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# Concentration behaviour of normalized ground states of the mass critical fractional Schrödinger equations with ring-shaped potentials

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We consider $L^2$-constraint minimizers of the mass critical fractional Schrödinger energy functional with a ring-shaped potential $V(x) = (|x| - M)^2$, where $M > 0$ and $x \in \mathbb{R}^2$. By analysing some new estimates on the least energy of the mass critical fractional Schrödinger energy functional, we obtain the concentration behaviour of each minimizer of the mass critical fractional Schrödinger energy functional when $a \nearrow a^* = \|Q\|_2^{2s}$, where $Q$ is the unique positive radial solution of $(-\Delta)^s u + su - |u|^{2s}u = 0$ in $\mathbb{R}^2$.

*Keywords:* Fractional Schrödinger equations; Normalized ground states; Ring-shaped potentials

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## 1. Introduction

In this paper, we study the following mass critical fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = \lambda u + a|u|^{2s}u, \quad \text{in} \mathbb{R}^2, \tag{1.1}$$

where $s \in (\frac{1}{2}, 1)$, $V : \mathbb{R}^2 \to \mathbb{R}$ is an external potential function, $\lambda \in \mathbb{R}$ and $a > 0$. It's well known that the fractional Laplacian $(-\Delta)^s (s \in (0, 1))$ can be defined by

$$(-\Delta)^s v(x) = C_{2,s} P.V. \int_{\mathbb{R}^2} \frac{v(x) - v(y)}{|x-y|^{2+2s}} \mathrm{d}y = C_{2,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(x)} \frac{v(x) - v(y)}{|x-y|^{2+2s}} \mathrm{d}y$$

for $v \in S(\mathbb{R}^2)$, where $P.V.$ denotes a Cauchy principal value, $S(\mathbb{R}^2)$ is the Schwartz space of rapidly decaying $C^\infty$ function, $B_\varepsilon(x)$ denotes an open ball of radius $\varepsilon$ centred at $x$ and the normalization constant $C_{2,s} = (\int_{\mathbb{R}^2} \frac{1-\cos(\zeta_1)}{|\zeta|^{2+2s}})^{-1}$(see e.g. [**5**, **17**, **24**] and reference therein). In fact, there are applications of operator $(-\Delta)^s$ in some areas such as fractional quantum mechanics, physics and chemistry, obstacle problems, optimization and finance, conformal geometry and minimal surfaces, please see [**1**, **2**, **12**, **13**, **15**, **18**] and the references therein for more details.

For equation (1.1), a direct choice is to search for solutions $u \in H^s(\mathbb{R}^2)$ by looking for critical points of the functional $I_{\lambda,a} : H^s(\mathbb{R}^2) \to \mathbb{R}$ defined by

$$I_{\lambda,a} = \frac{1}{2} \int_{\mathbb{R}^2} (|(-\Delta)^{\frac{s}{2}} u|^2 + (V(x) - \lambda)|u|^2) \, \mathrm{d}x - \frac{a}{2+2s} \int_{\mathbb{R}^2} |u|^{2+2s} \, \mathrm{d}x, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is fixed and the fractional Sobolev space $H^s(\mathbb{R}^2)$ can be defined as follows

$$H^s(\mathbb{R}^2) := \left\{ u \in L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x < \infty \right\},$$

endowed with the norm

$$\|u\|^2_{H^s(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2) \, \mathrm{d}x.$$

For this approach of finding solutions, we recommend the reader to see [**8**, **14**, **19**, **20**] for more details. Moreover, (1.1) is also the Euler–Lagrange equation of the following constrained minimization problem

$$e(a) := \inf_{u \in M} J_a(u), \tag{1.3}$$

where $J_a(u)$ is the mass critical fractional Schrödinger energy functional

$$J_a(u) = \int_{\mathbb{R}^2} (|(-\Delta)^{\frac{s}{2}} u|^2 + V(x)|u|^2) \, \mathrm{d}x - \frac{a}{1+s} \int_{\mathbb{R}^2} |u|^{2+2s} \, \mathrm{d}x, \quad u \in E. \tag{1.4}$$

Here we define

$$E := \left\{ u \in H^s(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)|u|^2 \, \mathrm{d}x < \infty \right\}, \tag{1.5}$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^2} (|(-\Delta)^{\frac{s}{2}} u|^2 + V(x) u^2) \, \mathrm{d}x,$$

and

$$M := \left\{ u \in E : \int_{\mathbb{R}^2} |u|^2 \, dx = 1 \right\}. \tag{1.6}$$

Very recently, when $V(x)$ has satisfied the following assumption

$$(V_1) \quad 0 \leqslant V(x) \in L^\infty_{loc}(\mathbb{R}^2), \quad \lim_{|x|\to\infty} V(x) = \infty \quad \text{and} \quad \inf_{x\in\mathbb{R}^2} V(x) = 0,$$

Du, Tian, Wang and Zhang [**6**] proved that for all $a \in [0, a^*)$, $e(a)$ has at least one minimizer and has no minimizers if $a \geqslant a^*$, where $a^* = \|Q\|_2^{2s}$ and $Q$ is the unique positive radial solution of

$$(-\Delta)^s u + su - |u|^{2s} u = 0, \quad \text{in } \mathbb{R}^2. \tag{1.7}$$

Moreover, they also obtained that for $a \in [0, a^*)$ small enough, $e(a)$ has a unique nonnegative minimizer.

When $s = 1$ and replace $a$ by $b$, system (1.1) reduces to the following Schrödinger equation

$$-\Delta u + V(x)u = \lambda u + b|u|^2 u, \quad \text{in } \mathbb{R}^2, \tag{1.8}$$

For problem (1.8), we can also consider the following constrained minimization problem

$$\bar{e}(b) := \inf_{u\in\bar{M}} \bar{J}_b(u), \tag{1.9}$$

where $\bar{J}_b(u)$ is defined by

$$\bar{J}_b(u) = \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) \, dx - \frac{b}{2} \int_{\mathbb{R}^2} |u|^4 \, dx, \quad u \in \bar{E}.$$

Here we define

$$\bar{E} := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)|u|^2 \, dx < \infty \right\},$$

and

$$\bar{M} := \left\{ u \in \bar{E} : \int_{\mathbb{R}^2} |u|^2 \, dx = 1 \right\}.$$

There are many works focusing on existence and nonexistence of minimizers for (1.9). For instance, in [**9**], Guo and Seiringer proved that there exists a critical value $b^* > 0$ such that (1.9) has at least one minimizer if $0 \leqslant b < b^*$, and (1.9) has no minimizers if $b \geqslant b^*$. Moreover, they also studied that the limit behaviour of minimizers for (1.9) as $b \nearrow b^*$. For more constrained minimization problem of (1.9), please see [**10**, **11**] and the references therein for more details.

The first purpose of this paper is to consider whether the minimizers of (1.3) are the ground states of (1.1) and whether the opposite is true? To solve these problems, we first give the definition of the ground states of (1.1). Let

$$S_{\lambda,a} := \left\{u \in E\backslash\{0\} : \langle I'_{\lambda,a}(u), \varphi\rangle = 0, \quad \forall \varphi \in E\right\}, \tag{1.10}$$

and

$$G_{\lambda,a} := \{u \in S_{\lambda,a} : I_{\lambda,a}(u) \leqslant I_{\lambda,a}(v), \quad \forall v \in S_{\lambda,a}\}. \tag{1.11}$$

We say $u \in E$ is a ground state of (1.1) if $u \in G_{\lambda,a}$. Moreover, for $a \in [0, a^*)$, we define

$$K_a := \{u_a : u_a \ \text{ is a minimizer of } e(a) \text{ in } (1.3)\}. \tag{1.12}$$

If $u_a \in K_a$, we can assume that $u_a$ is nonnegative since $J_a(u) \geqslant J_a(|u|)$. As mentioned in the above introduction, $u_a$ satisfies (1.1) with a suitable $\lambda = \lambda_a$ and we can define

$$a^{**} := \sup\{l > 0 : e(a) \ \text{ has a unique nonnegative minimizer for all } a \in [0, l)\}, \tag{1.13}$$

and $0 < a^{**} \leqslant a^*$. Now we state our first main result as follows.

Theorem 1.1. *Suppose that $(V_1)$ holds. Then, for all $a \in [0, a^{**})$ and for a.e. $a \in [a^{**}, a^*)$, all minimizers of $e(a)$ satisfy (1.1) with the same Lagrange multiplier $\lambda = \lambda_a$ and $K_a = G_{\lambda_a, a}$.*

Next, we focus on the concentration behaviour of nonnegative minimizers of (1.3) as $a \nearrow a^*$. To our best knowledge, currently only in [**6**], the authors have studied the concentration behaviour of nonnegative minimizers of (1.3) when $V(x) = h(x)\Pi_{i=1}^n |x - x_i|^{q_i}$, $q_i \in (0, 2\,s)$ and $C < h(x) < 1/C$ for some $C > 0$ and all $x \in \mathbb{R}^2$. Note that the method in [**6**] relies heavily on the fact that $V(x)$ has a finite number of minima $\{x_i \in \mathbb{R}^2, i = 1, \cdots, n\}$.

So what happens if $V(x)$ has infinitely many minima. Therefore, another main purpose of this paper is to study the concentration behaviour of mass critical fractional Schrödinger energy functional with a potential $V(x)$ with infinitely many minima. Therefore, in order to study this problem, it is assumed that $V$ satisfies the following explicit expression

$$V(x) = (|x| - M)^2, \quad \text{where} \quad M > 0, x \in \mathbb{R}^2. \tag{1.14}$$

Obviously, $V$ is a ring-shaped trapping potential and all points in set $\{x \in \mathbb{R}^2 : |x| = M\}$ are minima of $V(x)$. We state our main result as follows.

Theorem 1.2. *Let $V(x)$ be given by (1.14) and let $u_a$ be a nonnegative minimizer of (1.3) for $a < a^*$. For any given a sequence $\{a_k\}$ with $a_k \nearrow a^*$ as $k \to \infty$, there exists a subsequence, still denoted by $\{a_k\}$, such that each $u_{a_k}$ has a unique maximum point*

$x_k$ *and* $x_k \to y_0$ *as* $k \to \infty$ *for some* $y_0 \in \mathbb{R}^2$ *satisfying* $|y_0| = M > 0$. *Moreover,*

$$\lim_{k\to\infty} \frac{|x_k| - M}{(a^* - a_k)^{\frac{1}{2+2s}}} = 0, \tag{1.15}$$

*and*

$$(a^* - a_k)^{\frac{1}{2+2s}} u_{a_k}\left(x_k + (a^* - a_k)^{\frac{1}{2+2s}} x\right) \xrightarrow{k} \frac{\mu_0 Q\left(\frac{\mu_0|x|}{s^{\frac{1}{2s}}}\right)}{s^{\frac{1}{2s}}\|Q\|_2}, \tag{1.16}$$

*where* $\mu_0 > 0$ *satisfies*

$$\mu_0 = \left(\frac{s^{\frac{1}{s}}}{\|Q\|_2^{2-2s} M^2} \int_{\mathbb{R}^2} |y_0 \cdot x|^2 Q^2 \,\mathrm{d}x\right)^{\frac{1}{2+2s}}. \tag{1.17}$$

REMARK 1.3. Now, we list some of the difficulties encountered in this study.

(1) In order to prove theorem 1.2, we need to make a direct and accurate estimate of the mass critical fractional Schrödinger energy functional, however, similarly as the proof of lemma 5.1 in [**6**], we can only obtain estimate of the following type

$$C_1(a^* - a)^{\frac{2}{2+s}} \leqslant e(a) \leqslant C_2(a^* - a)^{\frac{1}{1+s}} \quad \text{as } a \nearrow a^*, \tag{1.18}$$

see lemma 4.1 in § 4. Therefore, we need to use the method in [**11**] to estimate the mass critical fractional Schrödinger energy functional so that the power $\frac{2}{2+s}$ on the left side of (1.18) decreases to $\frac{1}{1+s}$, see lemma 4.5 in § 4.

(2) Compared with the Gross–Pitaevskii equations studied in [**11**], our minimization solution sequence does not satisfy the property of exponential decay, so we need to analyse the decay property of the sequence, see lemma 5.1 in § 5.

The next theorem shows that we can determine exactly the coefficients estimated in lemma 4.5.

THEOREM 1.4. *Let* $V(x)$ *be given by* (1.14), *then the mass critical fractional Schrödinger energy* $e(a)$ *satisfies*

$$\lim_{a\nearrow a^*} \frac{e(a)}{(a^* - a)^{\frac{1}{1+s}}} = \frac{(s+1)\mu_0^{2s}}{s\|Q\|_2^{2s}}, \tag{1.19}$$

*where* $\mu_0$ *is given in* (1.17).

Ultimately, we consider the occurrence of a symmetry breaking for the minimizers of $e(a)$. We have

THEOREM 1.5. *Let* $V(x)$ *be given by* (1.14). *Then there exist two positive constant* $a_*$ *and* $a_{**}$ *satisfying* $a_{**} \leqslant a_* < a^*$ *such that*

(i) $e(a)$ *has a unique nonnegative minimizer which is radially symmetric about the origin if* $a \in [0, a_{**})$.

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(ii) $e(a)$ *has infinitely many different nonnegative minimizers, which are not radially symmetric if* $a \in [a_*, a^*)$.

Throughout this paper, we shall make use of the following notations.

- For $\rho > 0$ and $z \in \mathbb{R}^2$, $B_\rho(z)$ denotes the ball of radius $\rho$ centred at $z$.
- The symbol $\rightharpoonup$ denotes weak convergence and the symbol $\to$ denotes strong convergence.
- $L^q(\mathbb{R}^2)$ denotes the usual Lebesgue space with norm $\|u\|_q := (\int_{\mathbb{R}^2} |u|^q \, \mathrm{d}x)^{\frac{1}{q}}$, $1 \leqslant q \leqslant \infty$.
- For any $x \in \mathbb{R}^2$, arg $x$ be the angle between $x$ and the positive $x$-axis, and $\langle x, y\rangle$ be the angle between the vectors $x$ and $y$.
- $C$, $C_i$ $(i = 1, 2, 3 \cdots)$ denotes various positive constants which may vary from one line to another and which is not important for the analysis of the problem.

The paper is organized as follows. In § 2, we present some preliminaries results. In § 3, we will prove theorem 1.1. In § 4, we will establish some preparatory energy estimates. Section 5 is devoted to proving theorems 1.2, 1.4 and 1.5.

## 2. Preliminaries

In this section, we give some lemmas which will be frequently used throughout the rest of the paper. First, we give the fractional Gagliardo–Nirenberg–Sobolev inequality. Taking $N = 2$ and $p = 2 + 2s$ in (1.13) in [6], we have

LEMMA 2.1. *For* $2 < p < 2_s^* = \frac{2}{1-s}$, *the fractional Gagliardo–Nirenberg–Sobolev inequality*

$$\int_{\mathbb{R}^2} |u|^{2+2s} \, \mathrm{d}x \leqslant \frac{1+s}{\|Q\|_2^{2s}} \left(\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x\right) \left(\int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x\right)^s, \quad u \in H^s(\mathbb{R}^2), \tag{2.1}$$

*is attained at a function* $Q(x)$ *with the following properties:*

(i) $Q(x)$ *is radial, positive, and strictly decreasing in* $|x|$.

(ii) $Q(x)$ *is a solution of the fractional Schrödinger equation* (1.7).

(iii) $Q(x)$ *belongs to* $H^{2s+1}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$ *and satisfies*

$$\frac{C_1}{1+|x|^{2+2s}} \leqslant Q(x) \leqslant \frac{C_2}{1+|x|^{2+2s}}, \quad x \in \mathbb{R}^2. \tag{2.2}$$

From (1.7) and (2.1), it follows that

$$\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} Q|^2 \, \mathrm{d}x = \int_{\mathbb{R}^2} |Q|^2 \, \mathrm{d}x = \frac{1}{1+s} \int_{\mathbb{R}^2} |Q|^{2+2s} \, \mathrm{d}x. \tag{2.3}$$

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Let $\eta : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $\eta(x) = 1$ for $|x| \leqslant 1$, $\eta(x) = 0$ for $|x| \geqslant 2$, $0 \leqslant \eta \leqslant 1$ and $|\nabla\eta| \leqslant 2$. Define

$$Q_\tau(x) = \eta(x/\tau)Q(x), \quad x \in \mathbb{R}^2, \tag{2.4}$$

for any $\tau > 0$, where $Q(x)$ is given in lemma 2.1. According to lemma 3.2 in [6], we have the following result.

Lemma 2.2. *Let $s \in (0, 1)$. Then the following estimate holds true:*

$$\int_{\mathbb{R}^4} \frac{|Q_\tau(x) - Q_\tau(y)|^2}{|x-y|^{2+2s}} \mathrm{d}x\mathrm{d}y \leqslant \int_{\mathbb{R}^4} \frac{|Q(x) - Q(y)|^2}{|x-y|^{2+2s}} \mathrm{d}x\mathrm{d}y + O(\tau^{-4s}),$$

*as $\tau \to \infty$.*

From [3], we can deduce the following compactness result.

Lemma 2.3. *Suppose that $(V_1)$ hold. Then the embedding $E \hookrightarrow L^q(\mathbb{R}^2)$ is compact for all $q \in [2, 2_s^*)$.*

By [19], we know that the following vanishing lemma for fractional Sobolev space.

Lemma 2.4. *Assume that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and it satisfies*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_\rho(y)} |u_n(x)|^2 \,\mathrm{d}x = 0,$$

*where $\rho > 0$. Then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$.*

## 3. Normalized ground states

In this section, we give the proof of theorem 1.1. First, we give a smoothness result about $e(a)$.

Lemma 3.1. *Suppose that $(V_1)$ holds. Then, for $a \in (0, a^*)$, the left and right derivatives of $e(a)$ always exist in $[0, a^*)$ and satisfy*

$$e'_-(a) = -\frac{1}{1+s}\alpha_a \quad \textit{and} \quad e'_+(a) = -\frac{1}{1+s}\beta_a,$$

*where*

$$\alpha_a := \inf\left\{\int_{\mathbb{R}^2} |u_a|^{2+2s} \,\mathrm{d}x : u_a \in K_a\right\},$$

$$\beta_a := \sup\left\{\int_{\mathbb{R}^2} |u_a|^{2+2s} \,\mathrm{d}x : u_a \in K_a\right\},$$

*and $K_a$ is given by (1.12).*

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*Proof.* By $(V_1)$, lemma 2.1 and the definition of $e(a)$, we have

$$0 = \inf_{x\in\mathbb{R}^2} V(x) \leqslant e(a) \leqslant e(0) = \delta_1, \quad \text{for } a \in [0, a^*), \tag{3.1}$$

where $\delta_1$ is the first eigenvalue of $(-\Delta)^s + V(x)$ in $E$. Moreover, lemma 2.1 also shows that

$$\begin{aligned} e(a) &= \int_{\mathbb{R}^2} (|(-\Delta)^{\frac{s}{2}} u_a|^2 + V(x)|u_a|^2)\, \mathrm{d}x - \frac{a}{1+s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\, \mathrm{d}x \\ &\geqslant \frac{a^*}{1+s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\, \mathrm{d}x - \frac{a}{1+s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\, \mathrm{d}x \\ &= \frac{a^*-a}{1+s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\, \mathrm{d}x. \end{aligned} \tag{3.2}$$

From (3.1)–(3.2), it follows that

$$\int_{\mathbb{R}^2} |u_a|^{2+2s}\, \mathrm{d}x \leqslant \frac{1+s}{a^*-a} e(a) \leqslant \frac{1+s}{a^*-a}\delta_1 \quad \text{for } a \in [0,\, a^*). \tag{3.3}$$

For any $a_1,\, a_2 \in [0,\, a^*)$, it is easy to see that

$$e(a_1) \geqslant e(a_2) + \frac{a_2-a_1}{1+s}\int_{\mathbb{R}^2} |u_{a_1}|^{2+2s}\, \mathrm{d}x, \quad \forall u_{a_1} \in K_{a_1}, \tag{3.4}$$

and

$$e(a_2) \geqslant e(a_1) + \frac{a_1-a_2}{1+s}\int_{\mathbb{R}^2} |u_{a_2}|^{2+2s}\, \mathrm{d}x, \quad \forall u_{a_2} \in K_{a_2}. \tag{3.5}$$

By (3.4)–(3.5), we get

$$\lim_{a_2\to a_1} e(a_2) = e(a_1),$$

which implies that

$$e(a) \in C([0, a^*), \mathbb{R}^+). \tag{3.6}$$

By using (3.4)–(3.5), for $u_{a_1} \in K_{a_1}$ and $u_{a_2} \in K_{a_2}$, we have

$$\frac{a_2-a_1}{1+s}\int_{\mathbb{R}^2} |u_{a_1}|^{2+2s}\, \mathrm{d}x \leqslant e(a_1) - e(a_2) \leqslant \frac{a_2-a_1}{1+s}\int_{\mathbb{R}^2} |u_{a_2}|^{2+2s}\, \mathrm{d}x. \tag{3.7}$$

Without loss of generality, we set $0 < a_1 < a_2 < a^*$, from (3.7), we can see that

$$-\frac{1}{1+s}\int_{\mathbb{R}^2} |u_{a_2}|^{2+2s}\, \mathrm{d}x \leqslant \frac{e(a_2)-e(a_1)}{a_2-a_1} \leqslant -\frac{1}{1+s}\int_{\mathbb{R}^2} |u_{a_1}|^{2+2s}\, \mathrm{d}x, \tag{3.8}$$

for $\forall u_{a_i} \in K_{a_i}$, $i = 1, 2$. This shows that

$$-\frac{1}{1+s}\inf_{u_{a_2}\in K_{a_2}}\int_{\mathbb{R}^2} |u_{a_2}|^{2+2s}\, \mathrm{d}x \leqslant \frac{e(a_2)-e(a_1)}{a_2-a_1} \leqslant -\frac{1}{1+s}\beta_{a_1}. \tag{3.9}$$

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By (3.3) and lemma 2.3, we know that $\{u_{a_2}\}$ is bounded in $E$. Up to a subsequence, we may assume that there exists $u \in E$ such that

$$\begin{cases} u_{a_2} \rightharpoonup u, & \text{in } E \text{ as } a_2 \searrow a_1, \\ u_{a_2} \to u, & \text{in } L^q(\mathbb{R}^2) \text{ for } q \in [2, 2_s^*). \end{cases}$$

By (3.6), we deduce that

$$e(a_1) = \lim_{a_2 \searrow a_1} e(a_2) = \lim_{a_2 \searrow a_1} J_{a_2}(u_{a_2}) \geqslant J_{a_1}(u) \geqslant e(a_1),$$

which implies that

$$u_{a_2} \to u \in E \quad \text{and} \quad u \in K_{a_1}.$$

Thus, (3.9) shows that

$$\begin{aligned} &-\frac{1}{1+s}\int_{\mathbb{R}^2} |u|^{2+2s}\,\mathrm{d}x \leqslant \liminf_{a_2 \searrow a_1} \frac{e(a_2) - e(a_1)}{a_2 - a_1} \\ &\leqslant \limsup_{a_2 \searrow a_1} \frac{e(a_2) - e(a_1)}{a_2 - a_1} \leqslant -\frac{1}{1+s}\beta_{a_1}. \end{aligned} \tag{3.10}$$

Moreover, by the definition of $\beta_a$, we get

$$\int_{\mathbb{R}^2} |u|^{2+2s}\,\mathrm{d}x \leqslant \beta_{a_1}. \tag{3.11}$$

In view of (3.10)–(3.11), we can obtain that

$$e'_+(a_1) = -\frac{1}{1+s}\beta_{a_1}.$$

Similarly, we also have

$$e'_-(a_1) = -\frac{1}{1+s}\alpha_{a_1}.$$

□

*Proof of theorem* 1.1. By using (3.7), we can see that

$$\begin{aligned} |e(a_1) - e(a_2)| &\leqslant \frac{1}{1+s}|a_2 - a_1| \max\left\{\int_{\mathbb{R}^2} |u_{a_1}|^{2+2s}\,\mathrm{d}x, \int_{\mathbb{R}^2} |u_{a_2}|^{2+2s}\,\mathrm{d}x\right\} \\ &\leqslant \frac{1}{1+s}\max\{\beta_{a_1}, \beta_{a_2}\}|a_2 - a_1|, \quad \forall a_1, a_2 \in [0, a^*), \end{aligned}$$

which implies that $e(a)$ is locally Lipschitz continuous in $[0,\, a^*)$. Thus, by using Rademacher's theorem, we know that $e(a)$ is differentiable for a.e. $a \in [0,\, a^*)$. Moreover, lemma 3.1 shows that $e'(a)$ exists for all $a \in [0,\, a^{**})$ and a.e. $a \in [a^{**},\, a^*)$ and

$$e'(a) = -\frac{1}{1+s}\int_{\mathbb{R}^2} |u|^{2+2s}\,\mathrm{d}x, \quad \forall u \in K_a. \tag{3.12}$$

Thus, we know that all minimizers of $e(a)$ have the same $L^{2+2s}(\mathbb{R}^2)$-norm. For $a \in [0,\, a^*)$, taking each $u_a \in K_a$ such that $e'(a)$ satisfies (3.12), then $u_a$ satisfies

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(1.1) for some Lagrange multiplier $\lambda_a \in \mathbb{R}$. By (1.1) and (3.12), we have

$$\lambda_a = e(a) - \frac{sa}{1+s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x = e(a) + sae'(a), \tag{3.13}$$

which implies that all minimizers of $e(a)$ satisfy equation (1.1) with the same Lagrange multiplier $\lambda_a$, that is,

$$(-\Delta)^s u_a + V(x)u_a = \lambda_a u_a + a u_a^{2s+1}, \quad \text{in } \mathbb{R}^2 \tag{3.14}$$

For $\forall \varphi \in E$, by (1.2) and (3.14), we deduce that

$$\begin{aligned}\langle I'_{\lambda_a,a}(u_a), \varphi\rangle = \int_{\mathbb{R}^2} (-\Delta)^{\frac{s}{2}} u_a (-\Delta)^{\frac{s}{2}}\varphi\,\mathrm{d}x + \int_{\mathbb{R}^2} (V(x)-\lambda_a)u_a\varphi\,\mathrm{d}x \\ - a\int_{\mathbb{R}^2} u_a^{1+2s}\varphi\,\mathrm{d}x = 0,\end{aligned}$$

which implies that $u_a \in S_{\lambda_a,a}$. Now, we show that $G_{\lambda_a,a}$ is nonempty. In fact, according to the definition of $G_{\lambda_a,a}$, we only need to show that for any $u \in S_{\lambda_a,a}$ one has

$$I_{\lambda_a,a}(u) \geqslant I_{\lambda_a,a}(u_a). \tag{3.15}$$

Let $u_a \in K_a$ and $u \in S_{\lambda_a,a}$, we have

$$\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} (V(x)-\lambda_a)|u|^2\,\mathrm{d}x = a\int_{\mathbb{R}^2} |u|^{2+2s}\,\mathrm{d}x, \tag{3.16}$$

which implies that

$$I_{\lambda_a,a}(u) = \frac{a}{2}\int_{\mathbb{R}^2} |u|^{2+2s}\,\mathrm{d}x - \frac{a}{2+2s}\int_{\mathbb{R}^2} |u|^{2+2s}\,\mathrm{d}x = \frac{sa}{2+2s}\int_{\mathbb{R}^2} |u|^{2+2s}\,\mathrm{d}x > 0. \tag{3.17}$$

Similarly, we also have

$$I_{\lambda_a,a}(u_a) = \frac{sa}{2+2s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x. \tag{3.18}$$

Define

$$\hat{u} = \frac{1}{\sqrt{\sigma}}u \quad \text{with} \quad \sigma = \int_{\mathbb{R}^2} |u|^2\,\mathrm{d}x.$$

Clearly, $\int_{\mathbb{R}^2} |\hat{u}|^2\,\mathrm{d}x = 1$. Thus, we get that $J_a(\hat{u}) \geqslant J_a(u_a)$. This shows that

$$I_{\lambda_a,a}(\hat{u}) = \frac{1}{2}J_a(\hat{u}) - \frac{\lambda_a}{2}\int_{\mathbb{R}^2} |\hat{u}|^2\,\mathrm{d}x \geqslant \frac{1}{2}J_a(u_a) - \frac{\lambda_a}{2}\int_{\mathbb{R}^2} |u_a|^2\,\mathrm{d}x = I_{\lambda_a,a}(u_a). \tag{3.19}$$

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Moreover, by using (3.16), we deduce that

$$
\begin{aligned}
I_{\lambda_a,a}(\hat{u}) &= \frac{1}{2\sigma}\int_{\mathbb{R}^2}[|(-\Delta)^{\frac{s}{2}}u|^2+(V(x)-\lambda_a)|u|^2]\,\mathrm{d}x \\
&\quad -\frac{a}{(2+2s)\sigma^{1+s}}\int_{\mathbb{R}^2}|u|^{2+2s}\,\mathrm{d}x \\
&= \frac{a}{2+2s}\frac{1}{\sigma}\left(1+s-\frac{1}{\sigma^s}\right)\int_{\mathbb{R}^2}|u|^{2+2s}\,\mathrm{d}x. \qquad (3.20)
\end{aligned}
$$

Let $g(\sigma) := \frac{1}{\sigma}(1+s-\frac{1}{\sigma^s})$. It is easy to check that $g(\sigma)$ achieves the unique global maximum at $\sigma = 1$. By (3.17), (3.19) and (3.20), we deduce that

$$
\begin{aligned}
I_{\lambda_a,a}(u_a) \leqslant I_{\lambda_a,a}(\hat{u}) &= \frac{a}{2+2s}g(\sigma)\int_{\mathbb{R}^2}|u|^{2+2s}\,\mathrm{d}x \\
&\leqslant \frac{a}{2+2s}g(1)\int_{\mathbb{R}^2}|u|^{2+2s}\,\mathrm{d}x \\
&= \frac{sa}{2+2s}\int_{\mathbb{R}^2}|u|^{2+2s}\,\mathrm{d}x = I_{\lambda_a,a}(u),
\end{aligned}
$$

which shows that (3.15) holds. Hence, $G_{\lambda_a,\,a}$ is nonempty.

Next, we prove that $K_a = G_{\lambda_a,\,a}$. For any given $a \in [0,\,a^*)$, consider any $u_a \in K_a$ and $u \in G_{\lambda_a,\,a}$. Clearly, $u \in S_{\lambda_a,\,a}$. From (3.17)–(3.20), it follows that

$$
\begin{aligned}
&\frac{sa}{2+2s}\int_{\mathbb{R}^2}|u|^{2+2s}\,\mathrm{d}x = I_{\lambda_a,a}(u) \leqslant I_{\lambda_a,a}(u_a) \\
&\leqslant I_{\lambda_a,a}(\hat{u}) = \frac{a}{2+2s}\frac{1}{\sigma}\left(1+s-\frac{1}{\sigma^s}\right)\int_{\mathbb{R}^2}|u|^{2+2s}\,\mathrm{d}x, \qquad (3.21)
\end{aligned}
$$

which implies that

$$
\sigma^{1+s}-\frac{1+s}{s}\sigma^s+\frac{1}{s}\leqslant 0. \qquad (3.22)
$$

Define $h(\sigma) := \sigma^{1+s}-\frac{1+s}{s}\sigma^s+\frac{1}{s}$. Taking the derivative of $h(\sigma)$, we have

$$
\begin{cases}
h'(\sigma) < 0, & 0<\sigma<1,\\
h'(\sigma) = 0, & \sigma = 1,\\
h'(\sigma) > 0, & \sigma > 1.
\end{cases} \qquad (3.23)
$$

By (3.22)–(3.23), we know that $\sigma = 1$, that is, $\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}x = 1$. Therefore, (3.21) shows that

$$
I_{\lambda_a,a}(u) = I_{\lambda_a,a}(u_a) \quad \text{and} \quad J_a(u) = J_a(u_a).
$$

This implies that $u \in K_a$ and $u_a \in G_{\lambda_a,\,a}$. □

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## 4. Estimates in the energy $e(a)$ as $a \nearrow a^*$

In this section, we mainly establish the following estimates on the energy $e(a)$ as $a \nearrow a^*$.

LEMMA 4.1. *Let $V(x)$ be given by (1.14). Then there exist two constants $C_1$, $C_2 > 0$, independent of $a$, $s$, such that*

$$C_1(a^* - a)^{\frac{2}{2+s}} \leqslant e(a) \leqslant C_2(a^* - a)^{\frac{1}{1+s}}, \quad \text{as } a \nearrow a^*. \tag{4.1}$$

*Proof.* For any $\kappa > 0$ and $u \in E$ with $\|u\|_2^2 = 1$, by lemma 2.1, we have

$$\begin{aligned} J_a(u) &\geqslant \int_{\mathbb{R}^2} (|x| - M)^2 |u|^2 \, \mathrm{d}x + \frac{a^* - a}{1+s} \int_{\mathbb{R}^2} |u|^{2+2s} \, \mathrm{d}x \\ &= \kappa + \int_{\mathbb{R}^2} [(|x| - M)^2 - \kappa]|u|^2 \, \mathrm{d}x + \frac{a^* - a}{1+s} \int_{\mathbb{R}^2} |u|^{2+2s} \, \mathrm{d}x \\ &\geqslant \kappa - \frac{s}{(1+s)(a^* - a)^{\frac{1}{s}}} \int_{\mathbb{R}^2} [\kappa - (|x| - M)^2]_+^{1+\frac{1}{s}} \, \mathrm{d}x, \end{aligned} \tag{4.2}$$

where $[\cdot]_+ = \max\{0, \cdot\}$ denotes the positive part. Taking the variable $r = M + \sqrt{\kappa} \sin\theta$ with $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$, by direct computation, we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^2} [\kappa - (|x| - M)^2]_+^{1+\frac{1}{s}} \, \mathrm{d}x = 2\pi \int_{M-\sqrt{\kappa}}^{M+\sqrt{\kappa}} [\kappa - (r - M)^2]^{1+\frac{1}{s}} r dr \\ &= 2\pi\kappa^{1+\frac{1}{s}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{2+\frac{2}{s}} (M + \sqrt{\kappa}\sin\theta)\sqrt{\kappa}\cos\theta d\theta \leqslant C\kappa^{\frac{2+3s}{2s}}, \end{aligned} \tag{4.3}$$

for $\kappa > 0$ small enough. Taking $\kappa = (\frac{a^* - a}{2C^s})^{\frac{2}{2+s}}$ in (4.2) and (4.3), we have

$$\begin{aligned} J_a(u) &\geqslant \left(\frac{a^* - a}{2C^s}\right)^{\frac{2}{2+s}} - \frac{sC}{(1+s)(a^* - a)^{\frac{1}{s}}} \left(\frac{a^* - a}{2C^s}\right)^{\frac{2+3s}{s(2+s)}} \\ &\geqslant (a^* - a)^{\frac{2}{2+s}} \frac{1}{2^{\frac{2}{2+s}} C^{\frac{2s}{2+s}}} \left(1 - \frac{s}{1+s} \frac{1}{2^{\frac{1}{s}}}\right). \end{aligned} \tag{4.4}$$

Clearly, we know that $1 - \frac{s}{1+s}\frac{1}{2^{\frac{1}{s}}} > 0$ since $s \in (\frac{1}{2}, 1)$. Hence, (4.4) shows that

$$e(a) \geqslant C_1(a^* - a)^{\frac{2}{2+s}}, \text{as } a \nearrow a^*.$$

On the other hand, set a cut-off function $\eta \in C_0^\infty(\mathbb{R}^2)$ such that $\eta(x) = 1$ for $|x| \leqslant 1$, $\eta(x) = 0$ for $|x| \geqslant 2$, $0 \leqslant \eta \leqslant 1$ and $|\nabla\eta| \leqslant 2$. Define

$$u(x) := A_{R,\tau} \frac{\tau}{\|Q\|_2} \eta\left(\frac{x - x_0}{R}\right) Q(\tau(x - x_0)), \tag{4.5}$$

where $x_0 \in \mathbb{R}^2$, $R$, $\tau > 0$ and $A_{R,\tau} > 0$ is chosen so that $\int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x = 1$. First, we show that $\lim_{R\tau\to\infty} A_{R,\tau} = 1$. In fact, by (4.5) and lemma 2.1, we can see that

$$\frac{1}{A_{R,\tau}^2} = \frac{1}{\|Q\|_2^2} \int_{\mathbb{R}^2} \eta^2\left(\frac{x}{R\tau}\right) Q^2(x) \, \mathrm{d}x = 1 + O((R\tau)^{-2-4s}), \text{as } R\tau \to \infty. \tag{4.6}$$

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Now, taking $R = 1$ in (4.5), from lemma 2.1, lemma 2.2 and (4.6), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u|^2 \,\mathrm{d}x - \frac{a}{1+s} \int_{\mathbb{R}^2} |u|^{2+2s} \,\mathrm{d}x \\
&\leqslant \frac{\tau^{2s}}{\|Q\|_2^2} \left[ \int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} Q|^2 \,\mathrm{d}x - \frac{a}{(1+s)\|Q\|_2^{2s}} \int_{\mathbb{R}^2} |Q|^{2+2s} \,\mathrm{d}x + O(\tau^{-4s}) \right] \\
&= \frac{\tau^{2s}}{(1+s)\|Q\|_2^2} \left[ \left(1 - \frac{a}{\|Q\|_2^{2s}}\right) \int_{\mathbb{R}^2} |Q|^{2+2s} \,\mathrm{d}x + O(\tau^{-4s}) \right]. \qquad (4.7)
\end{aligned}$$

Moreover, from lemma 2.1, by direct computation, we get

$$\int_{\mathbb{R}^2} (|x| - M)^2 |u|^2 \,\mathrm{d}x \leqslant \frac{C}{\tau^2} \int_{\mathbb{R}^2} |x|^2 |Q|^2 \,\mathrm{d}x \leqslant \frac{C}{\tau^2}. \qquad (4.8)$$

In view of (4.7) and (4.8), we can deduce that

$$e(a) \leqslant C\tau^{2s}(a^* - a) + \frac{C}{\tau^2} + O(\tau^{-4s}). \qquad (4.9)$$

Taking $\tau = (a^* - a)^{-\frac{1}{2+2s}}$ in (4.9), we get

$$e(a) \leqslant C_2 (a^* - a)^{\frac{1}{1+s}}, \text{as } a \nearrow a^*.$$

□

Similar to the proofs of lemma 2.2 and 2.3 in [**11**], we can obtain the following two main results, which are extensions of the classical local problem in [**11**] to the nonlocal problem.

LEMMA 4.2. *Let* $V(x)$ *be given by* (1.14) *and suppose* $u_a$ *is a nonnegative minimizer of* (1.3), *then there exists a constant* $K > 0$, *independent of* $a$, $s$, *such that*

$$0 < K(a^* - a)^{-\frac{s}{2+2s}} \leqslant \int_{\mathbb{R}^2} |u_a|^{2+2s} \,\mathrm{d}x \leqslant \frac{1}{K}(a^* - a)^{-\frac{s}{1+s}}, \ \textit{as } a \nearrow a^*. \qquad (4.10)$$

*Proof.* From (4.2), it follows that

$$e(a) = J_a(u_a) \geqslant \frac{a^* - a}{1+s} \int_{\mathbb{R}^2} |u_a|^{2+2s} \,\mathrm{d}x,$$

which implies that the upper bounded of (4.10) since lemma 4.1.

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Moreover, for any $0 < b < a < a^*$, we have

$$e(b) \leqslant J_b(u_a) = e(a) + \frac{a-b}{1+s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x.$$

Then, lemma 4.1 shows that

$$\frac{1}{1+s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x \geqslant \frac{e(b)-e(a)}{a-b} \geqslant \frac{C_1(a^*-b)^{\frac{2}{2+s}} - C_2(a^*-a)^{\frac{1}{1+s}}}{a-b}. \tag{4.11}$$

Taking $b = a - C_3(a^*-a)^{\frac{2+s}{2+2s}}$ in (4.11), where $C_3 > 0$ is large enough such that $C_1 C_3^{\frac{2}{2+s}} > 2C_2$. Then, we can see that

$$\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x \geqslant C(a^*-a)^{-\frac{s}{2+2s}},$$

which implies that the lower bounded of (4.10). □

Lemma 4.3. *Let $V(x)$ be given by (1.14) and suppose $u_a$ is a nonnegative minimizer of (1.3), and set*

$$\epsilon_a^{-2s} := \int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u_a|^2\,\mathrm{d}x. \tag{4.12}$$

*Then, we have*

(i) $\epsilon_a \to 0$ *as* $a \nearrow a^*$.

(ii) *There exist a sequence $\{y_{\epsilon_a}\} \subset \mathbb{R}^2$ and positive constants $R_0$, $\eta$ such that the sequence*

$$w_a(x) := \epsilon_a u_a(\epsilon_a x + \epsilon_a y_{\epsilon_a}) \tag{4.13}$$

*satisfies*

$$\liminf_{a\nearrow a^*} \int_{B_{R_0}(0)} |w_a|^2\,\mathrm{d}x \geqslant \eta > 0. \tag{4.14}$$

(iii) *The sequence $\{\epsilon_a y_{\epsilon_a}\}$ is bounded uniformly for $\epsilon_a \to 0$. Moreover, for any sequence $\{a_k\}$ with $a_k \nearrow a^*$, there exists a convergent subsequence, still denoted by $\{a_k\}$, such that*

$$\bar{x} := \epsilon_{a_k} y_{\epsilon_{a_k}} \to x_0, \text{ as } a_k \nearrow a^*, \tag{4.15}$$

*for some $x_0 \in \mathbb{R}^2$ being a global minimum point of $V(x)$, i.e., $|x_0| = M > 0$. Furthermore, we also have*

$$w_{a_k} \xrightarrow{k} \frac{\beta_1}{s^{\frac{1}{2s}}\|Q\|_2} Q\left(\frac{\beta_1}{s^{\frac{1}{2s}}}|x - \bar{y}_0|\right) \tag{4.16}$$

*in $H^s(\mathbb{R}^2)$ for some $\bar{y}_0 \in \mathbb{R}^2$ and $\beta_1 > 0$.*

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*Proof.* ($i$) By lemma 2.1 and lemma 4.1, we deduce that

$$\int_{\mathbb{R}^2} V(x)|u_a|^2\,\mathrm{d}x \leqslant e(a) \leqslant C_2(a^*-a)^{\frac{1}{1+s}}, \text{as } a \nearrow a^*, \tag{4.17}$$

and

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u_a|^2\,\mathrm{d}x - \frac{a}{1+s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x \\ &= \epsilon_a^{-2s} - \frac{a}{1+s}\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x \leqslant e(a) \overset{a\nearrow a^*}{\longrightarrow} 0. \end{aligned} \tag{4.18}$$

Lemma 4.2 implies that

$$\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x \to +\infty, \text{ as } a \nearrow a^*. \tag{4.19}$$

By (4.18)–(4.19), we have

$$0 \leqslant \frac{\epsilon_a^{-2s}}{\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x} - \frac{a}{1+s} \leqslant \frac{e(a)}{\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x} \to 0, \text{ as } a \nearrow a^*,$$

that is,

$$\frac{\epsilon_a^{-2s}}{\int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x} \to \frac{a^*}{1+s}, \quad \text{as } a \nearrow a^*,$$

which implies that

$$0 < \frac{1}{m}\epsilon_a^{-2s} \leqslant \int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x \leqslant m\epsilon_a^{-2s}, \quad \text{as } a \nearrow a^*, \tag{4.20}$$

where $m = \max\{\frac{2}{a^*}, a^*\}$. Thus, from (4.20) and lemma 4.2, there exist $C_3$, $C_4 > 0$ such that

$$C_3(a^*-a)^{-\frac{s}{2+2s}} \leqslant \epsilon_a^{-2s} \leqslant C_4(a^*-a)^{-\frac{s}{1+s}}, \quad \text{as } a \nearrow a^*, \tag{4.21}$$

which implies that $\epsilon_a \to 0$ as $a \nearrow a^*$.

($ii$) Set

$$\tilde{w}_a(x) := \epsilon_a u_a(\epsilon_a x). \tag{4.22}$$

By (4.12) and (4.20), we have

$$\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} \tilde{w}_a|^2\,\mathrm{d}x = \int_{\mathbb{R}^2} |\tilde{w}_a|^2\,\mathrm{d}x = 1, \quad \frac{1}{m} \leqslant \int_{\mathbb{R}^2} |\tilde{w}_a|^{2+2s}\,\mathrm{d}x \leqslant m. \tag{4.23}$$

Next, we show that there exist a sequence $\{y_{\epsilon_a}\} \subset \mathbb{R}^2$ and $R_0$, $\eta > 0$ such that

$$\liminf_{\epsilon_a \to 0} \int_{B_{R_0}(y_{\epsilon_a})} |\tilde{w}_a|^2\,\mathrm{d}x \geqslant \eta > 0. \tag{4.24}$$

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Suppose by contradiction, for any $R > 0$, there exists a sequence $\{\tilde{w}_{a_k}\}$ with $a \nearrow a^*$ such that

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} |\tilde{w}_{a_k}|^2 \,\mathrm{d}x = 0.$$

From lemma 2.4, we get that $\tilde{w}_{a_k} \xrightarrow{k} 0$ in $L^r(\mathbb{R}^2)$ for $2 < r < 2_s^*$. Hence, $\tilde{w}_{a_k} \xrightarrow{k} 0$ in $L^{2+2s}(\mathbb{R}^2)$, which contradicts with (4.23). Therefore, from (4.22) and (4.24), we have

$$\liminf_{a\nearrow a^*} \int_{B_{R_0}(0)} |w_a|^2 \,\mathrm{d}x \geqslant \eta > 0.$$

(*iii*) From (4.13) and (4.17), it follows that

$$\int_{\mathbb{R}^2} (|x| - M)^2 |u_a|^2 \,\mathrm{d}x = \int_{\mathbb{R}^2} (|\epsilon_a x + \epsilon_a y_{\epsilon_a}| - M)^2 |w_a|^2 \,\mathrm{d}x \to 0, \quad \text{as } a \nearrow a^*. \tag{4.25}$$

Now, we prove that

$$\lim_{\epsilon_a\to 0} |\epsilon_a y_{\epsilon_a}| = M. \tag{4.26}$$

Indeed, assume by contradiction that there exist a constant $\alpha > 0$ and a subsequence $\{a_n\}$ with $a_n \nearrow a^*$ as $n \to \infty$ such that

$$\epsilon_n := \epsilon_{a_n} \to 0, \quad \text{and} \quad ||\epsilon_n y_{\epsilon_n}| - M| \geqslant \alpha > 0, \quad \text{as } n \to \infty.$$

By (4.14) and Fatou's lemma, we can see that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (|\epsilon_n x + \epsilon_n y_{\epsilon_n}| - M)^2 |w_{a_n}|^2 \,\mathrm{d}x \geqslant \frac{\alpha^2}{2}\eta > 0,$$

which gives a contradiction by (4.25). Thus, (4.26) shows that $\{\epsilon_a y_{\epsilon_a}\}$ is bounded uniformly as $\epsilon_a \to 0$ and (4.15) holds true.

Next, we prove that (4.16) holds. Since $u_a$ is a nonnegative minimizer of (1.3), we have

$$(-\Delta)^s u_a + (|x| - M)^2 u_a = \lambda_a u_a + a u_a^{2s+1}, \quad \text{in } \mathbb{R}^2, \tag{4.27}$$

where $\lambda_a \in \mathbb{R}$ is a Lagrange multiplier. Moreover, we also have

$$\lambda_a = e(a) - \frac{sa}{1+s} \int_{\mathbb{R}^2} |u_a|^{2+2s} \,\mathrm{d}x. \tag{4.28}$$

From (4.20), (4.28) and lemma 4.1, we can see that there exist $C_5, C_6 > 0$, independent of $a$, $s$, such that

$$-C_5 < \epsilon_a^{2s} \lambda_a < -C_6 < 0, \quad \text{as } a \nearrow a^*.$$

By (4.13) and (4.27), we deduce that

$$(-\Delta)^s w_a + \epsilon_a^{2s} (|\epsilon_a x + \epsilon_a y_{\epsilon_a}| - M)^2 w_a = \epsilon_a^{2s} \lambda_a w_a + a w_a^{2s+1}, \quad \text{in } \mathbb{R}^2. \tag{4.29}$$

Passing if necessary to a subsequence of $\{a_k\}$, still denoted by $\{a_k\}$, we may assume that

$$\begin{cases} \epsilon_k^{2s}\lambda_{a_k} \xrightarrow{k} -\beta_1^2 < 0, & \text{for some } \beta_1 > 0, \\ w_{a_k} \overset{k}{\rightharpoonup} w_0 \geqslant 0, & \text{in } H^s(\mathbb{R}^2). \end{cases}$$

From the boundedness of $\{\epsilon_a y_{\epsilon_a}\}$, by passing to the weak limit of (4.29), we get

$$(-\Delta)^s w_0 = -\beta_1^2 w_0 + a^* w_0^{2s+1}, \quad \text{in } \mathbb{R}^2. \tag{4.30}$$

Clearly, (4.14) implies that $w_0 \not\equiv 0$. Similar argument to the proof of proposition 4.4 in [**21**], we know that $w_0 \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$. Then, by lemma 3.2 in [**7**], we have

$$(-\Delta)^s w_0(x) = -\frac{1}{2}C(2, s)\int_{\mathbb{R}^2} \frac{w_0(x+y) + w_0(x-y) - 2w_0(x)}{|x-y|^{2+2s}} \mathrm{d}x\mathrm{d}y, \quad \forall x \in \mathbb{R}^2.$$

Next, we show that $w_0 > 0$. Assume by contradiction that there exists $x_0 \in \mathbb{R}^2$ such that $w_0(x_0) = 0$, then we can see that

$$(-\Delta)^s w_0(x_0) = -\frac{1}{2}C(2, s)\int_{\mathbb{R}^2} \frac{w_0(x_0+y) + w_0(x_0-y)}{|x_0-y|^{2+2s}} \mathrm{d}x\mathrm{d}y < 0,$$

since $w_0 \geqslant 0$ and $w_0 \not\equiv 0$. However, it is easy to see that

$$(-\Delta)^s w_0(x_0) = -\beta_1^2 w_0(x_0) + a^* w_0^{2s+1}(x_0) = 0,$$

which gives a contradiction. Hence $w_0 > 0$ for all $x \in \mathbb{R}^2$. Now, by (4.30) and $Q$ is the unique positive radial solution of (1.7), we can deduce that

$$w_0(x) = \frac{\beta_1}{s^{\frac{1}{2s}}\|Q\|_2} Q\left(\frac{\beta_1}{s^{\frac{1}{2s}}}|x - \bar{y_0}|\right), \text{for some } \bar{y_0} \in \mathbb{R}^2. \tag{4.31}$$

By simple computation, we know that $\|w_0\|_2^2 = 1$. From the norm preservation, we get that $w_{a_k} \xrightarrow{k} w_0$ in $L^2(\mathbb{R}^2)$. Hence, by the boundedness of $\{w_{a_k}\}$ in $H^s(\mathbb{R}^2)$, we have

$$w_{a_k} \xrightarrow{k} w_0, \text{ in } L^p(\mathbb{R}^2) \text{ for } p \in [2, 2_s^*).$$

Therefore, in view of (4.29) and (4.30), we know that $w_{a_k} \xrightarrow{k} w_0$ in $H^s(\mathbb{R}^2)$, and thus (4.16) holds. □

LEMMA 4.4. *Under the assumptions of lemma 4.3, and let $\{a_k\}$ be given by lemma 4.3-(iii). Then, for any $R > 0$, there exists $C(R) > 0$, independent of $a_k$, $s$, such that*

$$\lim_{\epsilon_{a_k}\to 0} \frac{1}{\epsilon_{a_k}^2}\int_{B_R(0)} (|\epsilon_{a_k} x + \epsilon_{a_k} y_{\epsilon_{a_k}}| - M)^2 |w_{a_k}|^2 \,\mathrm{d}x \geqslant C(R).$$

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*Proof.* The proof is parallel to lemma 2.4 in [**11**], for the reader's convenience, we give a brief proof. From lemma 4.3, we know that $\epsilon_{a_k} y_{\epsilon_{a_k}} \xrightarrow{k} x_0$ with $|x_0| = M > 0$. Hence. we get

$$y_{\epsilon_{a_k}} \xrightarrow{k} \infty, \tag{4.32}$$

which implies that

$$\frac{2x \cdot y_{\epsilon_{a_k}}}{|x|^2 + |y_{\epsilon_{a_k}}|^2} \xrightarrow{k} 0, \quad \text{uniformly for} x \in B_R(0). \tag{4.33}$$

Without loss of generality, we may assume that $x_0 = (M, 0)$. Then, arg $y_{\epsilon_{a_k}} \xrightarrow{k} 0$. By setting $0 < \theta < \frac{\pi}{16}$ small enough, we get that

$$-\theta < arg y_{\epsilon_{a_k}} < \theta, \text{as } \epsilon_{a_k} \to 0. \tag{4.34}$$

Let

$$\begin{aligned} \Delta^1_{\epsilon_{a_k}} &:= \left\{ x \in B_R(0) : \sqrt{|x|^2 + |y_{\epsilon_{a_k}}|^2} \leqslant \frac{M}{\epsilon_{a_k}} \right\} \\ &= \left\{ x \in B_R(0) : |x|^2 \leqslant \left(\frac{M}{\epsilon_{a_k}}\right)^2 - |y_{\epsilon_{a_k}}|^2 \right\}, \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} \Delta^2_{\epsilon_{a_k}} &:= \left\{ x \in B_R(0) : \sqrt{|x|^2 + |y_{\epsilon_{a_k}}|^2} > \frac{M}{\epsilon_{a_k}} \right\} \\ &= \left\{ x \in B_R(0) : \left(\frac{M}{\epsilon_{a_k}}\right)^2 - |y_{\epsilon_{a_k}}|^2 < |x|^2 < R^2 \right\}. \end{aligned} \tag{4.36}$$

Clearly, $B_R(0) = \Delta^1_{\epsilon_{a_k}} \cup \Delta^2_{\epsilon_{a_k}}$ and $\Delta^1_{\epsilon_{a_k}} \cap \Delta^2_{\epsilon_{a_k}} = \emptyset$. Next, we consider the following two cases.

**Case 1:** $|\Delta^1_{\epsilon_{a_k}}| \geqslant \frac{\pi R^2}{2}$. It is easy to check that $B_{\frac{R}{\sqrt{2}}}(0) \subset \Delta^1_{\epsilon_{a_k}}$. Let

$$\Delta^1 := \left(B_{\frac{R}{\sqrt{2}}}(0) \backslash B_{\frac{R}{2}}(0)\right) \cap \left\{ x : \frac{\pi}{2} + 2\theta < arg x < \frac{3\pi}{2} - 2\theta \right\} \subset \Delta^1_{\epsilon_{a_k}}.$$

By simple computation, we get

$$|\Delta^1| = \frac{(\pi - 4\theta)R^2}{8}. \tag{4.37}$$

By (4.34), we have

$$x \cdot y_{\epsilon_{a_k}} = |x||y_{\epsilon_{a_k}}| \cos\langle x, y_{\epsilon_{a_k}} \rangle < 0, \text{ for } x \in \Delta^1, \tag{4.38}$$

and

$$|\cos\langle x, y_{\epsilon_{a_k}} \rangle| > -\cos(\frac{\pi}{2} + \theta), \quad \text{for } x \in \Delta^1. \tag{4.39}$$

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Moreover, by (4.33) and the Taylor expansion, we have

$$\begin{aligned}
\frac{1}{\epsilon_{a_k}^2}(|\epsilon_{a_k}x+\epsilon_{a_k}y_{\epsilon_{a_k}}|-M)^2 &= \left(|x+y_{\epsilon_{a_k}}|-\frac{M}{\epsilon_{a_k}}\right)^2\\
&= \left|\sqrt{|x|^2+|y_{\epsilon_{a_k}}|^2}\sqrt{1+\frac{2x\cdot y_{\epsilon_{a_k}}}{|x|^2+|y_{\epsilon_{a_k}}|^2}}-\frac{M}{\epsilon_{a_k}}\right|^2\\
&= \left|\sqrt{|x|^2+|y_{\epsilon_{a_k}}|^2}\left(1+\frac{x\cdot y_{\epsilon_{a_k}}}{|x|^2+|y_{\epsilon_{a_k}}|^2}+O\left(\frac{1}{|y_{\epsilon_{a_k}}|^2}\right)\right)-\frac{M}{\epsilon_{a_k}}\right|^2\\
&= \left|\sqrt{|x|^2+|y_{\epsilon_{a_k}}|^2}+\frac{x\cdot y_{\epsilon_{a_k}}}{\sqrt{|x|^2+|y_{\epsilon_{a_k}}|^2}}-\frac{M}{\epsilon_{a_k}}+O\left(\frac{1}{|y_{\epsilon_{a_k}}|}\right)\right|^2. \qquad (4.40)
\end{aligned}$$

From (4.35), (4.38), (4.39) and (4.40), it follows that

$$\begin{aligned}
&\sqrt{|x|^2+|y_{\epsilon_{a_k}}|^2}+\frac{x\cdot y_{\epsilon_{a_k}}}{\sqrt{|x|^2+|y_{\epsilon_{a_k}}|^2}}-\frac{M}{\epsilon_{a_k}}+O\left(\frac{1}{|y_{\epsilon_{a_k}}|}\right)\\
&\leqslant \frac{x\cdot y_{\epsilon_{a_k}}}{\sqrt{|x|^2+|y_{\epsilon_{a_k}}|^2}}+O\left(\frac{1}{|y_{\epsilon_{a_k}}|}\right)\leqslant \frac{x\cdot y_{\epsilon_{a_k}}}{2\sqrt{|x|^2+|y_{\epsilon_{a_k}}|^2}}\\
&\leqslant \frac{|x||y_{\epsilon_{a_k}}|\cos\left(\frac{\pi}{2}+\theta\right)}{2\sqrt{|x|^2+|y_{\epsilon_{a_k}}|^2}},\ \text{for } x\in\Delta^1. \qquad (4.41)
\end{aligned}$$

Hence, by (4.32), (4.39) and (4.40), we have

$$\frac{1}{\epsilon_{a_k}^2}(|\epsilon_{a_k}x+\epsilon_{a_k}y_{\epsilon_{a_k}}|-M)^2 \geqslant \frac{\cos^2(\frac{\pi}{2}+\theta)|x|^2}{8},\ \text{for } x\in\Delta^1,$$

which implies that

$$\begin{aligned}
&\lim_{\epsilon_{a_k}\to 0}\frac{1}{\epsilon_{a_k}^2}\int_{B_R(0)}(|\epsilon_{a_k}x+\epsilon_{a_k}y_{\epsilon_{a_k}}|-M)^2|w_{a_k}|^2\,\mathrm{d}x\\
&\geqslant \lim_{\epsilon_{a_k}\to 0}\frac{1}{\epsilon_{a_k}^2}\int_{\Delta^1}(|\epsilon_{a_k}x+\epsilon_{a_k}y_{\epsilon_{a_k}}|-M)^2|w_{a_k}|^2\,\mathrm{d}x\\
&\geqslant \frac{\cos^2\frac{11\pi}{20}}{8}\int_{\Delta^1}|x|^2|w_0|^2\,\mathrm{d}x := C(R)>0,
\end{aligned}$$

where $\theta=\frac{\pi}{20}$.

**Case 2:** $|\Delta^2_{\epsilon_{a_k}}|\geqslant\frac{\pi R^2}{2}$. Clearly, $B_0(R)\backslash B_{\frac{R}{\sqrt{2}}}(0)\subset\Delta^2_{\epsilon_{a_k}}$. Set

$$\Delta^2 := \left(B_R(0)\backslash B_{\frac{R}{\sqrt{2}}}(0)\right)\cap\left\{x: -\frac{\pi}{2}+2\theta<arg x<\frac{\pi}{2}-2\theta\right\}\subset\Delta^2_{\epsilon_{a_k}}.$$

The rest of the proof is very similar to the case 1, we omit it. □

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LEMMA 4.5. *There exist two positive constants $C_7$ and $C_8$, independent of $a$, $s$, such that*

$$C_7(a^*-a)^{\frac{1}{1+s}} \leqslant e(a) \leqslant C_8(a^*-a)^{\frac{1}{1+s}}, \text{ as } a \nearrow a^*.$$

*Proof.* By lemma 4.1, it suffices to prove that there exists a $C>0$, independent of $a$, $s$, such that

$$e(a) \geqslant C(a^*-a)^{\frac{1}{1+s}}, \text{ as } a \nearrow a^*. \tag{4.42}$$

From lemma 4.3, we know that for any sequence $\{a_k\}$ with $a_k \nearrow a^*$, there exists a convergent subsequence, still denoted by $\{a_k\}$, such that $w_{a_k} \to w_0 > 0$ in $L^{2+2s}(\mathbb{R}^2)$, where $w_0$ satisfies (4.31). Thus, there exists $M_1>0$, independent of $a_k$, $s$, such that

$$\int_{\mathbb{R}^2} |u_{a_k}|^{2+2s}\,\mathrm{d}x \geqslant M_1, \text{ as } a_k \nearrow a^*. \tag{4.43}$$

Lemma 4.4 shows that there exists $M_2>0$, independent of $a_k$, $s$, such that

$$\int_{B_1(0)} (|\epsilon_k x + \epsilon_k y_{\epsilon_k}| - M)^2 |w_{a_k}|^2\,\mathrm{d}x \geqslant M_2\epsilon_k^2, \text{ as } a_k \nearrow a^*. \tag{4.44}$$

In view of (4.43)–(4.44), we deduce that

$$\begin{aligned} e(a_k) = J_{a_k}(u_{a_k}) &= \frac{1}{\epsilon_k^{2s}}\left[\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} w_{a_k}|^2\,\mathrm{d}x - \frac{a^*}{1+s}\int_{\mathbb{R}^2} |w_{a_k}|^{2+2s}\,\mathrm{d}x\right] \\ &\quad + \frac{a^*-a_k}{(1+s)\epsilon_k^{2s}}\int_{\mathbb{R}^2} |w_{a_k}|^{2+2s}\,\mathrm{d}x + \int_{\mathbb{R}^2} (|\epsilon_k x + \epsilon_k y_{\epsilon_k}| - M)^2 |w_{a_k}|^2\,\mathrm{d}x \\ &\geqslant \frac{a^*-a_k}{(1+s)\epsilon_k^{2s}} M_1 + M_2\epsilon_k^2 \\ &\geqslant \left\{\frac{M_1}{1+s}\left[\frac{sM_1}{(1+s)M_2}\right]^{-\frac{s}{1+s}} + M_2\left[\frac{sM_1}{(1+s)M_2}\right]^{\frac{1}{1+s}}\right\}(a^*-a_k)^{\frac{1}{1+s}}, \end{aligned}$$

as $a_k \nearrow a^*$ and here the last equality is achieved at

$$\epsilon_k = \left[\frac{sM_1(a^*-a_k)}{(1+s)M_2}\right]^{\frac{1}{2+2s}}.$$

Thus, (4.42) holds for the subsequence $\{a_k\}$. Actually, the above argument can be carried out for any subsequence $\{a_k\}$ satisfying $a_k \nearrow a^*$, which then implies that (4.42) holds for all $a \nearrow a^*$. □

Now, by using lemma 4.5, instead of using lemma 4.1 in the proof of lemma 4.2, and taking $b = a - C_3(a^*-a)$, we have

COROLLARY 4.6. *Let $V(x)$ be given by (1.14) and suppose $u_a$ is a nonnegative minimizer of (1.3), then there exists a constant $M>0$, independent of $a$, $s$, such that*

$$0 < M(a^*-a)^{-\frac{s}{1+s}} \leqslant \int_{\mathbb{R}^2} |u_a|^{2+2s}\,\mathrm{d}x \leqslant \frac{1}{M}(a^*-a)^{-\frac{s}{1+s}}, \text{ as } a \nearrow a^*.$$

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## 5. Concentration behaviour

In this last section we study the concentration behaviour of normalized ground states and give the proofs of theorems 1.2, 1.4 and 1.5. Let $u_a$ is a nonnegative minimizer of (1.3), we define

$$\varepsilon_a := (a^* - a)^{\frac{1}{2+2s}}. \tag{5.1}$$

By lemma 2.1, we deduce that

$$e(a) \geqslant \left(1 - \frac{a}{a^*}\right) \int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u_a|^2 \, \mathrm{d}x + \int_{\mathbb{R}^2} (|x| - M)^2 |u_a|^2 \, \mathrm{d}x.$$

Hence, from lemma 4.5, it follows that

$$\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} u_a|^2 \, \mathrm{d}x \leqslant C\varepsilon_a^{-2s} \quad \text{and} \quad \int_{\mathbb{R}^2} (|x| - M)^2 |u_a|^2 \, \mathrm{d}x \leqslant C\varepsilon_a^2. \tag{5.2}$$

Similar to the proof of (4.14), for $\varepsilon_a$ given by (5.1), we get that there exist a sequence $\{y_{\varepsilon_a}\} \subset \mathbb{R}^2$ and $R_0$, $\eta > 0$ such that

$$\liminf_{a \nearrow a^*} \int_{B_{R_0}(0)} |w_a|^2 \, \mathrm{d}x \geqslant \eta > 0, \tag{5.3}$$

where

$$w_a(x) := \varepsilon_a u_a(\varepsilon_a x + \varepsilon_a y_{\varepsilon_a}). \tag{5.4}$$

Moreover, by (5.2) and corollary 4.6, we have

$$\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} w_a|^2 \, \mathrm{d}x \leqslant C \quad \text{and} \quad M \leqslant \int_{\mathbb{R}^2} |w_a|^{2+2s} \, \mathrm{d}x \leqslant \frac{1}{M}. \tag{5.5}$$

Lemma 5.1. *For any given sequence $\{a_k\}$ with $a_k \nearrow a^*$, let $\varepsilon_k := \varepsilon_{a_k} = (a^* - a_k)^{\frac{1}{2+2s}} > 0$, $u_k(x) = u_{a_k}(x)$ be a nonnegative minimizer of (1.3), and $w_k := w_{a_k} \geqslant 0$ be defined by (5.4). Then, there is a subsequence, still denoted by $\{a_k\}$, such that*

$$z_k := \varepsilon_k y_{\varepsilon_k} \xrightarrow{k} y_0, \quad \text{for some } y_0 \in \mathbb{R}^2 \text{ and } |y_0| = M. \tag{5.6}$$

*Moreover, for any $\rho > 0$ small enough, we have*

$$u_k(x) = \frac{1}{\varepsilon_k} w_k\left(\frac{x - z_k}{\varepsilon_k}\right) \xrightarrow{k} 0. \quad \text{for any } x \in B_\rho^c(y_0). \tag{5.7}$$

*Proof.* We divide the proof into four steps. **Step 1.** By (4.27) and (5.4), we get

$$(-\Delta)^s w_k + \varepsilon_k^{2s}(|\varepsilon_k x + \varepsilon_k y_{\varepsilon_k}| - M)^2 w_k = \varepsilon_k^{2s} \lambda_k w_k + a_k w_k^{2s+1}, \text{ in } \mathbb{R}^2, \tag{5.8}$$

where $\lambda_k \in \mathbb{R}^2$ is a Lagrange multiplier. Similar to the proof of lemma 4.3-($iii$), we know that (5.6) holds.

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**Step 2.** For all $k$, we assume that $v_k \geqslant 0$ satisfies

$$(-\Delta)^s v_k - \varepsilon_k^{2s}\lambda_k v_k = a_k w_k^{2s+1}, \text{ in } \mathbb{R}^2. \tag{5.9}$$

Next, we prove that $\|v_k\|_\infty \leqslant C$, for all $k$. In fact, (5.8) shows that

$$(-\Delta)^s w_k - \varepsilon_k^{2s}\lambda_k w_k \leqslant a_k w_k^{2s+1}, \text{ in } \mathbb{R}^2. \tag{5.10}$$

From (5.9) and (5.10), it is easy to see that

$$0 \leqslant w_k \leqslant v_k, \text{ a.e. in } \mathbb{R}^2 \text{ and for all } k. \tag{5.11}$$

For $\beta \geqslant 1$ and $T > 0$, let

$$\varphi(t) = \begin{cases} 0, & t \leqslant 0, \\ t^\beta, & 0 < t < T, \\ \beta T^{\beta-1}(t-T) + T^\beta, & t \geqslant T. \end{cases}$$

Clearly, $\varphi$ is convex and Lipschitz continuous, we get

$$(-\Delta)^s \varphi(v_k) \leqslant \varphi'(v_k)(-\Delta)^s v_k, \tag{5.12}$$

in the weak sense. By using Sobolev inequality, (5.11), (5.12), the fact $\lambda_k < 0$, $\varphi'(v_k)\varphi(v_k) \leqslant \beta v_k^{2\beta-1}$, $v_k\varphi'(v_k) \leqslant \beta\varphi(v_k)$ and integrating by parts, we deduce that

$$\begin{aligned} \|\varphi(v_k)\|_{2_s^*}^2 &\leqslant C\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}}\varphi(v_k)|^2 \,\mathrm{d}x = C\int_{\mathbb{R}^2} \varphi(v_k)(-\Delta)^s\varphi(v_k)\,\mathrm{d}x \\ &\leqslant C\int_{\mathbb{R}^2} \varphi(v_k)\varphi'(v_k)(-\Delta)^s v_k \,\mathrm{d}x \\ &= C\int_{\mathbb{R}^2} \varphi(v_k)\varphi'(v_k)(\varepsilon_k^{2s}\lambda_k v_k + a_k w_k^{2s+1})\,\mathrm{d}x \\ &\leqslant C\int_{\mathbb{R}^2} \varphi(v_k)\varphi'(v_k)(1 + v_k^{2_s^*-1})\,\mathrm{d}x \\ &= C\left(\int_{\mathbb{R}^2} \varphi(v_k)\varphi'(v_k)\,\mathrm{d}x + \int_{\mathbb{R}^2} \varphi(v_k)\varphi'(v_k)v_k^{2_s^*-1}\,\mathrm{d}x\right) \\ &\leqslant C\beta\left(\int_{\mathbb{R}^2} v_k^{2\beta-1}\,\mathrm{d}x + \int_{\mathbb{R}^2} (\varphi(v_k))^2 v_k^{2_s^*-2}\,\mathrm{d}x\right), \end{aligned} \tag{5.13}$$

where $C > 0$ independent of $k$ and $\beta$.

Note that $\beta \geqslant 1$ and that $\varphi(v_k)$ is linear when $v_k \geqslant T$, then we have

$$\begin{aligned} \int_{\mathbb{R}^2} (\varphi(v_k))^2 v_k^{2_s^*-2}\,\mathrm{d}x &= \int_{\{v_k \leqslant T\}} (\varphi(v_k))^2 v_k^{2_s^*-2}\,\mathrm{d}x + \int_{\{v_k > T\}} (\varphi(v_k))^2 v_k^{2_s^*-2}\,\mathrm{d}x \\ &\leqslant T^{2\beta-2}\int_{\mathbb{R}^2} v_k^{2_s^*}\,\mathrm{d}x + C\int_{\mathbb{R}^2} v_k^{2_s^*}\,\mathrm{d}x < +\infty, \end{aligned}$$

which implies that $\int_{\mathbb{R}^2} (\varphi(v_k))^2 v_k^{2_s^*-2}\,\mathrm{d}x$ is well defined for every $T$.

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Now, we let $\beta$ in (5.13) such that $2\beta - 1 = 2_s^*$ and define $\beta_1 = \frac{2_s^*+1}{2}$. Let $R > 0$ be fixed later, by Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} (\varphi(v_k))^2 v_k^{2_s^*-2}\,\mathrm{d}x \\
&= \int_{\{v_k \leqslant R\}} (\varphi(v_k))^2 v_k^{2_s^*-2}\,\mathrm{d}x + \int_{\{v_k > R\}} (\varphi(v_k))^2 v_k^{2_s^*-2}\,\mathrm{d}x \\
&\leqslant R^{2_s^*-1} \int_{\{v_k \leqslant R\}} \frac{(\varphi(v_k))^2}{v_k}\,\mathrm{d}x \\
&\quad + \left(\int_{\{v_k > R\}} v_k^{2_s^*}\,\mathrm{d}x\right)^{\frac{2_s^*-2}{2_s^*}} \left(\int_{\mathbb{R}^2} (\varphi(v_k))^{2_s^*}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}}. \qquad (5.14)
\end{aligned}$$

Similar to the proof of lemma 4.3-($iii$), we know that $\{v_k\}$ converges strongly in $H^s(\mathbb{R}^2)$, then $\{v_k\}$ converges strongly in $L^{2_s^*}(\mathbb{R}^2)$, so we can choose $R$ sufficiently large such that

$$\left(\int_{\{v_k > R\}} v_k^{2_s^*}\,\mathrm{d}x\right)^{\frac{2_s^*-2}{2_s^*}} \leqslant \frac{1}{2C\beta_1}. \qquad (5.15)$$

From (5.13)–(5.15), it follows that

$$\left(\int_{\mathbb{R}^2} (\varphi(v_k))^{2_s^*}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \leqslant 2C\beta_1 \left(\int_{\mathbb{R}^2} v_k^{2_s^*}\,\mathrm{d}x + R^{2_s^*-1} \int_{\mathbb{R}^2} \frac{(\varphi(v_k))^2}{v_k}\,\mathrm{d}x\right). \qquad (5.16)$$

Thus, by applying $\varphi(v_k) \leqslant v_k^{\beta_1}$ and letting $T \to \infty$, we have

$$\left(\int_{\mathbb{R}^2} v_k^{2_s^*\beta_1}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \leqslant 2C\beta_1 \left(\int_{\mathbb{R}^2} v_k^{2_s^*}\,\mathrm{d}x + R^{2_s^*-1} \int_{\mathbb{R}^2} v_k^{2_s^*}\,\mathrm{d}x\right) < \infty,$$

which implies that

$$v_k \in L^{2_s^*\beta_1}(\mathbb{R}^2). \qquad (5.17)$$

Assume that $\beta > \beta_1$. By taking $T \to \infty$ in (5.13), we can see that

$$\left(\int_{\mathbb{R}^2} v_k^{2_s^*\beta}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \leqslant C\beta \left(\int_{\mathbb{R}^2} v_k^{2\beta-1}\,\mathrm{d}x + \int_{\mathbb{R}^2} v_k^{2\beta+2_s^*-2}\,\mathrm{d}x\right). \qquad (5.18)$$

Let

$$v_k^{2\beta-1} = v_k^l v_k^m,$$

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where $l = \frac{2_s^*(2_s^*-1)}{2(\beta-1)}$ and $m = 2\beta - 1 - l$. Moreover, $\beta > \beta_1$ implies that $0 < l$, $m < 2_s^*$, by using Young's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^2} v_k^{2\beta-1}\,\mathrm{d}x &\leqslant \frac{l}{2_s^*}\int_{\mathbb{R}^2} v_k^{2_s^*}\,\mathrm{d}x + \frac{2_s^*-l}{2_s^*}\int_{\mathbb{R}^2} v_k^{\frac{2_s^* m}{2_s^*-l}}\,\mathrm{d}x\\ &\leqslant \int_{\mathbb{R}^2} v_k^{2_s^*}\,\mathrm{d}x + \int_{\mathbb{R}^2} v_k^{2\beta+2_s^*-2}\,\mathrm{d}x\\ &\leqslant C\left(1+\int_{\mathbb{R}^2} v_k^{2\beta+2_s^*-2}\,\mathrm{d}x\right). \end{aligned} \tag{5.19}$$

In view of (5.18) and (5.19), we have

$$\left(\int_{\mathbb{R}^2} v_k^{2_s^*\beta}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \leqslant C\beta\left(1+\int_{\mathbb{R}^2} v_k^{2\beta+2_s^*-2}\,\mathrm{d}x\right), \tag{5.20}$$

which shows that

$$\left(1+\int_{\mathbb{R}^2} v_k^{2_s^*\beta}\,\mathrm{d}x\right)^{\frac{1}{2_s^*(\beta-1)}} \leqslant (C\beta)^{\frac{1}{2(\beta-1)}}\left(1+\int_{\mathbb{R}^2} v_k^{2\beta+2_s^*-2}\,\mathrm{d}x\right)^{\frac{1}{2(\beta-1)}}. \tag{5.21}$$

Iterating this argument, we obtain

$$\begin{aligned}&\left(1+\int_{\mathbb{R}^2} v_k^{2_s^*\beta_{i+1}}\,\mathrm{d}x\right)^{\frac{1}{2_s^*(\beta_{i+1}-1)}}\\ &\quad\leqslant (C\beta_{i+1})^{\frac{1}{2(\beta_{i+1}-1)}}\left(1+\int_{\mathbb{R}^2} v_k^{2_s^*\beta_i}\,\mathrm{d}x\right)^{\frac{1}{2(\beta_i-1)}}, \end{aligned} \tag{5.22}$$

where

$$2\beta_{i+1} + 2_s^* - 2 = 2_s^*\beta_i \quad \text{and} \quad \beta_{i+1} = (\frac{2_s^*}{2})^i(\beta_i - 1) + 1.$$

Setting $C_{i+1} = C\beta_{i+1}$ and

$$K_i = \left(1+\int_{\mathbb{R}^2} v_k^{2_s^*\beta_i}\,\mathrm{d}x\right)^{\frac{1}{2_s^*(\beta_i-1)}}.$$

We can see that there exists a constant $C > 0$ independent of $i$ , such that

$$K_{i+1} \leqslant \Pi_{i=2}^{i+1} C_i^{\frac{1}{2(\beta_i-1)}} K_1 \leqslant CK_1.$$

Hence, we have

$$\|v_k\|_\infty \leqslant C, \quad \text{for all } k.$$

**Step 3.** We prove that $w_k(x) \to 0$ as $|x| \to \infty$ uniformly in $k$.

In fact, we rewrite problem (5.9) as follows

$$(-\Delta)^s v_k + v_k = h_k(x), \quad x \in \mathbb{R}^2,$$

where $h_k(x) = v_k + \varepsilon_k^{2s}\lambda_k v_k + a_k w_k^{2s+1}$. Thus, step 2 shows that $h_k \in L^\infty(\mathbb{R}^2)$ and is uniformly bounded. From interpolation inequality and $\{v_k\}$ converges strongly in

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$H^s(\mathbb{R}^2)$, we know that $h_k \to h$ in $L^q(\mathbb{R}^2)$ for $q \in [2, +\infty)$. Thus, by [8], we deduce that

$$v_k = \int_{\mathbb{R}^2} \mathcal{K}(x-y)h_k(y)\mathrm{d}y,$$

where $\mathcal{K}$ is a Bessel potential and it satisfies

($\mathcal{K}_1$) $\mathcal{K}$ is positive, radially symmetric and smooth in $\mathbb{R}^2\backslash\{0\}$.

($\mathcal{K}_2$) There exists a $C>0$ such that $\mathcal{K}(x) \leqslant \frac{C}{|x|^{2+2s}}$ for $x \in \mathbb{R}^2\backslash\{0\}$.

($\mathcal{K}_3$) $\mathcal{K} \in L^r(\mathbb{R}^2)$ for $r \in [1, \frac{1}{1-s})$.

Now, for any $\zeta > 0$, we have

$$\begin{aligned} 0 \leqslant v_k &\leqslant \int_{\mathbb{R}^2} \mathcal{K}(x-y)|h_k(y)|\mathrm{d}y \\ &= \int_{\{|x-y| \geqslant \frac{1}{\zeta}\}} \mathcal{K}(x-y)|h_k(y)|\mathrm{d}y + \int_{\{|x-y| < \frac{1}{\zeta}\}} \mathcal{K}(x-y)|h_k(y)|\mathrm{d}y. \end{aligned}$$

By step 1 and ($\mathcal{K}_2$), we can see that

$$\int_{\{|x-y| \geqslant \frac{1}{\zeta}\}} \mathcal{K}(x-y)|h_k(y)|\mathrm{d}y \leqslant C\zeta \int_{\{|x-y| \geqslant \frac{1}{\zeta}\}} \frac{1}{|x-y|^{2+2s}}\mathrm{d}y = C\zeta^{2s}. \tag{5.23}$$

Moreover, by using Hölder's inequality and ($\mathcal{K}_3$), we deduce that

$$\begin{aligned} &\int_{\{|x-y| < \frac{1}{\zeta}\}} \mathcal{K}(x-y)|h_k(y)|\mathrm{d}y \\ &\leqslant \int_{\{|x-y| < \frac{1}{\zeta}\}} \mathcal{K}(x-y)|h_k - h|\mathrm{d}y + \int_{\{|x-y| < \frac{1}{\zeta}\}} \mathcal{K}(x-y)|h|\mathrm{d}y \\ &\leqslant \left(\int_{\mathbb{R}^2} |\mathcal{K}|^2 \mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |h_k - h|^2 \mathrm{d}y\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^2} |\mathcal{K}|^2 \mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\{|x-y| < \frac{1}{\zeta}\}} |h|^2 \mathrm{d}y\right)^{\frac{1}{2}}, \end{aligned}$$

which implies that there exist $K_0 \in \mathbb{N}$ and $R_0 > 0$ independent of $\zeta > 0$ such that

$$\int_{\{|x-y| < \frac{1}{\zeta}\}} \mathcal{K}(x-y)|h_k(y)|\mathrm{d}y \leqslant \zeta, \forall k \geqslant K_0 \text{ and } |x| \geqslant R_0, \tag{5.24}$$

where we have used the fact $s > \frac{1}{2}$ so that $2 < \frac{1}{1-s}$ and $(\int_{\{|x-y| < \frac{1}{\zeta}\}} |h|^2 \mathrm{d}y)^{\frac{1}{2}} \to 0$ as $|x| \to \infty$. Thus, by (5.23) and (5.24), we know that

$$\int_{\mathbb{R}^2} \mathcal{K}(x-y)|h_k(y)|\mathrm{d}y \leqslant C\zeta^{2s} + \zeta, \quad \forall k \geqslant K_0 \text{ and } |x| \geqslant R_0.$$

On the other hand, for all $k \in \{1, 2, \cdots, K_0 - 1\}$, there exists $R_k > 0$ such that

$$\left(\int_{\{|x-y| < \frac{1}{\zeta}\}} |h_k|^2 \mathrm{d}y\right)^{\frac{1}{2}} \leqslant \zeta, \text{ as } |x| \geqslant R_k,$$

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which implies that

$$\begin{aligned}\int_{\mathbb{R}^2}\mathcal{K}(x-y)|h_k(y)|\mathrm{d}y &\leqslant C\zeta^{2s}+\int_{\{|x-y|<\frac{1}{\zeta}\}}\mathcal{K}(x-y)|h_k(y)|\mathrm{d}y\\ &\leqslant C\zeta^{2s}+\|\mathcal{K}\|_2\left(\int_{\{|x-y|<\frac{1}{\zeta}\}}|h_k|^2\mathrm{d}y\right)^{\frac{1}{2}}\\ &\leqslant C(\zeta^{2s}+\zeta).\end{aligned}$$

Thus, setting $R=\max\{R_0,\, R_1,\, \cdots R_{K_0-1}\}$, we conclude that

$$0\leqslant v_k\leqslant\int_{\mathbb{R}^2}\mathcal{K}(x-y)|h_k(y)|\mathrm{d}y\leqslant C(\zeta^{2s}+\zeta),\quad \text{for all } |x|\geqslant R,$$

which implies

$$\lim_{|x|\to\infty}v_k(x)=0,\ \text{uniformly in } k. \tag{5.25}$$

From (5.11) and (5.25), it follows that

$$\lim_{|x|\to\infty}w_k(x)=0,\ \text{uniformly in } k. \tag{5.26}$$

**Step 4.** Combining step 2, step 3 and the proof of theorem 1.1 in [**22**], we can get that

$$w_k(x)\leqslant\frac{C}{1+|x|^{2+2s}},\ \text{for all } k. \tag{5.27}$$

For any $x\in B_\rho^c(y_0)$, (5.6) shows that

$$\frac{|x-z_k|}{\varepsilon_k}\geqslant\frac{|x-y_0|}{2\varepsilon_k}\geqslant\frac{\rho}{2\varepsilon_k}\xrightarrow{k}+\infty. \tag{5.28}$$

From (5.27) and (5.28), it follows that

$$\begin{aligned}u_k(x)=\frac{1}{\varepsilon_k}w_k\left(\frac{x-z_k}{\varepsilon_k}\right)&\leqslant\frac{1}{\varepsilon_k}\frac{C}{1+|\frac{x-z_k}{\varepsilon_k}|^{2+2s}}\\ &\leqslant\frac{1}{\varepsilon_k}\frac{C}{1+|\frac{\rho}{2\varepsilon_k}|^{2+2s}}\xrightarrow{k}0,\quad \forall x\in B_\rho^c(y_0).\end{aligned}$$

□

Inspired by [**25**], we now prove theorem 1.2.

*Proof of theorem 1.2.* Set $\varepsilon_k:=(a^*-a_k)^{\frac{1}{2+2s}}>0$, where $a_k\nearrow a^*$. Define $u_k(x):=u_{a_k}(x)$ is a nonnegative minimizer of (1.3). Moreover, we set $\bar{z}_k$ be any local

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maximum point of $u_k$. Clearly, we have

$$u_k(\bar{z}_k) \geqslant \left(\frac{-\lambda_k}{a_k}\right)^{\frac{1}{2s}} \geqslant C\varepsilon_k^{-1}. \tag{5.29}$$

Hence, from (5.29) and lemma 5.1, it follows that

$$\bar{z}_k \xrightarrow{k} y_0 \in \mathbb{R}^2 \text{ with } |y_0| = M. \tag{5.30}$$

Let

$$\bar{w}_k := \varepsilon_k u_k(\varepsilon_k x + \bar{z}_k). \tag{5.31}$$

By (5.8), we deduce that

$$(-\Delta)^s \bar{w}_k + \varepsilon_k^{2s}(|\varepsilon_k x + \bar{z}_k| - M)^2 \bar{w}_k = \varepsilon_k^{2s}\lambda_k \bar{w}_k + a_k \bar{w}_k^{2s+1}, \text{ in } \mathbb{R}^2. \tag{5.32}$$

Next, we prove that $\{\frac{\bar{z}_k - z_k}{\varepsilon_k}\} \subset \mathbb{R}^2$ is bounded uniformly in $k$. Assume by contradiction that $|\frac{\bar{z}_k - z_k}{\varepsilon_k}| \to \infty$ as $k \to \infty$. (5.27) shows that

$$u_k(\bar{z}_k) = \frac{1}{\varepsilon_k} w_k\left(\frac{\bar{z}_k - z_k}{\varepsilon_k}\right) \leqslant \frac{C}{\varepsilon_k}\frac{1}{1 + |\frac{\bar{z}_k - z_k}{\varepsilon_k}|^{2+2s}} = o(\varepsilon_k^{-1}), \text{ as } k \to \infty,$$

which implies a contradiction by (5.29). Thus, there exists $R_1 > 0$, independent of $k$, such that, $|\frac{\bar{z}_k - z_k}{\varepsilon_k}| < \frac{R_1}{2}$. By (5.3), we can see that

$$\begin{aligned} &\lim_{k\to\infty} \int_{B_{R_0+R_1}(0)} |\bar{w}_k|^2 \, \mathrm{d}x \\ &\quad = \lim_{k\to\infty} \int_{B_{R_0+R_1}\left(\frac{\bar{z}_k - z_k}{\varepsilon_k}\right)} |w_k|^2 \, \mathrm{d}x \geqslant \int_{B_{R_0+\frac{R_1}{2}}(0)} |w_k|^2 \, \mathrm{d}x \geqslant \eta > 0, \end{aligned} \tag{5.33}$$

where we have used the fact $\bar{w}_k(x) = w_k(x + \frac{\bar{z}_k - z_k}{\varepsilon_k})$. Similar to the argument of lemma 4.3-($iii$), we know that there exists a subsequence, still denoted by $\{\bar{w}_k\}$, of $\{\bar{w}_k\}$ such that

$$\begin{cases} \varepsilon_k^{2s}\lambda_k \xrightarrow{k} -\beta^2 < 0, & \text{for some } \beta > 0, \\ \bar{w}_k \xrightarrow{k} \bar{w}_0 \geqslant 0, & \text{in } H^s(\mathbb{R}^2), \end{cases} \tag{5.34}$$

where $\bar{w}_0$ satisfies

$$(-\Delta)^s \bar{w}_0 = -\beta^2 \bar{w}_0 + a^* \bar{w}_0^{2s+1}, \quad \text{in } \mathbb{R}^2. \tag{5.35}$$

Note from (5.33) that $\bar{w}_0 \not\equiv 0$. Thus, similar to the proof of lemma 4.3-($iii$), we know that $\bar{w}_0 > 0$ in $\mathbb{R}^2$. Since the origin is a critical point of $\bar{w}_k$, we get that the

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origin is also a critical point of $\bar{w}_0$. By (5.35) and $Q$ is the unique positive radial solution of (1.7), for the above $\beta > 0$, we can deduce that

$$\bar{w}_0(x) = \frac{\beta}{s^{\frac{1}{2s}}\|Q\|_2} Q\left(\frac{\beta}{s^{\frac{1}{2s}}}|x|\right). \tag{5.36}$$

Clearly, we know that $\bar{w}_k \geqslant (\frac{\beta^2}{2a^*})^{\frac{1}{2s}}$ at each local maximum point. Hence, lemma 5.1 implies that all the local maximum points of $\bar{w}_k$ stay in a finite ball in $\mathbb{R}^2$. By (5.26) and the definition of $\bar{w}_k$, we can get that

$$\|\bar{w}_k\|_\infty \leqslant C, \quad \text{uniformly as } k \to +\infty. \tag{5.37}$$

Next, we prove that $\{\bar{w}_k\}$ is bounded uniformly in $C^{2,\alpha}_{loc}(\mathbb{R}^2)$ for some $0 < \alpha < 1$. In fact, we rewrite (5.32) as follows

$$(-\Delta)^s \bar{w}_k(x) = f_k(x), \quad \text{in } \mathbb{R}^2, \tag{5.38}$$

where $f_k(x) = -\varepsilon_k^{2s}(|\varepsilon_k x + \bar{z}_k| - M)^2\bar{w}_k + \varepsilon_k^{2s}\lambda_k\bar{w}_k + a_k\bar{w}_k^{2s+1}$. Since $\varepsilon_k^{2s}(|\varepsilon_k x + \bar{z}_k| - M)^2$ is locally Lipschitz continuous in $\mathbb{R}^2$ and (5.37), we have

$$\|f_k(x)\|_\infty \leqslant C, \text{ uniformly as } k \to +\infty. \tag{5.39}$$

From (5.37), (5.38), (5.39) and lemma 2.3 in [**23**], we know that

$$\bar{w}_k \in C^{1,\alpha}(\mathbb{R}^2), \text{ for } \alpha < 2s - 1, \tag{5.40}$$

and

$$\|\bar{w}_k\|_{C^{1,\alpha}(\mathbb{R}^2)} \leqslant C(\|\bar{w}_k\|_\infty + \|f_k(x)\|_\infty) \leqslant C. \tag{5.41}$$

From (5.40) and (5.41), it follows that

$$\|f_k(x)\|_{C^1_{loc}(\mathbb{R}^2)} < C, \text{ uniformly as } k \to +\infty. \tag{5.42}$$

Thus, by (5.37), (5.38), (5.42) and lemma 4.4 in [**4**], we know that $\{\bar{w}_k\}$ is bounded uniformly in $C^{2,\alpha}_{loc}(\mathbb{R}^2)$ for some $0 < \alpha < 1$. Thus, we may assume that there exists $\hat{w}_0 \in C^{2,\alpha}_{loc}(\mathbb{R}^2)$ such that $\bar{w}_k \to \hat{w}_0$ in $C^2_{loc}(\mathbb{R}^2)$ as $k \to \infty$. Moreover, (5.34) shows that $\hat{w}_0 = \bar{w}_0$.

Since the origin is the only critical point of $\bar{w}_0$, then the content of the appeal discussion shows that all local maximum points of $\{\bar{w}_k\}$ must approach the origin and hence stay in a small ball $B_\varrho(0)$ as $k \to \infty$. Letting $\varrho > 0$ small enough such that $\bar{w}_0''(\tau) < 0$ for $0 < \tau < \varrho$. By lemma 4.2 in [**16**], we know that $\{\bar{w}_k\}$ has no critical points other than the origin. Therefore, we get that there exists a subsequence of $\{u_k\}$ concentrating at a unique global minimum point of potential $V(x) = (|x| - M)^2$ in $\mathbb{R}^2$.

Now, we turn to proving (1.15)–(1.17). In fact, by (5.31), we have

$$e(a_k) = J_{a_k}(u_k) = \frac{1}{\varepsilon_k^{2s}}\left[\int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}}\bar{w}_k|^2\,\mathrm{d}x - \frac{a^*}{1+s}\int_{\mathbb{R}^2}|\bar{w}_k|^{2+2s}\,\mathrm{d}x\right] + \frac{\varepsilon_k^2}{1+s}\int_{\mathbb{R}^2}|\bar{w}_k|^{2+2s}\,\mathrm{d}x + \int_{\mathbb{R}^2}(|\varepsilon_k x+\bar{z}_k| - |y_0|)^2|\bar{w}_k|^2\,\mathrm{d}x, \tag{5.43}$$

where $\bar{z}_k$ is the unique global maximum point of $u_k$, and $\bar{z}_k \to y_0 \in \mathbb{R}^2$ as $k \to \infty$ for some $|y_0| = M > 0$.

Next, we prove that $\{\frac{|\bar{z}_k|-|y_0|}{\varepsilon_k}\} \subset \mathbb{R}$ is bounded uniformly for $k \to \infty$. Assume by contradiction that there exists a subsequence $\{a_k\}$, still denoted by $\{a_k\}$ such that $|\frac{|\bar{z}_k|-|y_0|}{\varepsilon_k}| \to \infty$ as $k \to \infty$, by (5.33), for any $C > 0$, we have

$$\lim_{k\to\infty}\varepsilon_k^{-2}\int_{\mathbb{R}^2}(|\varepsilon_k x+\bar{z}_k| - |y_0|)^2|\bar{w}_k|^2\,\mathrm{d}x = \lim_{k\to\infty}\int_{\mathbb{R}^2}\left(|x+\frac{\bar{z}_k}{\varepsilon_k}| - \frac{|y_0|}{\varepsilon_k}\right)^2|\bar{w}_k|^2\,\mathrm{d}x \geqslant C. \tag{5.44}$$

From (5.43)–(5.44), it follows that

$$e(a_k) \geqslant C\varepsilon_k^2 = C(a^* - a_k)^{\frac{1}{1+s}},$$

holds for any $C > 0$, which implies a contradiction by lemma 4.5. Thus, there exists a subsequence $\{a_k\}$, still denoted by $\{a_k\}$ such that

$$\frac{|\bar{z}_k| - |y_0|}{\varepsilon_k} \to C_0,\ \text{as}\ k\to\infty, \tag{5.45}$$

for some constant $C_0$. Since $Q$ a radially symmetric function and polynomial decay as $|x| \to \infty$, we then deduce from (5.36) that

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{\varepsilon_k^2}&\int_{\mathbb{R}^2}(|\varepsilon_k x+\bar{z}_k| - |y_0|)^2|\bar{w}_k|^2\,\mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^2}\left(\frac{|\varepsilon_k x+\bar{z}_k| - |\bar{z}_k|}{\varepsilon_k} + \frac{|\bar{z}_k| - |y_0|}{\varepsilon_k}\right)^2|\bar{w}_k|^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left(\frac{y_0\cdot x}{|y_0|} + C_0\right)^2|\bar{w}_0|^2\,\mathrm{d}x \geqslant \int_{\mathbb{R}^2}\frac{|y_0\cdot x|^2}{|M|^2}|\bar{w}_0|^2\,\mathrm{d}x,\end{aligned} \tag{5.46}$$

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where the equality holds if and only if $C_0 = 0$. By (5.43) and (5.46), we have

$$\begin{aligned}\lim_{k\to\infty}\frac{e(a_k)}{(a^*-a_k)^{\frac{1}{1+s}}} &\geqslant \frac{1}{1+s}\|\bar{w}_0\|_{2+2s}^{2+2s}+\frac{1}{M^2}\int_{\mathbb{R}^2}|y_0\cdot x|^2|\bar{w}_0|^2\,\mathrm{d}x\\ &=\frac{1}{sa^*}\beta^{2s}+\frac{s^{\frac{1}{s}}}{\|Q\|_2^2\beta^2M^2}\int_{\mathbb{R}^2}|y_0\cdot x|^2|Q|^2\,\mathrm{d}x\\ &\geqslant\frac{1+s}{sa^*}\left(\frac{s^{\frac{1}{s}}}{\|Q\|_2^{2-2s}M^2}\int_{\mathbb{R}^2}|y_0\cdot x|^2|Q|^2\,\mathrm{d}x\right)^{\frac{s}{1+s}},\end{aligned}\tag{5.47}$$

where the equality is achieved at

$$\beta=\mu_0:=\left(\frac{s^{\frac{1}{s}}}{\|Q\|_2^{2-2s}M^2}\int_{\mathbb{R}^2}|y_0\cdot x|^2|Q|^2\,\mathrm{d}x\right)^{\frac{1}{2+2s}}.$$

We take

$$u(x)=\frac{\beta}{s^{\frac{1}{2s}}\varepsilon\|Q\|_2}Q\left(\frac{\beta}{s^{\frac{1}{2s}}}\frac{|x-y_0|}{\varepsilon}\right)$$

as a trial functional for $J_a$, and minimizes over $\beta > 0$. (5.47) shows that

$$\lim_{k\to\infty}\frac{e(a_k)}{(a^*-a_k)^{\frac{1}{1+s}}}=\frac{1+s}{sa^*}\left(\frac{s^{\frac{1}{s}}}{\|Q\|_2^{2-2s}M^2}\int_{\mathbb{R}^2}|y_0\cdot x|^2|Q|^2\,\mathrm{d}x\right)^{\frac{s}{1+s}}.\tag{5.48}$$

Therefore, from (5.48), we get the following several conclusions.

$(I)$ $\beta$ is unique, which is independent of the choice of the subsequence, and takes the value of $\mu_0$ as above.

$(II)$ $C_0 = 0$, that is, (1.15) holds.

Finally, by (5.30), (5.34) and (5.36), we have

$$(a^*-a_k)^{\frac{1}{2+2s}}u_{a_k}\left(x_k+(a^*-a_k)^{\frac{1}{2+2s}}x\right)\xrightarrow{k}\frac{\mu_0Q\left(\frac{\mu_0|x|}{s^{\frac{1}{2s}}}\right)}{s^{\frac{1}{2s}}\|Q\|_2}\text{ strongly in }H^s(\mathbb{R}^2),$$

that is, (1.16) holds. □

*Proof of theorem 1.4.* In fact, (5.48) shows that (1.19) holds for the subsequence $\{a_k\}$. Moreover, the proof of theorem 1.2 shows that (5.48) is correct for all $\{a_k\}$ with $a_k \nearrow a^*$. Therefore, (1.19) holds for all $a \nearrow a^*$. □

*Proof of theorem 1.5.* It then follows from theorem 1.2 that all nonnegative minimizers of $e(a)$ concentrate at any point on the ring $\{x \in \mathbb{R}^2 : |x| = M\}$. This further implies that there exists a $a_*$ satisfying $0 < a_* < a^*$ such that for any $a \in [a_*, a^*)$, $e(a)$ has infinitely many different nonnegative minimizers, each of which concentrates at a specific global minimum point of potential $V(x) = (|x| - M)^2$. However, recall from theorem 1.3-(ii) in [6] that $e(a)$ admits a unique nonnegative minimizer

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$u_a$ for all $a > 0$ being small enough ($a < a^*$), and noting that the trapping potential $V(x) = (|x| - M)^2$ ($M > 0$) is radially symmetric. Then similar to the argument of corollary 1.7 in [6], by rotation $u_a$ must be rotational symmetry with respect to the origin. □

## Data availability

All data, models, and code generated or used during the study appear in the submitted article.


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