$ implies $t \in [0, T]$ for any $x \in U$.*

Proof. The proof is straightforward. For more details, see [6]. □

Lemma 2.7. *Let $X \in \mathfrak{X}^1(M)$ and let Λ be a closed invariant set of X_t containing no singularity. Let U and T_0 be as given in Lemma 2.6, and take a neighborhood V of Λ such that $X_t(x) \in U$ for any $x \in \bar{V}$ and $0 \le t \le T_0$. Then, for any $\epsilon \in (0, 1)$ and*

$T_1 \in (0, T_0]$, there is $\epsilon' > 0$ such that for any $x, y \in V$, if an increasing continuous map $g : [0, T_1] \to \mathbb{R}$ satisfies $g(0) = 0$ and $d(X_t(x), X_{g(t)}(y)) \le \epsilon'$ for all $t \in [0, T_1]$, then $|g(T_1) - T_1| \le \epsilon T_1$.

Proof. The proof is straightforward. For more details, see [6]. □

If a dominated splitting $\mathcal{N}_\Lambda = \triangle^s \oplus \triangle^u$ is not a hyperbolic splitting for Ψ_t, then either $\triangle^s$ is not Ψ_t-contracting or $\triangle^u$ is not Ψ_t-expanding. Now we assume that $\triangle^s$ is not contracting. Then we have the following proposition.

Proposition 2.8. *Suppose that a closed invariant set $\Lambda \subset M$ admits a dominated splitting $\mathcal{N}_\Lambda = \triangle^s \oplus \triangle^u$ for the linear Poincaré flow Ψ_t. If the subbundle $\triangle^s$ is not contracting for Ψ_t, then there exists a point $b \in \Lambda$, called an obstruction point, such that $\|\Psi_t|_{\triangle^s(b)}\| \ge 1$ for all $t \ge 0$.*

Proof. Assume that for any $x \in \Lambda$, there is $t_x > 0$ with $\|\Psi_{t_x}|_{\triangle^s(x)}\| = \lambda_x < 1$. For any $x \in \Lambda$, we can find δ_x such that $\|\Psi_{t_x}|_{\triangle^s(y)}\| < \sqrt{\lambda_x}$ for $y \in B(x, \delta_x) \cap \Lambda$. Since Λ is compact, we can find a finite set $\{x_1, \dots, x_k\}$ such that

$$\Lambda \subset \bigcup_{i=1}^{k} B(x_i, \delta_{x_i}).$$

Let $K = \max\{t_{x_i} : i = 1, \dots, k\}$, $\lambda = \max\{\lambda_{x_i} : i = 1, \dots, k\}^{1/2K}$ and $C = \sup\{\|\Psi_t|_{\triangle^s(x)}\| : 0 \le t \le K, x \in \Lambda\} \cdot \lambda^{-K}$. Given any $x \in \Lambda$ and $t > 0$, we can find $0 \le t_1 \le K$ such that

$$\|\Psi_{t_1}|_{\triangle^s(x)}\| < \lambda^K \le \lambda^{t_1}.$$

Then we can find t_2 such that $0 < t_2 - t_1 \le K$ and

$$\|\Psi_{t_2 - t_1}|_{\triangle^s(X_{t_1}(x))}\| < \lambda^K \le \lambda^{t_2 - t_1}.$$

Hence, $\|\Psi_{t_2}|_{\triangle^s(x)}\| < \lambda^{t_2}$. Similarly, we can take $0 < t_1 < t_2 < \cdots < t_l < t$ with $t - t_l < K$ such that

$$\|\Psi_{t_i - t_{i-1}}|_{\triangle^s(X_{t_{i-1}})}\| < \lambda^k < \lambda^{t_i - t_{i-1}}.$$

Finally, we can verify that $\|\Psi_t|_{\triangle^s(x)}\| < C\lambda^t$. This contradicts the assumption that $\triangle^s$ is not Ψ_t-contracting. □

Now we will show in the following theorem that we can choose an 'adaptable point' of γ if $H_X(\gamma)$ is C^1-persistently expansive.

Proposition 2.9. *Let $T > 0, \eta > 0$ and γ' be as in Lemma 2.4. Then, for any $x \in \gamma'$ and a partition $0 = T_0 < T_1 < \cdots < T_\iota = \tau$ with $T \le T_i - T_{i-1} < 2T$ for any $i = 1, 2, \dots, \iota$, if the partition $\{T_i\}$ satisfies*

$$\frac{1}{\tau} \sum_{i=1}^{i=\iota} \log \|\Psi_{T_i - T_{i-1}}|_{\triangle^s(X_{T_{i-1}} x)}\| < -\eta,$$

$$\frac{1}{\tau} \sum_{i=1}^{i=\iota} \log m(\Psi_{T_i - T_{i-1}}|_{\triangle^u(X_{T_{i-1}} x)}) > \eta,$$

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denote by $\{T_i\}_{i\in\mathbb{Z}}$ the extension of $\{T_i\}_{0\le i\le\iota}$ satisfying $T_{i+k\iota} = T_i + k\tau$ for any integer i, k; then there exists an integer $0 \le i_0 < \iota$ such that

$$\begin{cases} \dfrac{1}{T_{i_0+k} - T_{i_0}} \displaystyle\sum_{j=1}^{j=k} \log \|\Psi_{T_{i_0+j}-T_{i_0+j-1}}|_{\triangle^s(X_{T_{i_0+j-1}}x)}\| \le -\eta, \\ \dfrac{1}{T_{i_0} - T_{i_0-k}} \displaystyle\sum_{j=1}^{j=k} \log m(\Psi_{T_{i_0-j+1}-T_{i_0-j}}|_{\triangle^u(X_{T_{i_0-j}}x)}) \ge \eta \end{cases} \tag{$\circledast$}$$

for any $k \ge 1$. Here the point $X_{T_{i_0}}(x)$ in γ which satisfies ($\circledast$) is said to be an 'adaptable point' of γ.

To prove Proposition 2.9, we need the following lemma.

Lemma 2.10. *Let $T > 0, \eta > 0$ and γ' be as in Lemma 2.4. For any $x \in \gamma'$ and a partition $0 = T_0 < T_1 < \cdots < T_\iota = \tau$ with $T \le T_i - T_{i-1} < 2T$ for any $i = 1, 2, \ldots, \iota$, if the partition $\{T_i\}$ satisfies the following inequality:*

$$\frac{1}{\tau} \sum_{i=1}^{i=\iota} \log \|\Psi_{T_i-T_{i-1}}|_{\triangle^s(X_{T_{i-1}}x)}\| < -\eta,$$

then there exists an integer $0 \le i_0 < \iota$ such that

$$\frac{1}{T_{i_0+k} - T_{i_0}} \sum_{j=1}^{j=k} \log \|\Psi_{T_{i_0+j}-T_{i_0+j-1}}|_{\triangle^s(X_{T_{i_0+j-1}}x)}\| \le -\eta$$

for any $k \ge 1$.

Proof. For each $k \in \mathbb{Z}$, we define $S(T_k)$ by

$$S(T_k) = \begin{cases} \displaystyle\sum_{j=0}^{k-1} \log \|\Psi_{T_{j+1}-T_j}|_{\triangle^s(X_{T_j}x)}\| & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ -\displaystyle\sum_{j=-k}^{-1} \log \|\Psi_{T_{j+1}-T_j}|_{\triangle^s(X_{T_j}x)}\| & \text{if } k < 0. \end{cases}$$

Then

$$S(T_\iota) < -\eta T_\iota \quad \text{and} \quad S(T_{l+i}) = S(T_l) + S(T_i)$$

for any $i \in \mathbb{Z}$. Hence,

$$S(T_{n\iota+i}) - S(T_i) < -\eta(T_{n\iota+i} - T_i).$$

We prove that the set

$$A = \{j_0 \in \mathbb{Z} : S(T_{j_0+k}) - S(T_{j_0}) < -\eta(T_{j_0+k} - T_{j_0}) \text{ for } k \ge 0\}$$

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is not empty. By contradiction, we assume that for $j \geq 0$, there exists $k_j > 0$ such that

$$S(T_{j+k_j}) - S(T_j) \geq -\eta(T_{j+k_j} - T_j).$$

Then we can find k_0 for 0, and k_{k_0} for $k_0, \ldots$. Let $0 < k_0 < k_1 < \cdots$ be a sequence such that

$$S(T_{k_i}) - S(T_{k_{i-1}}) \geq -\eta(T_{k_i} - T_{k_{i-1}})$$

for any $i \geq 0$. It is obvious that there exists $0 < j < i$ such that $k_i - k_j$ is a multiple of ι. The choice of the sequence means that

$$S(T_{k_i}) - S(T_{k_j}) \geq -\eta(T_{k_i} - T_{k_j}).$$

This contradicts the assumption. □

The same result respecting Δ^u can be obtained as follows. We can define $\tilde{S}(T_k)$ as

$$\tilde{S}(T_k) = -\sum_{j=0}^{k-1} \log m(\Psi_{T_i - T_{i-1}}|_{\Delta^u(X_{T_{i-1}}(x))}),$$

$$\tilde{S}(T_{-k}) = -\sum_{j=-k+1}^{0} \log m(\Psi_{T_i - T_{i-1}}|_{\Delta^u(X_{T_{i-1}}(x))})$$

for any $k > 0$. Then we can easily check that

$$\tilde{S}(T_\iota) < -\eta T_\iota \quad \text{and} \quad \tilde{S}(T_{l+i}) = \tilde{S}(T_l) + \tilde{S}(T_i)$$

for any integer i. Similarly, we can also prove that the set

$$B = \{j_0 \in \mathbb{Z} : \tilde{S}(T_{j_0}) - \tilde{S}(T_{j_0 - k}) < -\eta(T_{j_0} - T_{j_0 - k}) \text{ for } k \geq 0\}$$

is not empty.

Proof of Proposition 2.9. Let A and B be the sets which were obtained in the proof of Lemma 2.10. It is obvious that if $a \in A$ then $\pm\iota + a \in A$, and if $b \in B$ then $\pm\iota + b \in B$. We will prove that $A \cap B \cap [0, \iota)$ is not empty. If $A \cap B \cap [0, \iota) = \emptyset$, then there are $a \in A$ and $b \in B$ such that $b < a$ and $(b, a) \cap (A \cup B) = \emptyset$. Since $a - 1 \notin A$,

$$S(T_a) - S(T_{a-1}) \geq -\eta(T_a - T_{a-1}).$$

From

$$\frac{1}{T_a - T_{a-1}} (\log \|\Psi_t|_{\Delta^s(X_{T_{a-1}}(x))}\| - \log m(\Psi_t|_{\Delta^u(X_{T_{a-1}}(x))})) < -2\eta,$$

$$\tilde{S}(T_a) - \tilde{S}(T_{a-1}) < -\eta(T_a - T_{a-1}).$$

Similarly, if $a - 2 \notin A$,

$$S(T_a) - S(T_{a-2}) \geq -\eta(T_a - T_{a-2}).$$

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Suppose not. Then

$$S(T_a) - S(T_{a-2}) < -\eta(T_a - T_{a-2})$$

and

$$S(T_{a-1}) - S(T_{a-2}) < -\eta(T_{a-1} - T_{a-2}).$$

These two inequalities and $a \in A$ imply that $a - 2 \in A$. From

$$\frac{1}{T_a - T_{a-1}}(\log \| \Psi_{T_a - T_{a-1}}|_{\Delta^s(X_{T_{a-1}}(x))} \| - \log m(\Psi_{T_a - T_{a-1}}|_{\Delta^u(X_{T_{a-1}}(x))})) < -2\eta$$

and

$$\frac{1}{T_{a-1} - T_{a-2}}(\log \| \Psi_{T_{a-1} - T_{a-2}}|_{\Delta^s(X_{T_{a-2}}(x))} \| - \log m(\Psi_{T_{a-1} - T_{a-2}}|_{\Delta^u(X_{T_{a-2}}(x))})) < -2\eta,$$

$$\tilde{S}(T_a) - \tilde{S}(T_{a-2}) < -\eta(T_a - T_{a-2}).$$

Inductively, we can prove that for any $i \in [b, a)$,

$$\tilde{S}(T_a) - \tilde{S}(T_i) < -\eta(T_a - T_i).$$

These two inequalities and the fact that $b \in B$ imply that $a \in B$. This gives a contradiction with the assumption that $A \cap B = \emptyset$. Consequently, we can choose an integer $i_0 \in A \cap B \cap [0, \iota)$. Hence, we can see that the integer i_0 satisfies the conclusion of the proposition. □

To complete the proof of our main theorem, we introduce the notion of a quasi-hyperbolic orbit arc and the shadowing lemma as in [8].

DEFINITION 2.11. Let $\Lambda \subset M_X$ be a closed X_t-invariant set that has a continuous Ψ_t-invariant splitting $\mathcal{N}_\Lambda = \triangle^s \oplus \triangle^u$ with $\dim \triangle^s = p$, $1 \leq p \leq \dim M - 1$. For two real numbers $T > 0$ and $\eta > 0$, an orbit arc $(x, t) = X_{[0,t]}(x)$ is called an (η, T, p) *quasi-hyperbolic orbit arc* of X with respect to the splitting $\Delta^s \oplus \Delta^u$ if $[0, t]$ has a partition

$$0 = T_0 < T_1 < \cdots < T_l = t$$

such that $T \leq T_i - T_{i-1} < 2T, i = 1, \ldots, l$, with the following conditions:

(1) $(1/T_k) \sum_{j=1}^{k} \log \|\Psi_{T_j - T_{j-1}}|_{\Delta^s(X_{T_{j-1}}(x))}\| \leq -\eta$;

(2) $(1/(T_l - T_{k-1})) \sum_{j=k}^{l} \log m(\Psi_{T_j - T_{j-1}}|_{\Delta^u(X_{T_{j-1}}(x))}) \geq \eta$;

(3) $\log \|\Psi_{T_k - T_{k-1}} |_{\Delta^s(X_{T_{k-1}}(x))} \| - \log m(\Psi_{T_k - Tk-1} |_{\Delta^u(X_{T_{k-1}}(x))}) \leq -2\eta$ for $k = 1, \ldots, l$.

The first and second inequalities in the above definition involve a kind of contraction and expansion along the partition $(T_i)_{i=0}^l$ of the segment $[0, t]$. The third one is nothing but the dominated structure on the arc $X_{[0,t]}(x)$. The most important property, which was proved by Liao [8], of a quasi-hyperbolic orbit arc is that it can be shadowed by a periodic point if the two end points of the arcs are sufficiently close.

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PROPOSITION 2.12 (Generalized shadowing lemma [8]). *Let $X \in \mathfrak{X}^1(M)$ and let Λ be a closed invariant set containing no singularity. Assume that there exists a continuous Ψ_t-invariant splitting $\mathcal{N}_\Lambda = \triangle^s \oplus \triangle^u$ with $\dim \triangle^s = p$, $1 \le p \le \dim M - 1$. Then, for any $\eta > 0, T > 0$ and $\epsilon > 0$, there exists $\zeta > 0$ such that if (x, τ) is a (η, T, p)-quasi-hyperbolic orbit arc of X with respect to the splitting $\triangle^s \oplus \triangle^u$ and $d(X_\tau(x), x) < \zeta$, then there exist $y \in M$ and an orientation-preserving homeomorphism $g : [0, \tau] \to \mathbb{R}$ with $g(0) = 0$ such that $d(X_{g(t)}(y), X_t(x)) < \epsilon$ for any $t \in [0, \tau]$ and $X_{g(\tau)}(y) = y$.*

The following lemma is a flow version of Corollary 2.2 in [1].

LEMMA 2.13. *For any $0 < \eta < 1$ and any admissible neighborhood U of $H_X(\gamma)$, there is $\delta_\eta > 0$ such that for every adaptable point $x, y \in \bigcap_{t\in\mathbb{R}} X_t(U)$,*

$$d(x, y) < \delta_\eta \Rightarrow W^u(x) \pitchfork W^s(y) \neq \emptyset.$$

In particular, if x and y are periodic points, then they are hyperbolic, have the same index and are homoclinically related.

We will construct a pseudo orbit in $H_X(\gamma)$ which is a quasi-hyperbolic orbit arc of X.

PROPOSITION 2.14. *Assume that $H_X(\gamma)$ satisfies the hypotheses (P1)–(P3) in Proposition 2.5, and $\triangle^s$ is not contracting. Then, for any given constants $\delta > 0$ and $0 < \eta_1 < \eta_2 < \eta$, we can construct a $(\delta, 1)$-pseudo orbit $\{(x_i, t_i)\}_{i=0}^{n-1}$ in $H_X(\gamma)$ such that:*

(1) *x_0 is an adaptable point of γ';*
(2) *$X_{t_{n-1}}(x_{n-1}) = x_0$;*
(3) *$\frac{5}{4}T < t_i < \frac{7}{4}T$ for any $0 \le i \le n-1$;*
(4) *by denoting $T_k = t_0 + \cdots + t_{k-1}$ for all $1 \le k \le n$,*

$$\frac{1}{T_k}\sum_{i=0}^{k-1} \log \|\Psi_{t_i}|_{\triangle^s(x_i)}\| < -\eta_2,$$

$$\frac{1}{T_n - T_{n-k}}\sum_{i=1}^{k} \log m(\Psi_{t_{n-i}}|_{\triangle^u(x_{n-i})}) > \eta_2;$$

(5) $1/T_n \sum_{i=0}^{n-1} \log \|\Psi_{t_i}|_{\triangle^s(x_i)}\| > -\eta_1$.

PROOF. Since Δ^s is not contracting, there exists an obstruction point $b \in H_X(\gamma)$ such that $\|\Psi_t|_{\Delta^s(b)}\| \ge 1$ for all $t \ge 0$. Fix constants $\delta > 0$ and $0 < \eta_1 < \eta_2 < \eta$. We take a constant $0 < \epsilon < \delta$ small enough such that $d(x, y) < \epsilon$ implies that $d(X_t(x), X_t(y)) < \delta$ for any $0 \le t \le 2T$. Since $H_X(\gamma)$ is the homoclinic class containing γ, there exists $\gamma' \sim \gamma$ with period τ arbitrarily large such that $H_X(\gamma) \subset B(\gamma', \epsilon)$. We can assume that τ is big enough such that $\frac{5}{4}T < \tau/\iota < \frac{7}{4}T$, where ι is the integer part of $\tau/(3T/2)$. Now we take $x \in \gamma$ such that $d(x, y) < \epsilon < \delta$ and divide $[0, \tau]$ into ι parts:

$$0 < \frac{\tau}{\iota} < 2\frac{\tau}{\iota} < \cdots < (\iota - 1)\frac{\tau}{\iota} < \tau.$$

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From Proposition 2.9, there is $0 \le i_0 < \tau$ such that

$$\frac{1}{k\tau/\iota}\sum_{j=0}^{k-1}\log\|\Psi_{\tau/\iota}|_{\Delta^s(X_{(j+i_0)\tau/\iota}(x))}\| \le -\eta$$

and

$$\frac{1}{k\tau/\iota}\sum_{j=1}^{k}\log m(\Psi_{\tau/\iota}|_{\Delta^s(X_{(j+i_0)\tau/\iota}(x))}) \ge \eta$$

for any $k \ge 1$. Take an integer $s > 0$. Now we take $\bar{x}_i = X_{(i_0+i)\tau/\iota}(x)$ and $\bar{t}_i = \tau/\iota$ for $0 \le i < s\iota - i_0$. We know that $\bar{x}_0$ is an adaptable point of γ' and $X_{\tau/\iota}(\bar{x}_{s\iota-i_0-1}) = x$. Now, for $j \ge 0$, let

$$\bar{x}_{s\iota-i_0+j} = X_{(3T/2)j}(b) \quad \text{and} \quad \bar{t}_{s\iota-i_0+j} = \tfrac{3}{2}T.$$

By the property of the obstruction point b, we can choose $L > 0$ such that

$$\frac{1}{\bar{t}_0 + \cdots + \bar{t}_{s\iota-i_0+L-1}}\sum_{i=0}^{s\iota-i_0+L-1}\log\|\Psi_{\bar{t}_i}|_{\Delta^s(\bar{x}_i)}\| \ge -\frac{\eta_1+\eta_2}{2}$$

and

$$\frac{1}{\bar{t}_0 + \cdots + \bar{t}_{s\iota-i_0+l-1}}\sum_{i=0}^{s\iota-i_0+l-1}\log\|\Psi_{\bar{t}_i}|_{\Delta^s(\bar{x}_i)}\| < -\frac{\eta_1+\eta_2}{2}$$

for all $0 \le l < L$. It is easy to check that L is increasing as s is increasing. Since $H_X(\gamma) \subset B(\gamma', \epsilon)$, we can find $0 \le j_0 < \iota$ and a point $t' \in [j_0\tau/\iota, (j_0+1)\tau/\iota)$ such that

$$d(X_{(3/2)T}(\bar{x}_{s\iota-i_0+L-1}), X_{t'}(x)) < \epsilon.$$

Since $(j_0+1)\tau/\iota - t' < 2T$,

$$d(X_{(3/2)T+(j_0+1)\tau/\iota-t'}(\bar{x}_{s\iota-i_0+L-1}), X_{(j_0+1)\tau/\iota}(x)) < \delta.$$

To construct a $(\delta, 1)$-pseudo orbit, we can still let $\bar{x}_{s\iota-i_0+L-1} = X_{(L-1)\frac{3}{2}T}(b)$, but change $\bar{t}_{s\iota-i_0+L-1}$ to be $\frac{3}{2}T + (j_0+1)\tau/\iota - t'$. Then let

$$\bar{x}_{s\iota-i_0+L+i} = X_{(j_0+1+i)\frac{\tau}{\iota}}(x) \quad \text{and} \quad \bar{t}_{s\iota-i_0+L} = \frac{\tau}{\iota}$$

for any $0 \le i < 2\iota - j_0 + i_0$. Let

$$n = s\iota - i_0 + L + 2\iota - j_0 + i_0 = (s+2)\iota + L - j_0.$$

We can see that $(\bar{x}_i, \bar{t}_i)_{i=0}^{n-1}$ is a $(\delta, 1)$-pseudo orbit, and $X_{\bar{t}_{n-1}}(\bar{x}_{n-1}) = x_0$. Unfortunately, the constant $\bar{t}_{s\iota-i_0+L-1}$ may not be contained in $(\frac{5}{4}T, \frac{7}{4}T)$. Now we modify the $(\delta, 1)$-pseudo orbit $(\bar{x}_i, \bar{t}_i)_{i=0}^{n-1}$. We let $x_i = \bar{x}_i$ and $t_i = \bar{t}_i$ for $0 \le i < s\iota - i_0 + L - 8$ and $s\iota - i_0 + L \le i < n$. Then we put

$$x_{s\iota-i_0+L-8} = \bar{x}_{s\iota-i_0+L-8} \quad \text{and} \quad t_{s\iota-i_0+L-8} = \frac{3}{2}T + \frac{(j_0+1)\tau/\iota - t'}{8},$$

$$x_{s\iota-i_0+L-8+i} = X_{i(\frac{3}{2}T+((j_0+1)\frac{\tau}{\iota}-t')/8)} \quad \text{and} \quad t_{s\iota-i_0+L-8+i} = \frac{3}{2}T + \frac{(j_0+1)\tau/\iota - t'}{8}$$

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for $1 \le i \le 7$. Then we can see that $t_i \in (\frac{5}{4}T, \frac{7}{4}T)$ for every $0 \le i < n$. We will check that if τ and s are large enough, then $(x_i, t_i)_{i=0}^{n-1}$ is a desired $(\delta, 1)$-pseudo orbit. Let $K = \sup\{\|\Psi_t|_{N_x}\| : x \in H_X(\gamma), -2T \le T \le 2T\}$. To simplify the notation, we change $\bar{t}_{s\iota-i_0+L-1}$ back to be $\frac{3}{2}T$. Then we know that

$$\left|\log \frac{\|\Psi_{\bar{t}_{s\iota-i_0+L-8+i}}|_{\Delta^s(\bar{x}_{s\iota-i_0+L-8+i})}\|}{\|\Psi_{t_{s\iota-i_0+L-8+i}}|_{\Delta^s(x_{s\iota-i_0+L-8+i})}\|}\right| \le 2\log K$$

and

$$\left|\log \frac{m(\Psi_{\bar{t}_{s\iota-i_0+L-8+i}}|_{\Delta^u(\bar{x}_{s\iota-i_0+L-8+i})})}{m(\Psi_{t_{s\iota-i_0+L-8+i}}|_{\Delta^u(x_{s\iota-i_0+L-8+i})})}\right| \le 2\log K$$

for all $1 \le i \le 7$. Let $n_1 = s\iota - i_0 + L - 8$. By the choice of L,

$$\frac{1}{T_k}\sum_{i=0}^{k-1} \log \|\Psi_{t_i}|_{\Delta^s(x_i)}\| < -\frac{\eta_1 + \eta_2}{2}$$

for $0 < k \le n_1$. Hence,

$$\frac{1}{T_k}\sum_{i=0}^{k-1} \log \|\Psi_{t_i}|_{\Delta^s(x_i)}\| < \frac{-T_{n_1} \cdot \frac{\eta_1+\eta_2}{2} + (k - n_1)\log K}{T_k}$$

for $n_1 < k < n$. We know that $n - n_1 = 2\iota - j_0 + i_0 + 7 < 3\iota + 8$. Hence,

$$T_k - T_{n_1} < (3\iota + 8)2T \quad \text{and} \quad (k - n_1)\log K < (3\iota + 8)\log K$$

for any $n_1 < k < n$. So, if s is large enough,

$$\frac{1}{T_k}\sum_{i=0}^{k-1} \log \|\Psi_{t_i}|_{\Delta^s(x_i)}\| < -\eta_1.$$

By the choice of L,

$$\frac{1}{\bar{t}_0 + \cdots + \bar{t}_{n_1+7}}\sum_{i=0}^{n_1+7} \log \|\Psi_{\bar{t}_i}|_{\Delta^s(\bar{x}_i)}\| \ge -\frac{\eta_1 + \eta_2}{2}$$

and hence

$$\begin{aligned}\sum_{i=0}^{n_1} \log \|\Psi_{t_i}|_{\Delta^s(x_i)}\| &\ge -\frac{\eta_1 + \eta_2}{2}(\bar{t}_0 + \cdots + \bar{t}_{n_1+6}) - 7\log K \\ &> -\frac{\eta_1 + \eta_2}{2}T_{n_1} - 7\log K.\end{aligned}$$

Then

$$\frac{1}{T_n}\sum_{i=0}^{n_1} \log \|\Psi_{t_i}|_{\Delta^s(x_i)}\| > \frac{-(\eta_1 + \eta_2)/2T_{n_1} - 7\log K - (n - n_1)\log K}{T_n}.$$

If s is large enough,

$$\frac{1}{T_n}\sum_{i=0}^{n-1}\log\|\Psi_{t_i}|_{\Delta^s(x_i)}\| > -\eta_2.$$

Let $n_2 = s\iota - i_0 + L$. By the choice of x_0 and x_{n-1},

$$\frac{1}{T_n - T_{n-k}}\sum_{i=1}^{k}\log m(\Psi_{t_{n-i}}|_{\Delta^u(x_{n-i})}) > \eta > \eta_1$$

for any $1 \le k \le n - n_2$. By the choice of L,

$$\frac{1}{\bar{t}_{n_2-k} + \cdots + \bar{t}_{n_2-1}}\sum_{i=1}^{k}\log\|\Psi_{\bar{t}_{n_2-i}}|_{\Delta^s(\bar{x}_{n_2-i})}\| > -\frac{\eta_1+\eta_2}{2}$$

and hence

$$\frac{1}{\bar{t}_{n_2-k} + \cdots + \bar{t}_{n_2-1}}\sum_{i=1}^{k}\log m(\Psi_{\bar{t}_{n_2-i}}|_{\Delta^u(\bar{x}_{n_2-i})}) > 2\eta - \frac{\eta_1+\eta_2}{2} > \eta$$

for all $1 \le k \le n_2$. Hence,

$$\begin{aligned}\sum_{i=1}^{k}\log m(\Psi_{t_{n-i}}|_{\Delta^u(x_{n-i})}) &> \eta(T_n - T_{n_2}) + \eta(\bar{t}_{n-k} + \cdots + \bar{T}_{n_2-1}) - 14\log K\\ &> \eta(T_n - T_{n-k} - 2T) - 14\log K\end{aligned}$$

for all $n - n_2 < k < n$. Consequently,

$$\frac{1}{T_n - T_{n-k}}\sum_{i=1}^{k}\log m(\Psi_{t_{n-i}}|_{\Delta^u(x_{n-i})}) > \frac{\eta(T_n - T_{n-k} - 2T) - 14\log K}{T_n - T_{n-k}}$$

for all $n - n_2 < k < n$. We know that $n - n_2 > \iota$ and $T_n - T_{n-k} > \iota T$ if $k > n - n_2$. Finally, if we choose τ large enough, then

$$\frac{1}{T_n - T_{n-k}}\sum_{i=1}^{k}\log m(\Psi_{t_{n-i}}|_{\Delta^u(x_{n-i})}) > \eta_1$$

for all $n - n_2 < k < n$. This completes the proof. □

End of Proof of Main Theorem. Suppose that $H_X(\gamma)$ satisfies (P1)–(P3) of Proposition 2.5 and the bundle $\Delta^s(x)$ is not contracting for Ψ_t. By the shadowing property of $H_X(\gamma)$, it follows that for any neighborhood U of $H_X(\gamma)$ and constants $0 < \eta_1 < \eta_2 < \eta$ and $\epsilon > 0$, there exist $\gamma' \sim \gamma$, an adaptable point $p \in \gamma'$ and a shadowing point z which is ϵ close to p. Applying Lemma 2.13,

$$W^s(\mathrm{orb}(z)) \pitchfork W^u(\gamma') \neq \emptyset \quad \text{and} \quad W^u(\mathrm{orb}(z)) \pitchfork W^s(\gamma') \neq \emptyset.$$

Hence, $\mathrm{orb}(z) \subset H_X(\gamma)$. In the construction of the pseudo orbit in Proposition 2.14, we can make L be arbitrarily big, and so we can assume that the period of z is bigger than $\tilde{T}$. The property

$$\frac{1}{T_n}\sum_{i=0}^{n-1}\log\|\Psi_{T_{i+1}-T_i}|_{\Delta^s(X_{T_i(z)})}\| > -\eta_2$$

contradicts the assumption (P2) of Proposition 2.5. This completes the proof of our main theorem. □

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