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THE WEAK DROP PROPERTY ON CLOSED CONVEX SETS

PEI-KEE LIN and XINTAI YU

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Abstract

Recall a closed convex set C is said to have the weak drop property if for every weakly sequentially closed set A disjoint from C there exists $x \in A$ such that $\mathrm{co}(\{x\} \cup C) \cap A = \{x\}$. Giles and Kutzarova proved that every bounded closed convex set with the weak drop property is weakly compact. In this article, we show that if C is an unbounded closed convex set of X with the weak drop property, then C has nonempty interior and X is a reflexive space.

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Let C be a closed convex set. (Note: in this article, C always denotes a closed convex set.) A *drop* $D(x, C)$ determined by a point $x \notin C$ is the convex hull of the set $\{x\} \cup C$. C is said to have the *drop property* if for every nonempty closed set A disjoint from C, there exists a point $a \in A$ such that $D(a, C) \cap A = \{a\}$. Daneš [1] proved that if C is a bounded closed convex subset of X and A is a closed set at positive distance from C, then there exists an $a \in A$ such that $D(a, C) \cap A = \{a\}$. Modifying the above result, Kutzarova and Rolewicz [5] say that a nonempty proper closed convex set C has the *drop property* if for every nonempty closed set A disjoint from C, there exists a point $a \in A$ such that $D(a, C) \cap A = \{a\}$. A closed convex set C is said to have the *weak drop property* if for every weakly sequentially closed set A disjoint from C there is $a \in A$ such that $D(a, C) \cap A = \{a\}$. The weak drop property was introduced by Giles, Sims, and Yorke [3] (also see [4]). They proved that the unit ball has the weak drop property if and only if X is reflexive.

Recall that the *Kuratowski measure* of a closed set D is the number

$$\alpha(D) = \inf\{r : D \text{ is covered by a finite family of open sets with diameter } < r\}.$$

(Hence, if D is an unbounded set, then $\alpha(D) = \infty$.)

For a convex closed set C, $F(C)$ denotes the set of all bounded linear functionals $x^* \in X^*$, $x^* \neq 0$ which is bounded above on C. If $x^* \in F(C)$, and $\delta > 0$, the *slice* $S(x^*, C, \delta)$ is the set $\{x \in C : x^*(x) \geq M - \delta\}$, where $M = \sup\{x^*(x) : x \in C\}$. C is said to have *property* (α) if

$$\lim_{\delta\to 0} \alpha(S(x^*, C, \delta)) = 0$$

for every $x^* \in F(C)$. In [5], Kutzarova and Rolewicz proved that if a convex closed set C has the drop property, then C has property (α). Clearly, if C has property (α), then for every $x^* \in F(C)$ and $x_n \in S(C, x^*, 1/n)$, $\{x_n\}$ contains a convergent subsequence. Hence, we say a closed convex set has *weak property* (α) if for every $x^* \in F(C)$, and $x_n \in S(C, x^*, 1/n)$, $\{x_n\}$ has a weakly convergent subsequence. By James' Theorem, every bounded closed convex set with weak property (α) is weakly compact. On the other hand, one can easily show that every weakly compact convex set has the weak drop property and weak property (α). In [4], Giles and Kutzarova proved that every bounded closed convex set with the weak drop property is weakly compact. In this article, we prove that if C is not weakly compact, then C has the weak drop property if and only if X is reflexive and C has the weak property (α) (in this case, C cannot be bounded). First, we show that every closed convex set with the weak drop property has weak property (α). The proof is essentially same as the proof of (i) implies (ii) in [4, Theorem 3].

PROPOSITION 1. *Let C be a closed convex set. If C has the weak drop property, then C has weak property (α). Hence, if $x^* \in F(C)$, then x^* attains its supremum on C.*

PROOF. Recall a sequence $\{x_n\} \subseteq X \setminus C$ is called a *stream* if $x_{n+1} \in D(x_n, C) \setminus \{x_n\}$ for all $n > 1$. Clearly, every substream is a stream. Assume that C does not have weak property (α). We shall construct a stream which does not have any weakly convergent subsequence. Since C does not have weak property (α), there exist $x^* \in F(C)$ and a sequence $\{x_n\}$ such that

(i) $x_n \in S(x^*, C, 1/4^n)$;

(ii) no subsequence of $\{x_n\}$ converges weakly.

Let x_o be any element in X such that $x^*(x_o) = M+2$ where $M = \sup\{x^*(x) : x \in C\}$. For every $n \in \mathbb{N}$, let

$$y_n = \frac{1}{2^n}x_o + \sum_{i=1}^{n} \frac{1}{2^{n-i+1}}x_i.$$

Then $x^*(y_n) > M + 1/2^{n+1}$, and $\{y_n\}$ is a stream. If $\{y_n\}$ contains a subsequence $\{y_{n_k}\}$ which does not have any further weakly convergent subsequence, then $\{y_{n_k}\}$ is weakly sequentially closed and we are done. Hence, we may assume that every subsequence of $\{y_n\}$ has a further weakly convergent subsequence. So there exists an increasing sequence (n_k) such that both (y_{n_k}) and (y_{n_k+1}) converge weakly. But $y_{n_k+1} = \frac{1}{2}(y_{n_k} + x_{n_k+1})$. So (x_{n_k+1}) converges weakly and we get a contradiction. Therefore, C must have weak property (α).

REMARK 1. In [6], it has been proved that if C_1 and C_2 have the drop property, then $C_1 \cap C_2$ has the drop property. The same argument shows if C_1 and C_2 have the weak drop property, then $C_1 \cap C_2$ has the weak drop property.

In [5], it has been proved that if C is a noncompact set with the drop property, then the interior of C is nonempty. The same argument shows that a similar result holds if C is an unbounded closed convex set with the weak drop property. We present a proof here.

THEOREM 2. *If X contains a closed convex subset which is not weakly compact and which has the weak drop property, then X is reflexive.*

PROOF. Let C be a non-compact closed convex subset of X with the weak drop property. There exists a sequence $\{x_n\} \subseteq C$ such that $\{x_n\}$ does not have any weakly convergent subsequence. For any $x \notin C$, y_n is defined by

$$y_n = \frac{1}{2^n}x + \sum_{i=1}^{n} \frac{1}{2^{n-i+1}}x_i.$$

We call the sequence $\{y_n\}$ a *dyadic stream* generated x. We claim that every dyadic stream has a nonempty intersection with C.

Suppose this is not true. Then $\{y_n\}$ is a stream. Since $\{x_n\}$ does not contain any weakly convergent subsequence, by the proof of Proposition 1 there is a substream of $\{y_n\}$ which does not have any weakly convergent subsequence. This contradicts C having the weak drop property. We have proved our claim.

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For any $z \in X$ and $k \neq 0$, the *homothetic operation* $T_{z,k} : X \longrightarrow X$ is given by

$$T_{z,k}(x) = z + k(x - z).$$

Clearly, $T_{z,k}$ is a homeomorphism for every $z \in X$ and $k \neq 0$. Let $T_{\{x_1, x_2, \ldots, x_n\}} = T_{x_1,2} T_{x_2,2} \ldots T_{x_n,2}$. It is easy to see that $y_n \in C$ if and only if $x \in T_{\{x_1, x_2, \ldots, x_n\}}(C)$. So

$$X \setminus C = \bigcup_{n=1}^{\infty} T_{\{x_1, x_2, \ldots, x_n\}}(C).$$

By the Baire category theorem, $T_{\{x_1, x_2, \ldots, x_n\}}(C)$ contains an open set for some n. So C has nonempty interior.

Suppose that $\mathrm{int}(C) \neq \emptyset$. Without loss of generality, we may assume that $0 \in \mathrm{int}(C)$. So $0 \in \mathrm{int}(C \cap (-C))$. By Remark 1, $C \cap (-C)$ has the weak drop property. Since it is a symmetric set, by [5, Proposition 4] it is bounded. So X is a reflexive space.

REMARK 2. Kutzarova and Rolewicz [5] proved the following:

(i) If C is a closed convex set the with drop property, then $D(b, C)$ is closed for every $b \notin C$.

(ii) If C_1 and C_2 have property (α), then both $\mathrm{co}(C_1, C_2)$ and $C_1 + C_2$ have property (α).

(iii) Suppose that X is reflexive. Let C be an unbounded closed convex subset of X. If C has property (α) and if $\mathrm{int}(C) \neq \emptyset$, then C contains a ray $\{c + \lambda b : \lambda \geq 0\}$. Moreover, for any $x \in X$, there is $\beta > 0$ such that C contains the ray $\{x + \lambda b : \lambda \geq \beta\}$.

The same argument shows the above statements are still true if we replace property (α) by weak property (α).

It is natural to ask whether X is reflexive if X contains a closed convex subset C such that

(a) C is not weakly compact.

(b) C has the weak drop property.

The following theorem shows that this is true if C satisfies the following additional assumption

(c) $\mathrm{int}(C) \neq \emptyset$.

THEOREM 3. *Suppose that X contains a closed convex set C which satisfies the following conditions:*

(a) *C is not weakly compact.*
(b) *C has the weak drop property.*
(c) $\operatorname{int}(C) \neq \emptyset$.
Then X is a reflexive space.

PROOF. We claim that that C is w*-closed. Suppose the claim has been proved. Without loss of generality, we may assume that $0 \in \operatorname{int}(C)$. Then $C \cap (-C)$ is a bounded w*-closed set with nonempty interior. So X is reflexive.

If $x^{**} \in \partial \bar{C}^*$, then by Phelps Theorem [7] for any $\epsilon > 0$ there are $x_\epsilon^{**} \in \partial \bar{C}^*$ and $x^* \in X^*$ such that $\|x^{**} - x_\epsilon^{**}\| < \epsilon$ and

$$x_\epsilon^{**}(x^*) = \sup x^*(C) = \sup x^*(\bar{C}^*).$$

Hence, $x^* \in F(C)$. But C has weak property (α); so $D = \{x \in C : x^*(x) = \sup x^*(C)\}$ is weakly compact, and x_ϵ^{**} is an element of $D \subseteq C$. Since C is a closed set, this implies C contains x^{**}. We have proved our claim.

THEOREM 4. *Let C be any unbounded closed convex subset of X with a nonempty interior. Then C has the weak drop property if C has weak property (α).*

PROOF. By Theorem 3, X is reflexive. Hence, if C does not have the weak drop property, then there is a weakly closed stream $\{x_n\}$ which is disjoint with C. So $\{x_n\}$ does not contain any weakly convergent subsequence. Since C is an unbounded set with weak property (α), by Remark 2 (iii), there exists $b \neq 0$ such that for every $x \in X$ there is $\beta > 0$ such that C contains the ray $\{x + \lambda b : \lambda \geq \beta\}$. Let $\eta = \sup\{\beta : (\beta b + \{x_n : n \in \mathbb{N}\}) \cap C = \emptyset\}$. Note

(i) $\eta b + C \subseteq C$;
(ii) if $\eta b + x_n \in C$, then $\eta b + x_m \in C$ for every $m > n$.

Hence, if $a_i \geq 0$, $\sum_{i=1}^n a_i = 1$, and $\eta b + \sum_{i=1}^n a_i x_i \in \operatorname{int}(C)$, then

$$x_{n+1} + \eta b \in \operatorname{co}(\{\eta b + \sum_{i=1}^n a_i x_i\} \cup (\eta b + C)) \in \operatorname{int}(C).$$

This is impossible. So we have that $(\eta b + \operatorname{co}(x_n)) \cap \operatorname{int}(C) = \emptyset$. By the Hahn-Banach Theorem, there is $x^* \in X^*$ such that

$$\inf\{x^*(\eta b + x_n) : n \in \mathbb{N}\} = M = \sup\{x^*(c) : c \in C\}.$$

By the definition of η, there exists a sequnce $y_{n_k} \in C$ such that

$$\lim_{k\to\infty} \|\eta b + x_{n_k} - y_{n_k}\| = 0, \quad \text{and} \quad \lim_{k\to\infty} x^*(y_{n_k}) = M.$$

Since C has the weak property (α), $\{y_{n_k}\}$ contains a weakly convergent subsequence. This implies $\{x_n : n \in \mathbb{N}\}$ has a weakly convergent subsequence. This is a contradiction.

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Memphis State University
Memphis TN 38152
USA

East China Normal University
Shanghai
China