THE INFORMATION OF IMAGES AND ITS DEPENDENCE ON NOISE AND SPATIAL RESOLUTION

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1. INTRODUCTION

Radioastronomical observations are often incomplete in the sense that either the angular resolution desired or the signal-to-noise ratio required are not adequate. Attempts to fill in the missing information by either smoothing or deconvolving the observed data date back to Bracewell (1956, 1958). More recently the need for more sophisticated restoration procedures has increased since observations with radioastronomical interferometers frequently suffer from incomplete coverage of the u - v plane. Procedures like CLEAN (Högbom 1974, Schwarz 1978) and MEM (Wernecke 1977, Gull and Daniell 1978) have been developed to interpolate the unknown data in the u - v plane.

All these methods do not explicitly take into account the change of the signal-to-noise ratio and the angular resolution of the observed map during the restoration process. In the present paper we point out that the information of any observed temperature distribution is critically dependent on these parameters. An analytical expression is presented which gives the information measure I of a map as a function of angular and temperature resolution. Further, the relevance of I for convolutions, deconvolutions and nonlinear restorations of interferometer maps is discussed. We find that the information measure I is a useful quantity to judge the quality of any of those restorations.

2. THE INFORMATION MEASURE I

Observing a temperature distribution T on the sky with a radiotelescope affects T in two ways. T is convolved with a beam function f and contamination by statistical errors which are characterized by a noise function n.

$$T_{obs} = f * T + n$$

For any interpretation of such a map only structure is of interest which is statistically significant if compared to the noise and angular reso-

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lution. The information of a given map increases with signal-to-noise ratio, spatial resolution and observed area. Hence it is plausible that a measure for the information I contained in the map can be derived from the number of significant details. This number can be determined, if the observed solid angle A is measured in units of the beam solid angle

$$\Delta a = f_{max} \; / \int_{4\pi} f \, da \quad .$$

Then there are

$$m = A \, / \, \Delta a$$

locations in the map which may carry statistically independent details. Similarly, we measure the brightness temperature in units of the RMS noise level τ. Then the observed signal (assumed to be always positive) is decomposed into

$$l = \frac{\int_A T \, da}{\Delta a \quad \tau}$$

signal elements. The degrees of freedom m and l derived in this way can now be used to derive an expression for the information I contained in our map.

If P is the number of equally possible outcomes of an experiment, then the information I which is gained by observing one of these P cases is given by (Brillouin, 1963):

$$I = k \ln P \text{ (here we use } k = 1)$$

To derive P we count the total number of possible outcomes of the experiment by distributing the l undistinguishable signal elements over m degrees of freedom in position. From Bose - Einstein statistics we get

$$P = \frac{(m-1+l)!}{(m-1)!\,l!}$$

If m and l are large numbers we derive I by applying Stirlings formula

$$I = (m+l) \ln (m+l) - m \ln m - l \ln l \; .$$

After derivation of an analytical expression for the information content in a radioastronomical map we will study, how the information is changed by a manipulation of the data. For any reasonable transformation we require that the observed solid angle A and the total intensity $\int_A T_{obs} \, da$ remain unchanged. Thus the information I is merely influenced by changing the resolution limits Δa and τ. Both limits are affected in a complementary way: convolution increases Δa and decreases τ, deconvolution decreases Δa and increases τ. For linear transformations this complementarity is described by a relation $\Delta a \cdot \tau >$ const (Kalberla, 1978).

During any restoration procedure the information measure I changes in a well defined way. As an hypothesis we assume that I will become maximal for an optimal restoration. There is a simple example which shows that I indeed may be maximized. If the data are smoothed out completely by a convolution we obtain m = 1 and I ≐ 0. In the other extreme case of superresolution (Bracewell, 1958) the errors may be amplified such that they overcome the signal completely, thus 1 = 0 and I = 0 again.

3. COMPUTER TESTS.

To test, whether maximizing I eventually leads to an optimal restoration, some computer generated brightness temperature distributions have been degraded and restored. The fidelity of the restored distributions was measured by calculating the RMS deviation with respect to the model.

One dimensional test scans (delta functions, triangular, rectangular and gaussian distributions) have been convolved with a gaussian beam function, afterwards noise was added. In any case the scan length, the total intensity and the noise level have been adjusted in a way that for the corresponding degrees of freedom m >> 1. Then apparently a deconvolution is necessary to increase the information of the test scans. The restored scans T_{rest} have been calculated from (Kalberla, 1978)

$$T_{rest}(x) = q \cdot T_{obs}(x) - q \int_{scan} Q\ (x-x')\ T_{obs}(x')\ dx'$$

where

$$Q\ (x) = \lim_{n \to \infty} \sum_{i=1}^{n} (-qf)^{(i)}$$

is the resolving Kernel function. $f^{(i)}$ denotes i-fold convolution of f. The parameter q determines the strength of the restoration. The width Δa of the beam function decreases with increasing q, the amplification of the errors is proportional to q.

For any degraded model distribution a set of restored scans with different restoration parameters q has been calculated. For any of these restored scans the information I and the RMS deviation with respect to the model has been determined. Figure 1 shows two typical cases with different signal-to-noise ratios. The maximum of the information I is well correlated with the minimum of the RMS deviation between model and restored scan. Thus the information measure I is a useful quantity to determine the optimal restoration parameter q.

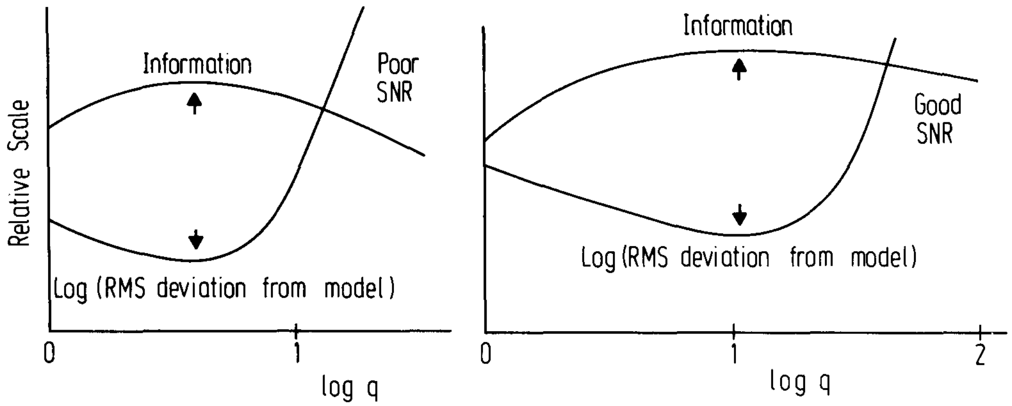

Fig. 1
Information I and RMS deviation between restored scan and model are plotted as a function of the restoration parameter q. The extrema of both curves are closely related.Two typical cases for poor and good signal-to-noise ratio are shown.

In a second set of computer tests aperture synthesis observations have been simulated. Figure 2 shows an example: A model for the brightness temperature distribution (A) was degraded by noise (B). After Fourier transformation into the u - v plane a subset of u - v data was erased. Figure 2 C shows the direct Fourier transform after eliminating 40% of the significant data. To obtain the restored map (D), the missing data have been replaced starting from the origin of the u - v plane proceeding to higher spatial frequencies. For this purpose a modified Gerchberg - Saxton algorithm (Gassmann, 1977 other references herein) was used. For any cycle of the iterative procedure the dirty map was convolved by a gaussian function. Then the remaining negative intensities have been forced to zero. From cycle to cycle the width of the smoothing function was decreased.

During this restoration procedure the RMS deviation between model and restored map was calculated.Simultaneously the information I was determined from the width of the smoothing function and the uncertainties which are introduced by the interpolation of the missing data points in the u - v plane.As for the one-dimensional test cases mentioned before maxima in I were found to correspond to minima in the RMS deviation between model and restored map. For some test models no extrema could be obtained but then both, information as well as RMS deviation, were found to converge.

These two cases can be distinguished by different m/l ratios. A maximum in I was obtained if m/l >> 1 which corresponds to typical aperture synthesis observations (e.g. for Westerbork maps

$m = \frac{A}{\Delta a} \approx 8000$ and l of the order of 1000).

Convergence was obtained for $m/l \lesssim 1$ which in general means very high signal-to-noise ratio.

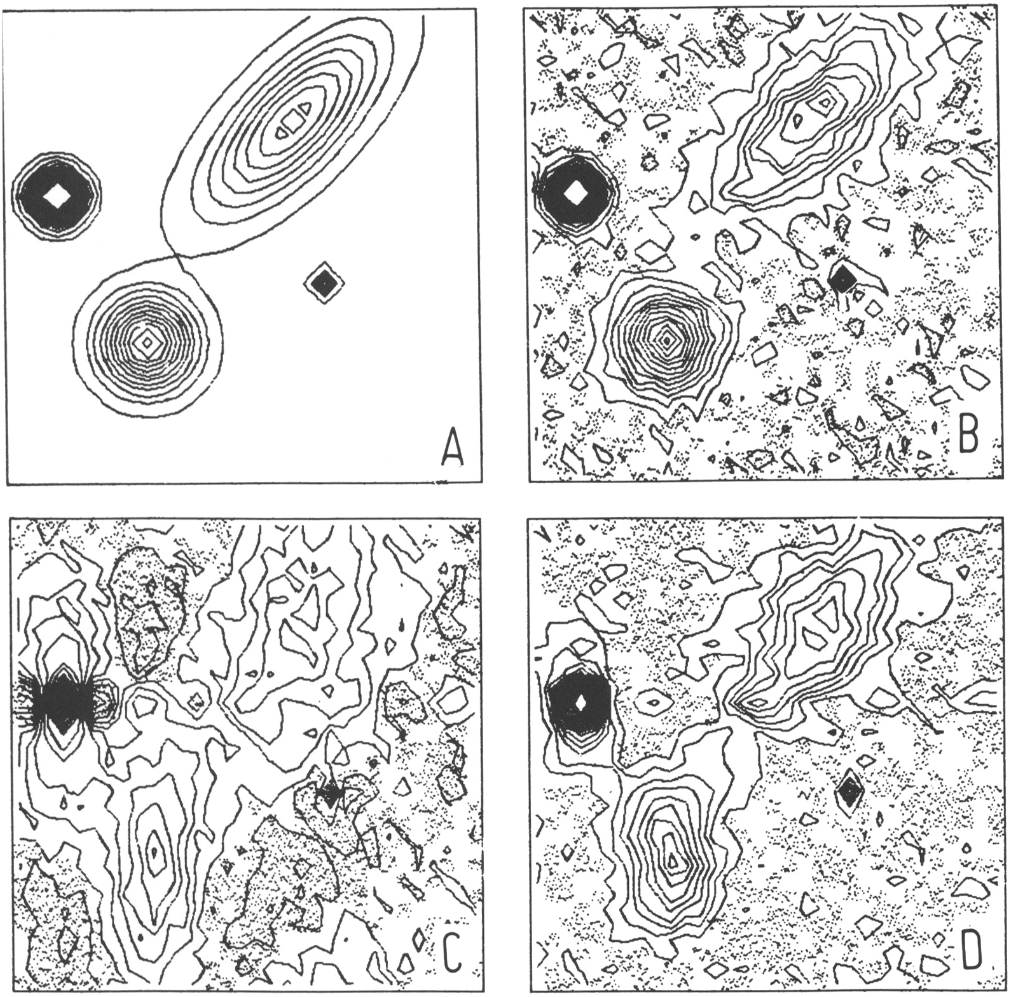

Fig. 2
Example for the restoration of a model distribution:
A) model B) model with noise, C) dirty map derived from an incomplete covered u - v plane, D) restored map.
Negative regions are shown shaded. The contour interval is constant.

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The purpose of the computer tests was to study whether or not there is any relation between the maximum in the RMS deviation between model and restored map. The results obtained so far suggest that such a relation exists for our restoration procedure. Although no tests with other restoration procedures have been made we propose that such a relation should hold in general. Otherwise the relevance of the information measure I has to be reconsidered

For a practical use of the information measure I in restoring observations it should be possible to predict those restoration parameters for which I becomes maximal. In order to do this we have to know how the restoration procedure changes the spatial resolution Δa and the temperature uncertainties τ (including the errors which are introduced by any interpolation process). While the change in resolution Δa for our procedure can be derived from the smoothing function which is applied to the map we cannot calculate yet the amplification of the errors τ. This problem needs further investigations.

4. DISCUSSION

It was shown that the information content I of a radioastronomical map is critically dependent on the resolution limits Δa and τ. During an restoration procedure both, spatial resolution Δa as well as temperature resolution τ are changed. Consequently the information I of a map is affected, too.

To test the hypothesis that maximizing I leads to an optimal restoration, computer tests have been performed. The restoration used here is a special one in the sense that it allows to calculate the resolution limits Δa and τ at any stage of the procedure. This is necessary for a simultaneous calculation of I. Whether or not other restoration procedures, e.g. CLEAN or MEM - methods, allow also to calculate I during the iterations has not been investigated.

To calculate the information measure I we only need the total intensity within the observed solid angle A and the resolution limits Δa and τ. Then maximizing I is possible without introducing further constraints to the data. Thus the information measure I can be considered as a new independent criterion to judge the quality of a map. It would probably be worth comparing the restoration methods mentioned above by applying them to the same set of observations and calculating the resulting information measure I for the restored maps.

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