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DISTRIBUTIVE PROJECTIVE LATTICES

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1. Introduction. Two basic unsolved problems of lattice theory are (1) the characterization of sublattices of free lattices and (2) the characterization of projective lattices. A solution to an important case of the first problem has been provided by Galvin and Jónsson [**3**], who characterize *distributive* sublattices of free lattices. In this paper, we solve the same case of the second problem by characterizing distributive projective lattices (Theorem 4.1). An interesting corollary is the verification for distributive lattices of the conjecture that a *finite* lattice is projective if and only if it is a sublattice of a free lattice.†

2. Preliminaries. We shall take the term "projective" to mean projective in the category of all lattices and lattice homomorphisms. If $f\colon A \to B$ and $g\colon B \to A$ are functions between sets A and B, we shall say for convenience that g *splits* f if $f \circ g$ is the identity map on B. Then, as in many algebraic categories, it is easy to show that a lattice P is projective if and only if any surjective lattice homomorphism $\varphi\colon L \to P$ can be split by some lattice homomorphism $\psi\colon P \to L$.

Interestingly, all countable lattices have the following useful pseudo-projective property.

2.1. LEMMA. *If P is any countable lattice and $\varphi\colon L \to P$ is any surjective lattice homomorphism, then there is an isotone (i.e., order-preserving) map $\mu\colon P \to L$ which splits φ.*

This fact is easily proved by a modification of the well-known "weaving" argument. (See Dean [**2**, Theorem 3].)

If L is a lattice and R, S are subsets of L, we shall write $R < S$ if $r < s$ for all $r \in R$ and $s \in S$. L is the *linear sum* of its sublattices L_α $(\alpha \in A)$ if $L = \bigcup_{\alpha \in A} L_\alpha$, and for any distinct $\alpha, \beta \in A$, either $L_\alpha < L_\beta$ or $L_\beta < L_\alpha$. L is (linearly) *indecomposable* if L is not the linear sum of any two (non-empty) sublattices. It is not difficult to show that any lattice L is the linear sum of uniquely determined indecomposable sublattices L_α, which we shall call the (linear) *components* of L.

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†An affirmative solution of the full conjecture is to appear, implicitly, in McKenzie's paper [**4**, proof of Theorem 6.3], where an essential portion of the argument is credited to B. Jónsson. A. Kostinsky has refined this argument to prove that every finitely generated sublattice of a free lattice is projective.

In this terminology, the result of Galvin and Jónsson can be expressed as follows.

2.2. THEOREM [**3**, Theorem 6]. *A distributive lattice L is a sublattice of a free lattice if and only if L is countable and each linear component of L is either*

(1) *a single element,*

(2) *an eight-element Boolean algebra, or*

(3) *the product of a countable chain and a two-element chain.*

3. Components and projectivity. We shall now show that the projectivity of a countable lattice can be determined by examining its components.

The following notation will be useful: If S is a subset of a lattice L, let S^* be the sublattice of L generated by S.

3.1. THEOREM. *A countable lattice P is projective if and only if all its linear components are projective.*

Proof. Suppose that P is projective. Let C be any component of P, and let $\varphi: L \to C$ be any surjective lattice homomorphism. We must show that φ can be split. Let P' be the lattice constructed from P by replacing the component C with a copy of L, while retaining the order of the components. Let $\varphi': P' \to P$ be the extension of φ which restricts to the identity map on all components of P' other than L. Now, φ' is a lattice homomorphism of P' onto P and so can be split by some homomorphism $\psi': P \to P'$. If ψ is the restriction of ψ' to C, then ψ splits φ, as required. C is therefore projective.

Conversely, suppose that all components of P are projective. Let a lattice L and a surjective lattice homomorphism $\varphi: L \to P$ be given. We must show that φ can be split by some lattice homomorphism. By Lemma 2.1, we know at least that there is an isotone map $\mu: P \to L$ which splits φ.

For any component C of P, let φ_C be the restriction of φ to $\mu(C)^*$. Then φ_C has image C, since $\varphi(\mu(C)^*) = \varphi(\mu(C))^* = C^* = C$. The assumed projectivity of C then implies that φ_C can be split by some lattice homomorphism $\psi_C: C \to \mu(C)^*$. Let $\psi: P \to L$ be the union of the maps ψ_C, taken over all components C of P. Then ψ does split φ. The only question is whether ψ is indeed a lattice homomorphism. By construction, ψ preserves lattice operations on any two elements c, d lying in the *same* component of P. Suppose that c and d lie respectively in different components C, D of P, with $C < D$. Then the isotonicity of μ yields $\mu(C)^* < \mu(D)^*$. In particular, $\psi(c) < \psi(d)$, so that lattice operations on c, d are again preserved. The proof is therefore complete.

Remark. Without the condition of countability, a lattice P with projective components may fail to be projective, even if P has as few as two components. In fact, if P is the linear sum of two infinite free lattices, not both countable, then P is not projective, and Lemma 2.1 fails as well. This follows from a modification of a proof given by Balbes and Horn [**1**, Theorem 3.13] for the analogous construction in the category of distributive lattices.

4. The characterization. By **2** we mean the two-element chain, with elements 0, 1. More generally, by a "lattice with 0 and 1" we mean a lattice with a least element (called 0) and a greatest element (called 1).

We can now state the promised characterization.

4.1. THEOREM. *A distributive lattice P is projective if and only if P is countable and each linear component of P is either*

(1) *a single element,*

(2) *an eight-element Boolean algebra, or*

(3) $K \times \mathbf{2}$, *where K is a countable chain with* 0 *and* 1.

The proof of this theorem will be preceded by two lemmas: In concert with Theorem 3.1, Lemma 4.2 will show the sufficiency of our conditions for the projectivity of P, and Lemma 4.3 will aid in showing the necessity.

4.2. LEMMA. *Lattices of types* (1), (2), (3), *as listed in Theorem* 4.1, *are themselves projective.*

Proof. A one-element lattice is trivially projective. If Q is a lattice of type (2) or (3) and a surjective lattice homomorphism $\varphi: L \to Q$ is given, we must show that there is a lattice homomorphism $\psi: Q \to L$ which splits φ. Lemma 2.1 enables us to start by choosing an isotone map $\mu: Q \to L$ which splits φ.

Suppose that Q is an eight-element Boolean algebra with atoms a_1, a_2, a_3. For $\{i, j, k\} = \{1, 2, 3\}$, let $y_i = \mu(a_j') \cdot \mu(a_k')$, where a_i' is the complement of a_i in Q. Define $\psi: Q \to L$ by setting $\psi(a_i) = y_i$, $\psi(a_1 a_2 a_3) = y_1 y_2 y_3$, and requiring ψ to preserve all sums of the a_i. Then ψ splits φ and is readily shown to be a lattice homomorphism, as required.

Suppose next that $Q = K \times \mathbf{2}$, where K is a countable chain with 0 and 1. Define $\psi: Q \to L$ by setting

$$\psi(x, 0) = \mu(x, 1) \cdot \mu(1, 0),$$
$$\psi(x, 1) = \psi(x, 0) + \mu(0, 1) \quad \text{for all } x \in K.$$

Then ψ splits φ and is easily shown to be a lattice homomorphism.

Q is therefore projective in each case, as asserted.

4.3. LEMMA. *If L is a lattice such that $L \times \mathbf{2}$ is projective, then L is a lattice with* 0 *and* 1.

Proof. Suppose that L has no greatest element. Let L' be L with a greatest element $1'$ adjoined, and define D to be the sublattice of $L \times L'$ consisting of all (a, b) with $a \leqq b$. Then the mapping $\varphi: D \to L \times \mathbf{2}$, defined as follows, is easily seen to be a surjective lattice homomorphism: $\varphi(a, b) = (a, 0)$ if $b < 1'$, $\varphi(a, b) = (a, 1)$ if $b = 1'$. Since $L \times \mathbf{2}$ is projective, there is a lattice homomorphism $\psi: L \times \mathbf{2} \to D$ which splits φ. Take any $a \in L$. Then $\psi(a, 0)$ must equal (a, b) for some b with $a \leqq b < 1'$. Choose c with $b < c < 1'$. Then $\psi(c, 0) = (c, d)$ with $c \leqq d$. Since ψ is a lattice homomorphism, we have $(a, b) = \psi(a, 0) = \psi(c, 0) \cdot \psi(a, 1) = (c, d) \cdot (a, 1') = (a, d)$, in contradiction of the relations $b < c \leqq d$. Hence L has a greatest element and, by duality, a least element.

Proof of Theorem 4.1. If P has components of only the three given types, then Lemma 4.2 shows that the components of P are projective. If in addition P is countable, then Theorem 3.1 yields the projectivity of P itself.

Conversely, if a distributive lattice P is projective, let $\varphi: F \rightarrow P$ be any homomorphism of a free lattice onto P, and let ψ split φ. Then ψ is an isomorphism of P with a sublattice of F. By Theorem 2.2, P is countable and each component of P is either a single element, an eight-element Boolean algebra, or $K \times \mathbf{2}$ for some countable chain K. It remains only to show that such a chain K must have elements 0 and 1. But by Theorem 3.1, each component is itself projective; Lemma 4.3 then implies that K does have least and greatest elements.

4.4. COROLLARY. *For a finite distributive lattice P, the following conditions are equivalent:*

(i) *P is projective;*

(ii) *P is a sublattice of a free lattice;*

(iii) *every linear component of P is either a single element, an eight-element Boolean algebra, or the product of a chain and* **2**.

For the proof, it is sufficient to note that Theorems 2.2 and 4.1 are identical in the case of finite lattices. These two theorems also show that the simplest example of a non-projective distributive sublattice of a free lattice is $N \times \mathbf{2}$, where N is the chain of natural numbers.

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